Marking in Combinatorial Constructions: Generating Functions and Limiting Distributions

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Abstract

There is a wide field of combinatorial constructions, especially in the combinatorial analyis of algorithms, where it is possible to find an explicit generating function $y(x) = \sum y_n x^n$ for the numbers y_n of objects of size n and the bivariate generating function $y(x,u) = \sum y_{nk} x^n u^k$ for the numbers y_{nk} of objects of size n where another parameter has value k. Formally this additional parameter is marked in the above combinatorial construction. The aim of this paper is to provide general methods to obtain the asymptotic limiting distribution of this additional parameter in objects of size n.

We are especially interested in local limit theorems, which involves estimating the coefficients of powers of generating functions. When $y(x)$ is a function with a logarithmic singularity, we derive uniform approximations for $[x^n]y(x)^k$ for $k \le \varepsilon n$; and as a byproduct, we obtain conditionnal limiting distributions for the number of trees in random mappings where the number of cycles is given.

Product schemas $y(x,u) = g(x)F(uw(x))$ are also considered: we show how the limiting distribution may be dictated either by $g(x)$, or by $F(uw(x))$, or should involve both g and F; and give many combinatorial applications.

This paper studies asymptotic distributions in bivariate analytic schemas of the form $y(x,u) = \sum y_{nk} x^n u^k$, associated to combinatorial structures.

The generating function approach to combinatorics (see e.g. [21]) rests on general mechanisms of correspondance between combinatorial constructions and functional operations. In this way, the structural description $\mathcal{A} = \Phi(\mathcal{B}, \mathcal{C}, ...)$ of a class of combinatorial objects is directly translated into an equation on generating functions $A(x) = F(B(x), C(x), ...)$. Additional parameters of structures can be handled with multivariate generating functions. Moreover, this correspondance preserves analycity, and asymptotics methods in Analysis can be used to estimate the coefficients of the resulting functions.

It should be mentioned that such combinatorial constructions quite frequently appear in the combinatorial analysis of algorithms. In fact a systematic description was motivated by this application ([21]).

The study of limiting distributions in combinatorial schemas, initiated by Bender [1], has been recently investigated by several authors [3, 4, 9, 12]. The aim is to classify limiting distributions appearing in combinatorics according to structural and analytic characteristics of combinatorial constructions.

Consider for example the two classical results on permutations : the limiting distribution of the number of cycles is Gaussian, whereas the number of cycles of fixed length l is asymptotically Poisson distributed. From a "combinatorial schema" stand point, the Gaussian law is a direct consequence (see [8]) of the construction of permutations as *Sets of Cycles of points*, which leads to the bivariate schema $y(x,u) = \exp(u \log \frac{1}{1-z})$. On the other hand, the Poisson law results from the product schema $y(x,u) = \exp(\log \frac{1}{1-z} - \frac{z^l}{l} + u\frac{z^l}{l})$, where the factor of dominant importance $\frac{1}{1-z}\exp(-\frac{z^l}{l})$ implies the discrete nature of the distribution, and the Set construction in $\exp(u\frac{z^l}{l})$ determines that it is Poisson (see section 4).

This paper is divided into four sections.

In Section 1, we review the general mechanisms of translating combinatorial constructions into generating functions, and marking parameters to get multivariate functions. Finally we show how various statistical characteristics of the parameter of interest will be obtained by extracting coefficients of generating functions.

Section 2 is devoted to the analytic background. We first present two powerful methods to evaluate the coefficients of generating functions, the Saddle Point Method for admissible functions, and Singularity Analysis for alg-log functions. We also introduce the notion of acceptable functions, and show some closure properties with respect to combinatorial constructions.

The rest of this section is concerned with asymptotic distributions. We review some global limit theorems (convergence in distribution) and local limit theorems (convergence in density) for a sequence of random variables, in the context of combinatorial schemas. Actually, structures generated by classical combinatorial constructions are very regular from a statistical point of view: usually both a local and a global theorem hold, together with exponential tail estimates and asymptotic expansions for all centralized moments.

In Section 3 we discuss the asymptotic behaviour of coefficients of powers of functions $[x^n]y(x)^k$, where k tends to infinity. For combinatorial structures, these results can be applied to derive asymptotic densities in schemas $F(uy(x))$. We concentrate on logarithmic functions. Using singularity analysis and a saddle point method, we obtain uniform approximations for $[x^n]y(x)^k$, for $k \leq \varepsilon n$. As a direct application we get the conditionnal limiting law of the number of trees in random mappings where the number of cycles m is given: this law is Gamma when m is of order $\log n$, and Gaussian when m is of order $(\log n)^M$, $M > 1$.

Section 4 studies product schemas of the form $y(x,u) = g(x)F(uw(x))$. There are cases where the factor $g(x)$ has no influence on the limiting distribution, i.e. the limiting distribution of $y(x,u)$ is the same as the limiting distribution of $F(uw(x))$.

Inversely, $g(x)$ may be of dominant importance, and dictate the limiting distribution of $y(x,u)$.

Some dominancy criteria on acceptable functions are proved. When g is regular at the singular curve of $F(uw(x))$, we show that the limiting distribution is either Gaussian or discrete. And when g is of dominant importance, the limiting distribution is shown to be either Gaussian or Gamma or discrete. Various examples arising from combinatorics are discussed, and we also investigate some interesting cases where neither $g(x)$ dominates nor is dominated in $y(x,u)$.

1 Combinatorial Background

1.1 Marking in Combinatorial Constructions

We will use the concept of combinatorial structures described in [21]. A combinatorial structure $\mathcal{C}$ is a set of objects $o \in \mathcal{C}$ with size $|o|$ such that the set $\mathcal{C}_n = \{o \in \mathcal{C} \,|\, |o| = n\}$ of objects of size n is finite. For any combinatorial structure we can adjoin an ordinary generating function

$$c(x) = \sum_{o \in \mathcal{C}} x^{|o|} = \sum_{n \geq 0} c_n x^n \tag{1}$$

and an exponential generating function

$$\hat{c}(x) = \sum_{o \in \mathcal{C}} \frac{x^{|o|}}{(|o|)!} = \sum_{n \geq 0} c_n \frac{x^n}{n!}, \tag{2}$$

where $c_n = |\mathcal{C}_n|$ denotes the number of objects o of size $|o| = n$. Usually the ordinary generating function is used for unlabelled structures and the exponential generating function for labelled structures. (For details see [21].)

The essential point is that there are combinatorial constructions (such as disjoint union, product, sequence of, set of, multiset of, substitution,...) which correspond to simple transformations for the generating functions. The following table provides a list of such admissible constructions.

Example 1 : Let $\mathcal{P}$ denote the set of planted plane rooted trees. In order to count the number of trees of size n we will use the ordinary generating function $p(x)$. Planted plane trees can be recursively described as a node followed by a sequence of planted plane trees, or in terms of combinatorial constructions $\mathcal{P} = \circ * \texttt{sequence}(\mathcal{P})$, where $\circ$ stands for the combinatorial structure of a node, with generating function $\circ(x) = x$. This translates to $p(x) = x/(1-p(x))$ which gives $p(x) = (1-\sqrt{1-4x})/2$. Now it is an easy exercise by using Lagrange's inversion formula or the binomial series expansion to obtain that the number p_n of planted plane trees is given by the Catalan number

$$p_n = [x^n]p(x) = \frac{1}{n}\binom{2n-2}{n-1}. \quad \square \tag{3}$$

	unlabelled case (ogf)	labelled case (egf)
$\mathcal{S} = \mathcal{A} + \mathcal{B}$	$s(z) = a(z) + b(z)$	$\hat{s}(z) = \hat{a}(z) + \hat{b}(z)$
$\mathcal{S} = \mathcal{A} * \mathcal{B}$	$s(z) = a(z) * b(z)$	$\hat{s}(z) = \hat{a}(z) * \hat{b}(z)$
$\mathcal{S} = \texttt{sequence}(\mathcal{A})$	$s(z) = \frac{1}{1-a(z)}$	$\hat{s}(z) = \frac{1}{1-\hat{a}(z)}$
$\mathcal{S} = \texttt{cycle}(\mathcal{A})$	$s(z) = \sum\limits_{k\geq 1} \frac{\varphi(k)}{k} \log \frac{1}{1-a(z^k)}$	$\hat{s}(z) = \log \frac{1}{1-\hat{a}(z)}$
$\mathcal{S} = \texttt{set}(\mathcal{A})$	$s(z) = \exp\left(\sum\limits_{k\geq 1} \frac{(-1)^{k+1}}{k} a(z^k)\right)$	$\hat{s}(z) = \exp(\hat{a}(z))$

Table: Admissible constructions

Example 2 : Let $\mathcal{A}$ denote the set of labelled rooted trees, called Cayley trees. Here we have to use exponential generating functions. Since we have proposed that our trees $\mathcal{A}$ are not plane, $\mathcal{A}$ can be described as a node followed by a set of $\mathcal{A}$: $\mathcal{A} = \circ * \texttt{set}(\mathcal{A})$ Hence we get for the exponentially generating function $\hat{a}(x) = xe^{\hat{a}(x)}$ and by Lagrange's inversion formula

$$a_n = \left[\frac{x^n}{n!}\right] \hat{a}(x) = n^{n-1}. \quad \square$$

The next step is to describe more complex combinatorial structures by the use of simple objects.

Example 3 : A very prominent example is that of random mappings $\mathcal{R} = \{\varphi \,|\, \varphi : \{1, \ldots, n\} \to \{1, \ldots, n\}, n \geq 1\}$. The graph of such a mapping consisting of the edges $(i, \varphi(i))$ may be considered as a set of cycles of Cayley trees:

From $\mathcal{R} = \texttt{set}(\texttt{cycle}(\mathcal{A}))$ we directly get

$$\hat{r}(x) = \exp(\log \frac{1}{1-\hat{a}(x)}) = \frac{1}{1-\hat{a}(x)}. \ \square$$

The advantage of such combinatorial constructions is that we can formally mark a parameter in the constructions, by a symbol like $[u]$. And this marking directly

Figure 1: Graph of the function mapping (1,2,3,4,5,6,7,8,9) to (4,7,2,8,4,4,2,1,4)

leads to the bivariate generating function for the number of objects according to their size and the value of the parameter of interest.

For example, if we are interested in the number of trees in graphs of random mappings we have to mark the trees in the combinatorial construction and obtain $\mathcal{R} = \mathtt{set}(\mathtt{cycle}([u]\mathcal{A}))$. Formally this leads to $\hat{r}(x,u) = (1-u\hat{a}(x))^{-1}$ which is exactly the generating function $\hat{r}(x,u) = \sum r_{nk} \frac{x^n}{n!} u^k$ of the numbers r_{nk} of random mappings with k trees in the graph representation. And by Lagrange's inversion formula

$$r_{nk} = [x^n]\hat{a}(x)^k = \frac{k}{n}[z^{n-k}]e^{nz} = \frac{k}{n}\frac{n^{n-k}}{(n-k)!}.$$

Furthermore we can iterate this procedure by marking trees and cycles. From $\mathcal{R} = \mathtt{set}([v]\mathtt{cycle}([u]\mathcal{A}))$ we obtain

$$\hat{r}(x,u,v) = \exp\left(v \log \frac{1}{1-u\hat{a}(x)}\right) = \frac{1}{(1-u\hat{a}(x))^v} \tag{4}$$

and the number r_{nkm} of random mappings with k trees and m cycles (components) in their graph representation can be calculated by

$$\begin{aligned} r_{nkm} &= \left[\frac{x^n}{n!}u^k v^m\right] e^{v \log \frac{1}{1-u\hat{a}(x)}} = \frac{n!}{m!}[x^n u^k] \log^m \frac{1}{1-u\hat{a}(x)} \\ &= \frac{n!}{m!}[w^k] \log^m \frac{1}{1-w}[x^n]\hat{a}(x)^k = \binom{n}{k} k n^{n-k-1} |s_{km}|, \end{aligned} \tag{5}$$

where we have used that

$$[w^k]\log^m \frac{1}{1-w} = \frac{m!}{k!}|s_{km}|, \tag{6}$$

with s_{km} being the Stirling numbers of the first kind.

1.2 Generating Functions and Limiting Distributions

The main purpose of this paper is to provide some methods to characterize the limiting distribution of some marked parameter in a combinatorial construction. In order to describe this procedure more precisely we have to introduce a (hypothetical) sequence of random variables. Let $y(x,\mathbf{u}) = \sum y_{n\mathbf{k}} x^n \mathbf{u}^{\mathbf{k}}$ be a generating function in x and in the m-dimensional variable $\mathbf{u} = (u_1, \ldots, u_m)$. (As usual we use the notation $\mathbf{u}^{\mathbf{k}}$ for the product $u_1^{k_1} \cdots u_m^{k_m}$, where $\mathbf{k} = (k_1, \ldots, k_m)$ is an m-dimensional multiindex of non-negative integers k_j.) Let $y_n = [x^n]y(x,\mathbf{1})$ and consider (discrete) random variables $\mathbf{X}_n = (X_{n1}, \ldots, X_{nm})$ satisfying

$$\mathbf{Pr}[\mathbf{X}_n = \mathbf{k}] = \frac{y_{n\mathbf{k}}}{y_n}. \tag{7}$$

Then the expected value $\mathbf{E}\mathbf{X}_n = (\mathbf{E}X_{n1}, \ldots, \mathbf{E}X_{nm})$ is given by $\mathbf{E}X_{nj} = \frac{1}{y_n}[x^n]y_{u_j}(x,\mathbf{1})$ and the covariance matrix $\mathbf{Cov}\mathbf{X}_n = (\mathbf{E}X_{nj}X_{nl} - \mathbf{E}X_{nj}\mathbf{E}X_{nl})$ can be evaluated by $\mathbf{E}X_{nj}^2 = \frac{1}{y_n}[x^n](y_{u_j u_j}(x,\mathbf{1}) + y_{u_j}(x,\mathbf{1}))$ and by $\mathbf{E}X_{nj}X_{nl} = \frac{1}{y_n}[x^n]y_{u_j u_l}(x,\mathbf{1})$ for $j \neq l$.

There are three typical cases for the limit distribution: $\mathbf{X}_n$ tends to a discrete limit distribution or $\mathbf{X}_n/\sqrt{|\mathbf{Cov}\mathbf{X}_n|}$ tends to a one-sided continuous limit distribution or $\mathbf{X}_n$ satisfies a central limit theorem, i.e. $(\mathbf{X}_n - \mathbf{E}\mathbf{X}_n)/\sqrt{|\mathbf{Cov}\mathbf{X}_n|}$ tends to a normal distribution. In any of these three cases it is possible to characterize the limiting distribution by using the characteristic function

$$\phi_{\mathbf{X}_n}(\mathbf{t}) = \mathbf{E}e^{i\mathbf{X_n}} = \frac{1}{y_n}\sum_{\mathbf{k}} y_{n\mathbf{k}} e^{i\mathbf{t}\mathbf{k}}. \tag{8}$$

This is one way to get some information about the limiting distribution: we get the distribution function of the limit distribution, i.e. a global limit theorem. A most accurate information can be obtained by explicit or uniform asymptotic formulas for the density (7). By using such formulas we do not get only a precise information for the density of $\mathbf{X}_n$ in the range around the expected value, i.e. a local limit theorem with error terms, but even for the range away from the expected value, the so-called tail. Therefore our main interest is to provide uniform multivariate asymptotic expansions for the coefficient $y_{n\mathbf{k}}$.

1.3 Extracting the Coefficient

Of course there is no general asymptotic formula nor a general method to get precise information about the coefficient $y_{n\mathbf{k}}$ of a general generating function $y(x,\mathbf{u})$. Therefore our first aim is to simplify the general case bearing in mind that $y(x,\mathbf{u})$ has been generated according to admissible constructions, where some parameters of interest have been marked. First suppose that $m = 1$ and that we only mark the parameter of interest once. Then $y(x,u)$ has the shape $y(x,u) = G(x, uw(x))$. Moreover assume that we have some information about $[z^k]G(x,z) = g_k(x)$. Then we only have to consider

$$y_{nk} = [x^n]g_k(x)w(x)^k$$

which is in many cases much easier to handle than a direct evaluation of y_{nk}.

For example, let $\hat{y}(x,u) = (1 - xe^{u\hat{a}(x)})^{-1}$ be the generating function of random mappings where we have marked the nodes of the Cayley trees with distance 1 to the cycle, then $G(x,z) = (1-xe^z)^{-1}$ is much easier to handle than $y(x,u)$.

A very important special case of this already simplified version is when $y(x,u)$ has the form $y(x,u) = g(x)F(uw(x))$. Here we just have to evaluate

$$y_{nk} = [x^n]g(x)w(x)^k[z^k]F(z). \tag{9}$$

Many important applications are covered by this case, as we shall see in Section 4.

If we have marked more than one parameter then the situation is quite similar. For example in the case $m = 2$ two important special cases are of the form $y(x,u_1,u_2) = g(x)F_1(u_1w_1(x))F_2(u_2w_2(x))$ leading to the coefficient

$$y_{nk_1k_2} = [x^n]g(x)w_1(x)^{k_1}w_2(x)^{k_2}[z_1^{k_1}]F_1(z_1)[z_2^{k_2}]F_2(z_2);$$

and of the form $y(x,u_1,u_2) = g(x)F(u_1G(u_2w(x)))$, where we have to calculate

$$y_{nk_1k_2} = [x^n]g(x)w(x)^{k_2}[z_1^{k_1}]F_1(z_1)[z_2^{k_2}]G(z_2)^{k_1}.$$

(Marking trees and cycles in random mappings has been of this shape with $g(x) \equiv 1$.)

The advantage of these simplifications is that these cases reduce to evaluating the coefficients of a power series in *one variable*. However this power series mostly consists of powers of functions. Such problems can be treated by analytic means to be presented in Section 3.

2 Analytic Background

2.1 Saddle Point Methods and Singularity Analysis

As indicated in the previous section we mainly have to evaluate coefficients of power series. In what follows we will always assume that the coefficients are non-negative and that we are dealing with convergent power series. Hence Cauchy's formula

$$y_n = [x^n]y(x) = \frac{1}{2\pi i}\int\limits_{|x|=r} \frac{y(x)}{x^{n+1}}\,dx \tag{10}$$

applies for $0 < r < R$, where $R > 0$ denotes the radius of convergence of $y(x)$. Since we have the a-priori information $y_n \ge 0$ we get $\max\limits_{|x|=r} y(x) = y(r)$ and so we can estimate $y_n \le y(r)r^{-n}$. The minimum of $y(r)r^{-n}$ is obtained for a saddle point $\zeta_n \in (0,R)$ of $y(x)x^{-n}$. Thus we have $y_n \le \frac{y(\zeta_n)}{\zeta_n^n}$, where $\zeta_n \in (0,R)$ is defined by

$$\mu_y(\zeta_n) = \left[\frac{\partial}{\partial v}\log y(\zeta_n e^v)\right]_{v=0} = \frac{\zeta_n y'(\zeta_n)}{y(\zeta_n)} = n. \tag{11}$$

There is a large scale of functions where $y(x)x^{-n}$ is concentrated at the saddle point ζ_n on the circle contour $|x| = \zeta_n$ such that a usual saddle point approximation in the sense of Laplace applies:

$$y_n \sim \frac{y(\zeta_n)\zeta_n^{-n}}{\sqrt{2\pi\sigma^2(\zeta_n)}}, \qquad (n \to \infty) \tag{12}$$

where $\sigma_y^2(\zeta)$ is defined by

$$\sigma_y^2(\zeta) = \left[\frac{\partial^2}{\partial v^2}\log y(\zeta e^v)\right]_{v=0} = \frac{\zeta^2 y''(\zeta)}{y(\zeta)} - \left(\frac{\zeta y'(\zeta)}{y(\zeta)}\right)^2 + \frac{\zeta y'(\zeta)}{y(\zeta)}. \tag{13}$$

Such functions are called admissible in the sense of Hayman [17]. For example, $y(x) = e^x$ or $y(x) = e^{e^x}$ is admissible, and there is a list of construction principles, how to build up admissible functions ([17]).

However, there is an important class of functions which contains no admissible ones. Let $y(x) = (1-x)^{-\alpha}$ $(\alpha > 0)$ and let us try to apply (12). The saddle point evaluates to $\zeta_n = \left(1 + \frac{\alpha}{n}\right)^{-1} \sim 1 - \frac{\alpha}{n}$. Hence we obtain

$$\frac{y(\zeta_n)\zeta_n^{-n}}{\sqrt{2\pi\sigma^2(\zeta_n)}} \sim \frac{n^{\alpha-1}}{\sqrt{2\pi}\alpha^{\alpha-\frac{1}{2}}e^{-\alpha}}, \tag{14}$$

whereas the real coefficient satisfies

$$y_n = \binom{\alpha+n-1}{n} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}. \tag{15}$$

Of course, (14) and (15) are different and hence $y(x) = (1-x)^{-\alpha}$ is not admissible. Nevertheless (14) is a proper formula to get a first insight into the behaviour of y_n. (By the way the constant $\sqrt{2\pi}\alpha^{\alpha-\frac{1}{2}}e^{-\alpha}$ is the first term in Stirling's approximation formula for $\Gamma(\alpha)$.)

However there exist methods to find the right asymptotic behaviour of y_n for functions like $y(x) = (1-x)^{-\alpha}$. In [11] a very powerful method, called Singularity Analysis, is developed. A typical theorem reads like the following one. The essential tool for the proof is the use of a part of an Hankel contour around the singularity.

Theorem 1 ([11]) *Let $y(x)$ be analytic in a region $|x| < R + \varepsilon$, $|\arg(x - R)| > \phi$ $(\varepsilon > 0,\ 0 \le \phi < \frac{\pi}{2})$ such that*

$$y(x) \sim \frac{1}{(1-x/R)^\alpha}\left(\log\frac{1}{1-x/R}\right)^\beta \left(\log\left(\frac{R}{x}\log\frac{1}{1-x/R}\right)\right)^\gamma \quad (x \to R) \tag{16}$$

for some real $\alpha \notin \{0, -1, -2, \ldots\}$ and real β, γ. Then

$$y_n = [x^n]y(x) \sim R^{-n}\frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log n)^\beta(\log\log n)^\gamma \qquad (n \to \infty). \tag{17}$$

Actually for any choice of α, β, γ the coefficients of $y(x)$ in (16) can be asymptotically expanded (see [11]). In the sequel we will denote functions which are linear combinations of type (16) as *alglog-functions.* Observe that only one term of such an alglog-function determines the asymtotic behaviour of the coefficients.

Although there is a big difference between admissible functions and alglog-functions they have exactly those properties in common which are useful for our purposes: They appear in combinatorial constructions, and it is always possible to get the asymptotics for the cofficients. Moreover, the essential information about the order of magnitude of the coefficients is contained in $y(\zeta_n)\zeta_n^{-n}$, where ζ_n denotes the saddle point defined in (11). We will introduce the notion of acceptable functions satisfying proper closure properties with respect to sum, product, sequence, set and cycle construction of combinatorial structures.

Definition 1 *A function $y(x) = \sum_{n\geq 0} y_n x^n$ with non-negative coefficients y_n is called* **acceptable** *if it is a finite sum of functions which are admissible or alglog-functions.*

Theorem 2

- *If $f(x)$ and $g(x)$ are acceptable then $y(x) = f(x) + g(x)$ is acceptable.*
- *If $f(x)$ and $g(x)$ are acceptable such that $f(x)$ and $g(x)$ are entire or their radii of convergence are different. Then $y(x) = f(x)g(x)$ is acceptable.*
- *If $f(x)$ is acceptable then $y_1(x) = (1-f(x))^{-1}$ and $y_2(x) = \log(1-f(x))^{-1}$ are acceptable, too.*
- *If $f(x)$ is acceptable such that for all contained alglog-functions $\alpha > 0$ then $y(x) = e^{f(x)}$ is acceptable.*

Proof. The first property is obvious.

If $f(x)$ and $g(x)$ are entire then they are finite sums of entire admissible functions. Since the product of admissible functions is again admissible it immediately follows that $y(x) = f(x)g(x)$ is acceptable. Now suppose that $f(x)$ has finite radius of convergenc r and $g(x)$ has radius of convergence $R > r$, i.e. $g(x)$ is analytic at $x = r$. Let $f_1(x)$ denote the alglog part and $f_2(x)$ the admissible part of $f(x)$ with radius of convergence r. Since analytic terms can be interpreted as alglog-functions, i.e. $\alpha = \beta = \gamma = 0$, we have $f(x) = f_1(x) + f_2(x)$. Now $f_1(x)g(x)$ is an alglog-function and $f_2(x)g(x)$ is the sum of admissible functions (compare with Lemma 1) which implies that $y(x) = f(x)g(x)$ is acceptable.

If $R \leq \infty$ denotes the radius of convergence of $f(x)$ and if we assume that $\lim_{x\to R} f(x) > 1$ then $y_1(x) = (1 - f(x))^{-1}$ and $y_2(x) = \log(1 - f(x))^{-1}$ are obviously alglog-functions. Now suppose that $\lim_{x\to R} f(x) \leq 1$. Then $f(x)$ is an alglog-function. This follows from the fact that any admissible function $h(x)$ with finite radius of convergence R satisfies (see [17]) $\lim_{x\to R}(R-x)\frac{xh'(x)}{h(x)} = \infty$ and consequently $\lim\limits_{x\to R} h(x) = \infty$. Furthermore, $f(x)$ and can be represented by

$$f(x) = f(R) - C(1-x/R)^{\alpha'} L\left(\frac{1}{1-x/R}\right)(1+o(1-x/R)) \tag{18}$$

as $x \to R$ for some $\alpha' > 0$. Hence (18) implies that $y_1(x)$ and $y_2(x)$ are alglog-functions for any choice of $f(R) \le 1$.

If $f(x)$ is admissible then $e^{f(x)}$ is admissible (see [17]). Furthermore, if $f(x)$ is of type (16) with $\alpha > 0$ it follows from [17, Theorem XII] that $e^{f(x)}$ is admissible, too. Thus $y(x) = e^{f(x)}$ is admissible if $f(x)$ is acceptable and contains no alglog-functions with $\alpha \le 0$. $\square$.

Note that not all acceptable functions satisfy all closure properties, e.g.

$$y(x) = \exp\left(\left(\log\frac{1}{1-x}\right)^{\beta}(1+o(1-x))\right)$$

need not be admissible or an alglog-function. However, much more can be proved about the product property than stated in Theorem 2. Obviously the product of two alglog-functions is again an alglog-function. Furthermore we can prove a very general property for the product of an alglog-funtion and an admissible function.

Lemma 1 *Let $f(x)$ be an alglog-function with radius of convergence R and $g(x)$ an admissible function with the same radius of convergence such that*

$$\frac{\log\sigma_g^2(x)}{\sigma_g^2(x)} = o((1-x/R)^2)$$

as $x \to R$. Then $y(x) = f(x)g(x)$ is admissible.

Proof. $g(x)$ is admissible if there is a function $\delta(r)$ $(0 < \delta(r) < \pi)$ such that $g(re^{i\vartheta}) \sim g(r)e^{i\vartheta\mu_g(r)-\frac{1}{2}\vartheta^2\sigma_g^2(r)}$ as $r \to R$ uniformly for $|\vartheta| \le \delta(r)$ and $g(re^{i\vartheta}) = o\left(g(r)/\sqrt{\sigma_g(r)}\right)$ as $r \to R$ uniformly for $\delta(r) \le |\vartheta| \le \pi$. Thus w.l.o.g. we can assume the following upper bound for $\delta(r)$:

$$\delta(r) = \mathcal{O}\left(\sqrt{\frac{\log\sigma_g^2(r)}{\sigma_g^2(r)}}\right) = o((1-r/R)).$$

If $f(x)$ is an alglog-function then $\mu_f(r) = \mathcal{O}(1-r/R)$, $\sigma_f^2(r) = \mathcal{O}((1-r/R)^2)$, and $f(re^{i\vartheta}) = f(r)e^{i\vartheta\mu_f(r)+\mathcal{O}(\vartheta^2)\sigma_f^2(r)} \sim f(r)$ for $|\vartheta| \le \delta(r)$. Hence

$$f(re^{i\vartheta})g(re^{i\vartheta}) \sim f(r)g(r)e^{i\vartheta\mu_g(r)-\frac{1}{2}\vartheta^2\sigma_g^2(r)} \sim f(r)g(r)e^{i\vartheta\mu_{fg}(r)-\frac{1}{2}\vartheta^2\sigma_{fg}^2(r)}$$

uniformly for $|\vartheta| \le \delta(r)$. The property $f(re^{i\vartheta})g(re^{i\vartheta}) = o\left(f(r)g(r)/\sqrt{\sigma_{fg}^2(r)}\right)$ follows immediately. Thus $f(x)g(x)$ is admissible. $\square$

2.2 Global and Local Limit Laws

In order to get a first insight to the asymptotic behaviour of a combinatorial parameter of interest we just have to have to get some information about mean value and variance. But these characteristic parameters are not sufficient to characterize the kind of limiting distribution. There are different ways to describe the convergence of the (artificial) random variable X_n to its limit. Weak convergence or convergence in distribution of sequences of random variables can be rewritten in terms of the charcteristic function

$$\phi_X(t) = \mathbf{E}e^{itX}. \tag{19}$$

Proposition 1 (any textbook) *A sequence of random variables X_n converges to the random variable X in distribution if and only if the sequence $\phi_{X_n}(t)$ of corresponding characteristic functions tends for any fixed $t \in \mathbf{R}$ to a function $\phi(t)$ which is continuous at $t = 0$. $\phi(t)$ equals the characteristic function $\phi_X(t)$ of X.*

Weak limit theorem can also be proved by using the moment generating function

$$M_X(t) = \mathbf{E}e^{tX} \tag{20}$$

instead of the characteristic function.

Proposition 2 (any textbook) *Let X_n be a sequence of random variables such that $M_{X_n}(t)$ exist in an interval $I = [-\vartheta, \vartheta]$ ($\vartheta > 0$) and assume that $\lim_{n\to\infty} M_{X_n}(t) = M(t)$ exists for all $t \in I$. Then X_n converges in distribution to a random variable X with $M_X(t) = M(t)$.*

It is interesting that the converse statement need not be true (see [5]). Furthermore it may happen that the moment generating $M_X(t)$ does not exist for $t \neq 0$.

However, we are mainly interested in local limit theorems, i.e. in providing uniform multivariate asymptotic expansions for the coefficient $y_{n\mathbf{k}} = [x^n\mathbf{u}^\mathbf{k}]y(x,\mathbf{u})$. In fact, in the case of combinatorial constructions we can always prove a local version where a global limit theorem applies. Moreover these local theorems can be obtained by the same methods, maybe a little bit modified. From our point of view local theorem give a much more precise insight to the structure of the parameter of interest. Usually we get more than a local expansion around the explected value but a better understanding of the behaviour away from the mean and $\mathcal{O}$-error terms. In the next section we will also show that a combination of a weak limit theorem and a local limit theorem provides asymptotic expansions for all (centralized) moments.

One aspect which will not be discussed here is the fact that $y_{nk} = [x^n u^k]y(x,u)$ is often log-concave as a function of k, especially if $y(x,u)$ is of the form $y(x,u) = (1-uy(x))^{-1}$ (see [19]).

For completeness we present a general theorem due to Gao and Richmond [12] which generalizes results by Bender and Richmond [1, 3].

Theorem 3 ([12]) *Let $y(x,\mathbf{u}) = \sum \varphi_n(\mathbf{u})x^n$ and suppose that $\varphi_n(\mathbf{u})$ satisfies*

$$\varphi_n(\mathbf{u}) \sim a_n A(\mathbf{u}) r(\mathbf{u})^{-n} n^{\alpha(\mathbf{u})} (\log n)^{\beta(\mathbf{u})} \tag{21}$$

uniformly in some neighbourhood of $\mathbf{u} = \mathbf{1}$, where $A(\mathbf{u})$ is uniformly continuous and $r(\mathbf{u})$, $\alpha(\mathbf{u})$, $\beta(\mathbf{u})$ have continuous third partial derivatives and $a_n > 0$ is an arbitrary sequence. Set

$$\begin{aligned}
\mu_r(\mathbf{t}) &= \left(-\frac{\partial}{\partial v_j}\log r(e^{\mathbf{v}})\right)_{1\le j\le m}, &\quad \Sigma_r(\mathbf{t}) &= \left(-\frac{\partial^2}{\partial v_j \partial v_l}\log r(e^{\mathbf{v}})\right)_{1\le j,l\le m},\\
\mu_\alpha(\mathbf{t}) &= \left(\frac{\partial}{\partial v_j}\alpha(e^{\mathbf{v}})\right)_{1\le j\le m}, &\quad \Sigma_\alpha(\mathbf{t}) &= \left(\frac{\partial^2}{\partial v_j \partial v_l}\alpha(e^{\mathbf{v}})\right)_{1\le j,l\le m},\\
\mu_\beta(\mathbf{t}) &= \left(\frac{\partial}{\partial v_j}\beta(e^{\mathbf{v}})\right)_{1\le j\le m}, &\quad \Sigma_\beta(\mathbf{t}) &= \left(\frac{\partial^2}{\partial v_j \partial v_l}\beta(e^{\mathbf{v}})\right)_{1\le j,l\le m},
\end{aligned}$$

where $\mathbf{t} = e^{\mathbf{v}} = (e^{v_1},\ldots,e^{v_m})$.

If $\Sigma_r(\mathbf{1})$ is nonsingular then the numbers $y_{n\mathbf{k}} = [x^n\mathbf{u}^{\mathbf{k}}]y(x,\mathbf{u})$ satisfy a central limit theorem with mean value $\sim n\cdot\mu_r(\mathbf{1})$ and covariance matrix $\sim n\cdot\Sigma_r(\mathbf{1})$.

If $\Sigma_r(\mathbf{1}) = \mathbf{0}$ and $\Sigma_\alpha(\mathbf{1})$ is nonsingular then the numbers $y_{n\mathbf{k}}$ satisfy a central limit theorem with mean value $\sim n\cdot\mu_r(\mathbf{1}) + \log n\cdot\mu_\alpha(\mathbf{1})$ and covariance matrix $\sim \log n\cdot\Sigma_\alpha(\mathbf{1})$.

Finally, if $\Sigma_r(\mathbf{1}) = \Sigma_\alpha(\mathbf{1}) = \mathbf{0}$ and $\Sigma_\beta(\mathbf{1})$ is nonsingular then the numbers $y_{n\mathbf{k}}$ satisfy a central limit theorem with mean value $\sim n\cdot\mu_r(\mathbf{1}) + \log n\cdot\mu_\alpha(\mathbf{1}) + \log\log n\cdot\mu_\beta(\mathbf{1})$ and covariance matrix $\sim \log\log n\cdot\Sigma_\beta(\mathbf{1})$.

For example, consider functions behaving like

$$y(x,\mathbf{u}) = g(s)\left(1-\frac{x}{r(\mathbf{u})}\right)^{\alpha(\mathbf{u})}\left(\frac{r(\mathbf{u})}{x}\log\left(-\frac{x}{r(\mathbf{u})}\right)^{-1}\right)^{\beta(\mathbf{u})}(1+R(x,\mathbf{u})) \tag{22}$$

in some neighbourhood of $\mathbf{u} = \mathbf{1}$, where $g(\mathbf{u})$ is continuous and nonzero, $r(\mathbf{u})$, $\alpha(\mathbf{u})$, $\beta(\mathbf{u})$ have continuous third partial derivatives, and either $\alpha(\mathbf{u}) = \text{const.}$ or $\alpha(\mathbf{u}) \notin \{0,-1,-2,\ldots\}$. The function $R(x,\mathbf{u})$ should be analytic for $x \ne r(\mathbf{u})$, $|x| \le r(\mathbf{u})+\delta$, $|\arg(x-r(\mathbf{u}))| \ge \theta$ ($\delta > 0$, $0 < \theta < \frac{1}{2}\pi$) and should satisfy $R(x,\mathbf{u}) = o(1)$ uniformly for $x \to r(\mathbf{u})$. Then an analogue of Theorem 1 (see [12]) implies that $\varphi_n(\mathbf{u})$ has the form (21).

These functions obviously contain functions of the type $y(x,u) = g(x,u)e^{uG(x)}$, (considered in [8]), where

$$G(x) = a\log\frac{1}{1-x/x_0} + C + o(|\log^{-1}(1-x/x_0)|) \qquad (x \to x_0)$$

for some real constants $a > 0, C$. In this case we get a central limit theorem with mean value $\sim a\log n$ and variance $\sim a\log n$.

In any (reasonable) case it is possible to prove even local limit theorem (assuming proper additional assumptions.) One of the first general theorems was obtained by Bender and Richmond [3]. (For the remaining cases see [12].)

Theorem 4 ([3]) *Let $y(x,\mathbf{u}) = \sum \varphi_n(\mathbf{u})x^n$ and suppose that $\varphi_n(\mathbf{u})$ satisfies*

$$\varphi_n(\mathbf{u}) \sim a_n A(\mathbf{u}) r(\mathbf{u})^{-n} \tag{23}$$

uniformly in some open set V containing a real point $\mathbf{u}_0 > \mathbf{0}$, where $A(\mathbf{u})$ is uniformly continuous and $\lambda(\mathbf{u})$ has uniformly continuous third-order partials. Let $K \subseteq V$ be a compact set of positive reals and assume further that $\Sigma_r(\mathbf{t})$ is non-singular for $\mathbf{t} \in K$ and that $\varphi_n(\mathbf{u})/\varphi_n(|\mathbf{u}|) = o\left(n^{-\frac{m}{2}}\right)$ for $|\mathbf{u}| \in V$ and $\mathbf{u} \notin V$. Then we have

$$y_{n\mathbf{k}} = \frac{\varphi_n(\mathbf{t}_0)\mathbf{t}_0^{-\mathbf{k}_0}}{\sqrt{(2\pi n)^m |\det(\Sigma(\mathbf{t}_0)|}} \left(\exp\left(\frac{1}{2n}(\mathbf{k}-\mathbf{k}_0)\Sigma_r(\mathbf{t}_0)^{-1}(\mathbf{k}-\mathbf{k}_0)'\right) + o(1)\right)$$

uniformly for $\mathbf{t} \in K$ and all $\mathbf{k} \geq \mathbf{0}$, where $\mu_r(\mathbf{t}_0) = \mathbf{k}_0/n$.

It should be mentioned that it is not always the case that the limiting distribution is Gaussian although the Gaussian distribution appears quite frequently. In [10] functions of the type $y(x,u) = F(uw(x))$ are discussed, where $w(x)$ has a local expansion of the form

$$w(x) = \tau - c(1-x/r)^\lambda + \sum_{p\geq 2} c_p(1-x/r)^{p\lambda}$$

with $0 < \lambda < 1$ and $c > 0$. In this case many different limiting distributions appear, e.g. hypergeometric distributions.

In Section 4 we will prove some general theorems concerning combinatorial constructions which provide other examples with non-Gaussian limiting distribution.

2.3 Exponential Tails

The only disadvantage of pure local limit laws is that they are usually not sufficient to prove convergence of moments or even to prove the corresponding weak limit theorem rigorously. However, if we are in the position to prove a local limit theorem there is mostly a weak limit theorem, too, especially in the context of combinatorial constructions. In this case we easily get exponential tail estimates which can be used to obtain asymptotic expansions for the moments.

Proposition 3 ([9]) *Let Y_n be a sequence of random variables and suppose that there is an interval $I = [-\vartheta_1, \vartheta_2], (\vartheta_1, \vartheta_2 > 0)$ such that the moment generating functions $M_{Y_n}(t) = \mathbf{E}e^{tY_n}$ exist for $t \in I$ and that $M_{Y_n}(t) \leq C$ for all n and all $t \in I$. Then*

$$\mathbf{Pr}[|Y_n| > k] < C\alpha^k \qquad \textit{uniformly for all } n,$$

in which $\alpha = e^{-\min(\vartheta_1,\vartheta_2)}$.

Corollary 1 *Let Y_n be as in Proposition 3 and $\eta > 0$. Then*

$$\int_{|Y_n|\ge\eta} |Y_n|^m dP = \mathcal{O}\left(\eta^m \alpha^\eta\right).$$

Proof. Let $F_n(x) = \mathbf{Pr}[Y_n \le x]$ denote the distribution function of Y_n. Then by Theorem 3 $|F_n(x) - 1| \le C\alpha^m$ for $x \ge 0$. Hence we obain

$$\begin{aligned}
&\int_{Y_n\ge\eta} Y_n^m dP = \int_\eta^\infty x^m \, dF_n(x) \\
&= \lim_{M\to\infty}\left(M^m(F_n(M)-1) - \eta^m(F_n(\eta)-1)\int_\eta^M m x^{m-1}(F_n(x)-1)\,dx \right) \\
&= \mathcal{O}\left(\eta^m\alpha^\eta\right)
\end{aligned}$$

which proves the Corollary.□

As an example we will apply this concept to random variables satisfying a central limit theorem. Typically such a situation appears in the context of combinatorial constructions when Theorems 3 and 4 can be applied. Usually we get $\lim_{n\to\infty} M_{Y_n}(t) = e^{\frac{1}{2}t^2}$ uniformly for $|t| \le \vartheta$ (for some $\vartheta > 0$), where $Y_n = (X_n - \mu n)/\sqrt{\sigma^2 n}$, and

$$\mathbf{Pr}[X_n = k] = \frac{1}{\sqrt{2\pi\sigma^2 n}}\left(\exp\left(-\frac{(k-\mu n)^2}{2\sigma^2 n}\right) + \mathcal{O}(n^{-\frac{1}{2}})\right)$$

uniformly for all $k \ge 0$ as $n \to \infty$. Hence we obtain

$$\begin{aligned}
\int_\Omega Y_n^{2m}\,dP &= (\sigma^2 n)^{-m}\int_\Omega (X_n-\mu n)^{2m}\,dP \\
&= \frac{1}{2\pi}\int_{-\eta}^{\eta} x^{2m}e^{-\frac{1}{2}x^2}\,dx + \mathcal{O}\left(\eta^{2m+1}n^{-\frac{1}{2}} + \eta^{2m}\alpha^\eta\right) \\
&= \frac{(2m)!}{2^m m!} + \mathcal{O}\left((\log n)^{2m+1}n^{-\frac{1}{2}}\right)
\end{aligned}$$

and

$$\int_\Omega Y_n^{2m+1}\,dP = (\sigma^2 n)^{-\frac{2m+1}{2}}\int_\Omega (X_n-\mu n)^{\frac{1}{2}(2m+1)}\,dP = \mathcal{O}\left((\log n)^{2m+2}n^{-\frac{1}{2}}\right),$$

in which $\eta = C\log n$ with a sufficiently large constant $C > 0$. Especially we get

$$\mathbf{E}X_n = \mu n + \mathcal{O}((\log n)^2) \qquad \text{and} \qquad \mathbf{V}X_n = \sigma^2 n + \mathcal{O}((\log n)^3\sqrt{n}).$$

This example shows that a combination of a weak limit theorem (in terms of the moment generating function) and a local limit theorem (in terms of a multivariate asymptotic expansion with error estimates) provide asymptotic expansions of all (centralized) moments with error terms. It is clear that this concept can be adapted to other types of limiting distribution.

Finally we want to mention that the trivial estimate $y_{nk} = [u^k]\varphi_n(u) \le \varphi_n(\zeta)\zeta^{-k}$, in which ζ equals the saddle point ζ_{nk} satisfying $\zeta_{nk}\varphi_n'(\zeta_{nk})/\varphi(\zeta_{nk}) = k$ or an approximation of ζ_{nk} may give better tail estimates than that of Proposition 3. For example, let $y(x,u) = (1 - x(1+u))^{-1}$. Then

$$y_{nk} \le \frac{n^n}{(n-k)^{n-k}k^k} \le 2^n \exp\left(-c\left(\frac{k-n/2}{n}\right)^2\right)$$

for some constant $c > 0$ which yields a quadratic exponentail tail estimate. Of course such a tail estimate is better and more interesting from a theoretic point of view. However, for our purposes it is as good as that provided in Proposition 3. It gives the same asymptotic expansion for the moments. Nevertheless is should be noted that we need no information about the moment generating function or about a weak limit theorem a priori. We can just work with a local limit theorem and these tail estimates to provide convergence of moments and a weak limit theorem a posteriori.

3 Powers of Functions

In order to show how powerful the saddle point method and singularity analysis are, let us discuss the asymptotic behaviour of coefficients of powers of functions $[x^n]y(x)^k$, where k tends to infinity.

If $\frac{n}{k} \in [a,b]$ for some $a > 0$ and $b < \infty$, then the saddle point ρ of $y(x)^k x^{-n}$ which is given by

$$\frac{\rho y'(\rho)}{y(\rho)} = \frac{n}{k} \tag{24}$$

(usually) varies in a finite interval. Hence the saddle point method (almost) always applies to obtain

$$[x^n]y(x)^k = \frac{y(\rho)^k}{\sqrt{2\pi k\sigma^2(\rho)}\rho^n}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \tag{25}$$

uniformly for $n \to \infty$, $\frac{n}{k} \in [a,b]$. (For details see [15, 16, 6].) It should be mentioned that D. Gardy [13] observed that (25) remains true for $n = o(k)$ as long as $n \to \infty$ (and $y_0, y_1 \ne 0$, where $y(x) = y_0 + y_1 x + \cdots$). Only the error term has to be modified. Furthermore for fixed $n \ge 0$ we have

$$[x^n]y(x)^k = \binom{k}{n} y_0^{k-n} y_1^n \left(1 + \mathcal{O}\left(\frac{1}{k}\right)\right). \tag{26}$$

The case $k = o(n)$ is more delicate and usually depends on the function to be considered. (Some results can be found in [13].)

3.1 Algebraic functions

In [6] it is shown that (25) remains true uniformly for $k \le Cn$ if $y(x)$ has an algebraic singularity.

Theorem 5 ([6]) *Let $y(x)$ has the singular expansion $y(x) = g(x) - h(x)\sqrt{1 - x/x_0}$, where $g(x)$ and $h(x)$ are analytic functions at $x = x_0$, then*

$$[x^n]y(x)^k = \frac{kg(x_0)^{k-1}h(x_0)}{2n^{\frac{3}{2}}\sqrt{\pi}x_0^n}\left(\exp\left(-\frac{k^2}{4n}\left(\frac{h(x_0)}{g(x_0)}\right)^2\right) + \mathcal{O}(n^{-\frac{1}{2}})\right), \tag{27}$$

uniformly for $k = \mathcal{O}(n \log^{-2} n)$.

This theorem can be proved either by using a Hankel contour or by applying the saddle point method locally for the inverse function.

Example 4 : As seen in section 1, the generating function for random mappings where trees (cyclic points) are marked is $\hat{r}(x,u) = (1 - u\hat{a}(x))^{-1}$. Since $\hat{a}(x)$ is the solution of the functional equation $\hat{a}(x) = xe^{\hat{a}(x)}$, $x_0 = \frac{1}{e}$ is the only singularity on the cycle of convergence and we have

$$\hat{a}(x) = 1 - \sqrt{2}\sqrt{1 - ex} + C_2(1 - ex) + C_3(1 - ex)^{\frac{3}{2}} + \cdots \tag{28}$$

Using Singularity analysis, it can easily be shown that the number T_n of trees in a random mapping of size n has mean value and variance of order $\sqrt{n}$:

$$\mu_n = \frac{[x^n]\hat{r}'_u(x,u)|_{u=1}}{[x^n]\hat{r}(x,1)} \sim \sqrt{\pi n/2}.$$

$$\sigma_n^2 = \frac{[x^n]\hat{r}''_u(x,u)|_{u=1}}{[x^n]\hat{r}(x,1)} - \mu_n^2 + \mu_n \sim n(2 - \pi/2)$$

And the probability that T_n has value k is $\mathbf{Pr}[T_n = k] = [x^n]\hat{a}^k(x)/[x^n]\hat{r}(x,1)$. Thus for $k = z\sqrt{n}$, an application of Theorem 5 shows that the number of trees is asymptotically Rayley distributed :

$$\mathbf{Pr}[T_n = z\sqrt{n}] = \frac{1}{\sqrt{n}} z \exp\left(-\frac{z^2}{2}\right) + o(1),$$

uniformly for $z \in [z_0, z_1]$, where $0 < z_0 < z_1 < \infty$ are arbitrary but fixed.

Of course, this only holds for those $z \in [z_0, z_1]$ such that $z\sqrt{n}$ is a positive integer.□

Example 4' : Now if we are interested in the (non cyclic) points at distance 1 to a cycle, the bivariate generating function is $\hat{r}_1(x,u) = (1 - x\exp(u\hat{a}(x)))^{-1}$. The number $T1_n$ of points at distance 1 to a cycle, in a random mapping of size n, has mean value and variance of order $\sqrt{n}$. And the probability that $T1_n$ has value k is

$$\mathbf{Pr}[T1_n = k] = \frac{[u^k x^n]\hat{r}_1(x,u)}{[x^n]\hat{r}_1(x,1)} = \frac{\frac{1}{k!}[x^n](\hat{a}^k(x)\sum_p x^p p^k)}{[x^n]\hat{r}_1(x,1)}.$$

Since $\sum_p x^p p^k$ has radius of convergence 1, the dominant singularity of $\hat{a}^k(x)\sum_p x^p p^k$ is e^{-1}. Using $\sum_p e^{-p} p^k \sim \Gamma(k+1)$, we finally have

$$\mathbf{Pr}[T1_n = z\sqrt{n}] \sim \frac{1}{\sqrt{n}} z \exp\left(-\frac{z^2}{2}\right),$$

uniformly for $z \in [z_0, z_1]$, where $0 < z_0 < z_1 < \infty$ are arbitrary but fixed.

Actually the limiting distribution of the number of points at distance d to a cycle is still Rayleigh, for any fixed d (see [18] and [7]).□

3.2 Logarithmic functions

When $y(x)$ is a function with a logarithmic singularity, we can also derive uniform approximations for $[x^n]y(x)^k$, for $k \leq \varepsilon n$. The technique of proof depends of the range of k. For $k = \mathcal{O}((\frac{\log n}{\log\log n})^2)$, we use a method akin to singularity analysis. And for $\log^{\frac{5}{3}} n \leq k \leq \varepsilon n$, singularity analysis does not apply anymore, but we can use a saddle point method locally for the inverse function.

As a direct application we obtain the conditionial limiting distribution of the number of trees in random mappings where the number of cycles m is given: for $m \sim \gamma \log n$, $(0 \leq \gamma < \infty)$, the number of trees is Gamma distributed, and for $m = \gamma_n \log n$, with $\gamma_n \to \infty$ and $\gamma_n = O(\log^M n)$, $M > 0$, the number of trees is normally distributed.

3.2.1 Theorems

The first theorem uses a Hankel contour around the singularity. A saddle point method would not be applicable. Heuristically we can say that the saddle point is too near the singularity.

Theorem 6 *Assume that $y(x)$ is analytic in a region $|x| < x_0 + \varepsilon$, $x \notin [x_0, x_0 + \varepsilon)$ $(\varepsilon > 0)$ such that $y(x) = -a\log(1 - \frac{x}{x_0}) + C + \mathcal{O}(|\log^{-2}(1 - x/x_0)|)$ for some real constants $a > 0$, C. Then we have*

$$[x^n]y(x)^k = \frac{a^k \log^k n}{n x_0^n \Gamma(\frac{k}{\log n})} \exp\left(\frac{C}{a}\frac{k}{\log n}\right)(1 + \mathcal{O}(k(\frac{\log\log n}{\log n})^2)) \tag{29}$$

uniformly for $k = \mathcal{O}((\frac{\log n}{\log\log n})^2)$.

Proof. For simplicity we will assume that the radius of convergence $x_0 = 1$. We will use Cauchy's formula for $y(x)^k$ for the following path of integration $\gamma = \gamma_1 \cup \gamma_2$:

$$\begin{aligned}
\gamma_1 &= \left\{ x = 1 + \frac{t}{n} \,\middle|\, |t| = 1, \Re t \le 0, \text{ or } 0 < \Re t \le \log^2 n, \Im t = \pm 1 \right\} \\
\gamma_2 &= \left\{ x \,\middle|\, |x| = \left| 1 + \frac{\log^2 n + i}{n} \right|, \arg(1 + \frac{\log^2 n + i}{n}) \le |\arg(x)| \le \pi \right\}.
\end{aligned}$$

Let $k = \mathcal{O}((\frac{\log n}{\log\log n})^2)$. Then we have for $x \in \gamma_1$

$$\begin{aligned}
y(x)^k x^{-(n+1)} &= (a\log n)^k \left(1 - \frac{\log(-t)}{\log n} + \frac{C}{a\log n} + \mathcal{O}(\log^{-2} n)\right)^k \left(1 + \frac{t}{n}\right)^{-n-1} \\
&= a^k (\log n)^k e^{-t} (-t)^{-\frac{k}{\log n}} \exp\left(\frac{kC}{a\log n}\right) \cdot \\
&\quad \cdot \left(1 + \mathcal{O}(\frac{|t|^2}{n}) + \mathcal{O}(k(\frac{\log(|t|+1)}{\log n})^2) = \mathcal{O}(\frac{k}{\log^2 n})\right).
\end{aligned}$$

Since

$$\frac{1}{2\pi i}\int_{\gamma'} e^{-t}(-t)^{-\frac{k}{\log n}}\, dt = \frac{1}{\Gamma(\frac{k}{\log n})} + \mathcal{O}(e^{-\log^2 n}),$$

where $\gamma' = \{t \,|\, |t| = 1, \Re t \le 0, \text{ or } 0 < \Re t \le \log^2 n, \Im t = \pm 1\}$, and

$$\int_{\gamma_2} \frac{y(x)^k}{x^{n+1}}\, dx = \mathcal{O}(e^{-\log^2 n})$$

we directly get (29). □

Corollary 2 *Under the same conditions for $y(x)$ as in Theorem 6, and for $\alpha \in \mathbf{R} \setminus \{-1, -2, \cdots\}$,*

$$[x^n]\frac{1}{(1 - x/x_0)^\alpha} y(x)^k = \frac{a^k \log^k n}{n^{1-\alpha} x_0^n \Gamma(\frac{k}{\log n} + \alpha)} \exp\left(\frac{C}{a}\frac{k}{\log n}\right)(1 + o(1))$$

uniformly for $k = \mathcal{O}((\frac{\log n}{\log\log n})^2)$.

Proof. We use the same contour as in Theorem 6, and

$$\frac{1}{(1 - x/x_0)^\alpha} y(x)^k x^{-(n+1)} \sim a^k (\log n)^k \frac{n^\alpha}{(-t)^\alpha} e^{-t}(-t)^{-\frac{k}{\log n}} \exp\left(\frac{kC}{a\log n}\right). \quad □$$

Assume for a moment that $y_0, y_1 \ne 0$ (otherwise you have to use an easy transformation). Then Gardy's result [13] can be applied for $n = o(k)$. Furthermore the saddle point method surely applies for the range $\frac{n}{k} \in [a, b]$ for $a > 0$ and $b < \infty$. Hence we only have to discuss a range of the following kind: $\log^{\frac{5}{3}} n \le k \le \varepsilon n$ for some $\varepsilon > 0$. It will turn out that a saddle point method locally applied for the inverse function can be used in this case.

Theorem 7 *Assume that $y(x)$ is analytic in a region $|x| < x_0 + \varepsilon$, $x \notin [x_0, x_0 + \varepsilon)$ ($\varepsilon > 0$) such that*

$$y(x) = a \log \frac{1}{1 - \frac{x}{x_0}} + C + g(\sqrt{1 - x/x_0}), \tag{30}$$

where $g(z)$ is analytic around $z = 0$ and satisfies $g(0) = 0$. Then we have (25) uniformly for $\log^{\frac{5}{3}} n \le k \le \varepsilon n$ for some $\varepsilon > 0$ with an error term $\mathcal{O}(\log^{-\frac{1}{4}} n)$.

Proof. We will assume that $x_0 = 1$ and $a = 1$. Let $\rho_{\frac{n}{k}}$ be the saddle point of $y(x)^k$ defined by (24). By elementary considerations we get $1 - \rho_{\frac{n}{k}} \sim (\frac{n}{k} \log \frac{n}{k})^{-1}$ and

$$\overline{\rho}_{\frac{n}{k}} = y(\rho_{\frac{n}{k}}) \sim \log \frac{n}{k} + \log\log \frac{n}{k}. \tag{31}$$

Let $z(w)$ be the inverse function of $y(x)$ locally in $\{x \in \mathbf{C} \,|\, |x - 1| < \varepsilon, x \notin [1, 1+\varepsilon]\}$. This inverse function surely exists by (30). Again we will use Cauchy's formula but now for the following path of integration $\gamma = \gamma_1 \cup \gamma_2$:

$$\begin{aligned} \gamma_1 &= \{x \,|\, |y(x)| = \overline{\rho}_{\frac{n}{k}}, |\arg(y(x))| \le \lambda_{\frac{n}{k}}\}, \\ \gamma_2 &= \{x \,|\, |x| = |z(\overline{\rho}_{\frac{n}{k}} e^{i\lambda_{\frac{n}{k}}})|, |\arg(z(\overline{\rho}_{\frac{n}{k}} e^{i\lambda_{\frac{n}{k}}}))| \le |\arg(x)| \le \pi\}, \end{aligned}$$

where $\lambda_{\frac{n}{k}} = C/\log\frac{n}{k}$ and $C > 0$ is sufficiently small to be specified in the sequel. Trivially we have

$$\left| \int_{\gamma_2} \frac{y(x)^k}{x^{n+1}}\, dx \right| = \mathcal{O}\left(\overline{\rho}^k_{\frac{n}{k}} |z(\overline{\rho}_{\frac{n}{k}} e^{i\lambda_{\frac{n}{k}}})|^{-n} \right) = \mathcal{O}\left(y(\rho_{\frac{n}{k}})^k \rho_{\frac{n}{k}} e^{-\frac{C}{2}k} \right). \tag{32}$$

For $x \in \gamma_1$ we will use the substitution

$$\int \frac{y(x)^k}{x^{n+1}}\, dx = \frac{k}{n} \int \frac{y(x)^{k-1}}{x^n} y'(x)\, dx - \frac{y(x)^k}{nx^n} = \frac{k}{n} \int \frac{w^{k-1}}{z(w)^n}\, dw - \frac{y(x)^k}{nx^n}$$

and the (transformed) path of integration $\overline{\gamma}_1 = \{w = \overline{\rho}_{\frac{n}{k}} e^{it} \,|\, |t| \le \lambda_{\frac{n}{k}}\}$. Thus we have to calculate

$$\int_{\overline{\gamma}_1} \frac{w^{k-1}}{z(w)^n}\, dw = \rho_{\frac{n}{k}}^{-n} y(\rho_{\frac{n}{k}})^k \int_{-\lambda_{\frac{n}{k}}}^{\lambda_{\frac{n}{k}}} \exp\left(\frac{\overline{\kappa}_2}{2} nt^2 - \sum_{m \ge 3} \frac{\overline{\kappa}_m}{m!} n(it)^m \right) i\, dt,$$

where $\overline{\kappa}_m = \frac{\partial^m}{\partial v^m} \log z(\overline{\rho}_{\frac{n}{k}} e^v)|_{v=0}$. Observe that $\overline{\kappa}_2 = -\frac{k^3}{n^3}\sigma^2(\rho_{\frac{n}{k}}) < 0$. We are now in the position to apply the usual saddle point method to get (25). We only have to check that

$$(-\overline{\kappa}_2)\frac{t^2}{4} > \left| \sum_{m \ge 3} \frac{\overline{\kappa}_m}{m!} (it)^m \right| \qquad \text{for } |t| \le \lambda_{\frac{n}{k}} \tag{33}$$

and that

$$\lim_{n\to\infty}(-\overline{\kappa}_2)n\lambda_{\frac{n}{k}}^2 = \infty. \tag{34}$$

From (31) we immediately get $(-\overline{\kappa}_2) \sim \overline{\rho}_{\frac{n}{k}}^2 e^{-\overline{\rho}_{\frac{n}{k}}} \sim \overline{\rho}_{\frac{n}{k}}\overline{\kappa}_1 \sim \frac{k}{n}\log\frac{n}{k}$ and

$$\begin{aligned}\sum_{m\ge 3}\frac{\overline{\kappa}_m}{m!}(it)^m &= \frac{t^3}{3!}\frac{\partial^3}{\partial v^3}\log z(\overline{\rho}_{\frac{n}{k}}e^v)|_{v=i\theta,0<\theta<t}\\ &= \mathcal{O}(t^3\overline{\rho}_{\frac{n}{k}}^3 e^{-\overline{\rho}_{\frac{n}{k}}}) = \mathcal{O}\left(t^3\frac{k}{n}\log^2\frac{n}{k}\right)\end{aligned} \tag{35}$$

for $|t| \le \lambda_{\frac{n}{k}}$. Hence (33) and (34) are satisfied for $k \ge \log^{\frac{5}{3}} n$. Hence we can proceed as follows to approximate the integral

$$\begin{aligned}&\int_{-\lambda_{\frac{n}{k}}}^{\lambda_{\frac{n}{k}}} \exp\left(\frac{\overline{\kappa}_2}{2}nt^2 - \sum_{m\ge 3}\frac{\overline{\kappa}_m}{m!}n(it)^m\right)dt\\ &= \int_{|t|\le\mu}\exp\left(\frac{\overline{\kappa}_2}{2}nt^2\right)\left(1+\mathcal{O}(n\overline{\rho}_{\frac{n}{k}}^3 e^{-\overline{\rho}_{\frac{n}{k}}}\mu^3)\right)dt\\ &\quad + \int_{\mu\le|t|\le\lambda_{\frac{n}{k}}}\exp\left(\frac{\overline{\kappa}_2}{2}nt^2 - \sum_{m\ge 3}\frac{\overline{\kappa}_m}{m!}n(it)^m\right)e^{it}\,dt\\ &= \int_{-\infty}^{\infty}\exp\left(\frac{\overline{\kappa}_2}{2}nt^2\right)dt\left(1+\mathcal{O}(\log^{-\frac{1}{4}} n)\right) + \mathcal{O}\left(\int_{|t|\ge\mu}\exp\left(\frac{\overline{\kappa}_2}{4}nt^2\right)dt\right),\end{aligned}$$

where $\mu = k^{-\frac{1}{3}}\log^{-\frac{3}{4}}(\frac{n}{k})$. This proves Theorem 7. □

Corollary 3 *Let $M > 0$ be an arbitrary but fixed integer and $y(x)$ as in Theorem 7. Then we have*

$$\begin{aligned}[x^n]y(x)^k &= \frac{ka^k\left(\log\left(\frac{n}{k}\log\frac{n}{k}\right)\right)^{k-\frac{1}{2}}}{\sqrt{2\pi n}x_0^n}\exp\left(\frac{k}{\log\frac{n}{k}}\sum_{j=0}^{M}\sum_{l=0}^{j}b_{jl}\frac{(\log\log\frac{n}{k})^l}{(\log\frac{n}{k})^j}\right)\cdot\\ &\quad\cdot\left(1+\mathcal{O}\left(k\frac{(\log\log\frac{n}{k})^{M+1}}{(\log\frac{n}{k})^{M+2}}\right)\right)\end{aligned} \tag{36}$$

uniformly for $k = \mathcal{O}(\log^{M+1} n)$, where $b_{00} = 1$ and b_{ij} are real constants.

Proof. For $k = \mathcal{O}(\log^{\frac{3}{5}} n)$ the asymptotic formula (36) follows from Theorem 6. Hence we only have to discuss the case $k \ge \log^{\frac{5}{3}} n$, where we have to apply Theorem 7. The essential point is to get a good approximation for the saddle point $\rho_{\frac{n}{k}}$.

By bootstrapping it is easy to show that for any integer $M \geq 0$ we have

$$\rho_{\frac{n}{k}} = 1 - \frac{1}{\frac{n}{k}\log\frac{n}{k}} \sum_{j=0}^{M}\sum_{l=0}^{j} c_{jl} \frac{(\log\log\frac{n}{k})^l}{(\log\frac{n}{k})^j} + \mathcal{O}\left(\frac{(\log\log\frac{n}{k})^{M+1}}{\frac{n}{k}(\log\frac{n}{k})^{M+2}}\right),$$

where $c_{00} = 1$ and c_{jl} are real constants. Hence

$$y(\rho_{\frac{n}{k}}) = a(\log\frac{n}{k} + \log\log\frac{n}{k}) + \sum_{j=0}^{M}\sum_{l=0}^{j} d_{jl} \frac{(\log\log\frac{n}{k})^l}{(\log\frac{n}{k})^j} + \mathcal{O}\left(\frac{(\log\log\frac{n}{k})^{M+1}}{(\log\frac{n}{k})^{M+1}}\right),$$

where $d_{00} = \frac{C}{a}$. Consequently we get

$$\begin{aligned} y(\rho_{\frac{n}{k}})^k &= a^k\left(\log\left(\frac{n}{k}\log\frac{n}{k}\right)\right)^k \exp\left(\frac{C}{a}\frac{k}{\log\frac{n}{k}}\sum_{j=0}^{M}\sum_{l=0}^{j} e_{jl}\frac{(\log\log\frac{n}{k})^l}{(\log\frac{n}{k})^j}\right) \cdot \\ &\quad \cdot\left(1 + \mathcal{O}\left(k\frac{(\log\log\frac{n}{k})^{M+1}}{(\log\frac{n}{k})^{M+2}}\right)\right) \end{aligned}$$

and

$$\sigma^2(\rho_{\frac{n}{k}}) = \left(\frac{n}{k}\right)^2 \log\frac{n}{k}\left(\sum_{j=0}^{M}\sum_{l=0}^{j} f_{jl}\frac{(\log\log\frac{n}{k})^l}{(\log\frac{n}{k})^j} + \mathcal{O}\left(\frac{(\log\log\frac{n}{k})^{M+1}}{(\log\frac{n}{k})^{M+1}}\right)\right),$$

where $e_{00} = f_{00} = 1$, which completes the proof of Corollary 3 $\square$

Corollary 4 *Under the same conditions for $y(x)$ as in Corollary 3, the saddle point method also applies for $[x^n](1 - x/x_0)^{-\alpha}y(x)^k$, and*

$$[x^n]\frac{1}{(1 - x/x_0)^\alpha}y(x)^k \sim \left(\frac{n}{k}\log\frac{n}{k}\right)^\alpha [x^n]y(x)^k$$

uniformly for $k = \mathcal{O}(\log^{M+1} n)$.

3.2.2 Trees and Cycles in Random Mappings

The probability that a random mapping of size n has k trees and m components (cycles) is given by (see section 1.1) :

$$\frac{r_{nkm}}{n^n} = \frac{\frac{1}{m!}[x^n]\hat{a}^k(x).[w^k]\log^m\frac{1}{1-w}}{[x^n](1 - \hat{a}(x))^{-1}}.$$

Using Stirling's approximation formula and Theorem 6, together with Example 4, we directly get

Theorem 8 *Let $m = \frac{1}{2}\log n + s$ and $k = x\sqrt{n}$. Then we have*

$$\frac{r_{nkm}}{n^n} = \frac{1}{\sqrt{\pi n \log n}} \exp\left(-\frac{s^2}{\log n}\right) \cdot x \exp\left(-\frac{x^2}{2}\right)\left(1 + \mathcal{O}(\log^{-\frac{1}{2}} n)\right) \tag{37}$$

uniformly for $|s| \le C \log^{\frac{1}{2}} n$ and $\frac{1}{C} \le x \le C$, where $C > 1$ is arbitrary but fixed.

This formula reveals that the number of trees and the number of cycles are almost independent in the range of interest. Furthermore the number of cycles satisfy a central limit theorem with mean value and variance $\sim \frac{1}{2}\log n$ and the number of trees is asymptotically Rayleigh distributed with mean value $\sim \sqrt{\frac{\pi}{2} n}$.

Similarly we can consider the conditional distribution of the number of trees where the number of cycles is given. Consider random mappings of size n being all equally likely, and let the random variables T_n denote the number of trees, and C_n the number of cycles. The conditional probability $\mathbf{Pr}[T_n = k \mid C_n = m]$ is

$$\frac{\mathbf{Pr}[T_n = k \text{ and } C_n = m]}{\mathbf{Pr}[C_n = m]} = \frac{[x^n u^k v^m] \frac{1}{(1-u\hat{a}(x))^v}}{[x^n] \frac{1}{(1-\hat{a}(x))}} \cdot \frac{[x^n] \frac{1}{(1-\hat{a}(x))}}{[x^n v^m] \frac{1}{(1-\hat{a}(x))^v}}$$

Thus

$$\mathbf{Pr}[T_n = k \mid C_n = m] = \frac{[x^n]\hat{a}^k(x).[w^k] \log^m \frac{1}{1-w}}{[x^n] \log^m \frac{1}{1-\hat{a}(x)}}. \tag{38}$$

Moreover, from the singular expansion of $\hat{a}(x)$ (28), we get

$$\log \frac{1}{1-\hat{a}(x)} = \frac{1}{2}\log\frac{1}{1-ex} - \frac{1}{2}\log 2 + g\left(\sqrt{1-ex}\right).$$

Using Theorem 6, or Theorem 7, according to the order of m provides easy proofs and extensions of results by Yu. L. Pavlov [20]

Theorem 9 ([20]) *If $m \to \infty$ and $m/\log n \to 0$ then*

$$\sqrt{n}\mathbf{Pr}[T/\sqrt{n} = z \,|\, C = m] = \sqrt{\frac{2}{\pi}} \exp\left(-\frac{z^2}{2}\right) + o(1) \tag{39}$$

uniformly for $z \in [z_0, z_1]$, where $0 < z_0 < z_1 < \infty$ are arbitrary but fixed.

If $n \to \infty$ and $m/\log n \to \gamma$ $(0 < \gamma < \infty)$ then

$$\sqrt{n}\mathbf{Pr}[T/\sqrt{n} = z \,|\, C = m] = \frac{1}{\sqrt{2\pi}} \frac{2^\gamma \Gamma(\gamma)}{\Gamma(2\gamma)} z^{2\gamma} \exp\left(-\frac{z^2}{2}\right) + o(1) \tag{40}$$

uniformly for $z \in [z_0, z_1]$, where $0 < z_0 < z_1 < \infty$ are arbitrary but fixed.

Sketch of the Proof. To evaluate $\mathbf{Pr}[T_n = k \mid C_n = m]$ in (38), use Theorem 5 to find $[x^n]\hat{a}^k(x)$, and Theorem 6 to find $[w^k]\log^m \frac{1}{1-w}$ and $[x^n]\log^m \frac{1}{1-\hat{a}(x)}$. The mean value and standard deviation are shown to be of order $\sqrt{n}$ using Corollary 2. $\square$

Next we extend Pavlov's result to the case where $m/\log n \to \infty$. Hence we get a precise information about the conditional density not only in a small range around the mean value.

Theorem 10 *Suppose that $\gamma_n = \frac{m}{\log n} \to \infty$ and that $\gamma_n = \mathcal{O}(\log^M n)$ for some fixed integer $M > 0$. Then*

$$\sqrt{n}\mathbf{Pr}\left[\frac{T - \sqrt{2\gamma_n n}}{\sqrt{n/2}} = z \,\middle|\, C = \gamma_n \log n\right] = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) + o(1) \qquad (41)$$

uniformly for $z \in [z_0, z_1]$, where $-\infty < z_0 < z_1 < \infty$ are arbitrary but fixed. Thus the conditional distribution is asymptotically normal with mean value $\sim \sqrt{2\gamma_n n}$ and variance $\sim \frac{n}{2}$.

Sketch of the Proof. Use Theorem 7 applied to

$$[w^k]\log^m \frac{1}{1-w} \qquad \text{and} \qquad [x^n]\log^m \frac{1}{1-\hat{a}(x)}.$$

The mean value and standard deviation can be evaluated using Corollary 4. $\square$

4 Product schemas $y(x,u) = g(x)F(uw(x))$

4.1 g is Dominated in $y(x,u)$

As indicated in Section 1.3 a very important special case for $y(x,u)$ is the form

$$y(x,u) = g(x)F(uw(x)). \qquad (42)$$

There are many cases where the factor $g(x)$ has actually no influence on the limiting distribution, i.e. the limiting distribution of $y(x,u)$ is the same as the limiting distribution of

$$\overline{y}(x,u) = F(uw(x)). \qquad (43)$$

In this case we will say that $g(x)$ is dominated in $y(x,u)$.

In order to be more precise we will introduce the notion of dominance of in a product of functions. As above we will use the notations

$$\mu_f(x) = \frac{xf'(x)}{f(x)} \qquad (44)$$

and

$$\sigma_f^2(x) = \frac{x^2 f''(x)}{f(x)} + \frac{xf'(x)}{f(x)} - \left(\frac{xf'(x)}{f(x)}\right)^2. \qquad (45)$$

Definition 2 *Let $f(x)$, $g(x)$ be (convergent) generating functions with non-negative coefficients. We say that $f(x)$ dominates $g(x)$ if*

$$[x^n]f(x)g(x) \sim g(\zeta_n)[x^n]f(x) \qquad (n \to \infty), \tag{46}$$

where ζ_n is the saddle point of $f(x)$ defined by $\mu_f(\zeta_n) = n$.

If the radius of convergence of the first factor is smaller than that of the second one then the first factor usually dominates.

Lemma 2 *Suppose that the coefficients of $f(x) = \sum_{n\geq 0} a_n x^n$ satisfy $\lim_{n\to\infty} a_n/a_{n+1} = R$ for some $R \in (0,\infty)$, that $\lim_{x\to R} \mu_f(x) = \infty$, and that the radius of convergence of $g(x)$ is greater than R. Then $f(x)$ dominates $g(x)$.*

Proof. By [2] $[x^n]f(x)g(x) \sim g(R)[x^n]f(x) \quad (n \to \infty)$. Hence Lemma 2 follows from $\zeta_n \to R$.□

If both factors have the same radius of convergence or are entire then the situation is obviously much more involved. However, for acceptable functions we easily find a necessary condition for dominance.

Lemma 3 *Let $f(x)$, $g(x)$ be alglog-functions or admissible functions such that $f(x)g(x)$ is an alglog-function or admissible. If $\mu_g(x) = o\left(\sqrt{\sigma_f^2(x)}\right)$ as $x \to R$ (in which $R \in (0,\infty]$ is the radius of convergence of $f(x)g(x)$) then $f(x)$ dominates $g(x)$.*

Proof. For alglog-functions the dominance condition can easily be checked.

Now suppose that $f(x)$ and $f(x)g(x)$ are admissible. Hence by [17]

$$[x^n]f(x) \sim \frac{f(\zeta_n)\zeta_n^{-n}}{\sqrt{2\pi\sigma_f^2(\zeta_n)}} \tag{47}$$

and

$$[x^n]f(x)g(x) \sim \frac{f(\zeta_n)g(\zeta_n)\zeta_n^{-n}}{\sqrt{2\pi(\sigma_f^2(\zeta_n)+\sigma_g^2(\zeta_n))}} \left(\exp\left(-\frac{\mu_g(\zeta_n)^2}{2(\sigma_f^2(\zeta_n)+\sigma_g^2(\zeta_n))}\right) + o(1)\right). \tag{48}$$

Since $\sigma_g^2(\zeta_n)) = o(\mu_g(\zeta_n)^2) = o(\sigma_f^2(\zeta_n))$, (47) and (48) prove

$$[x^n]f(x)g(x) \sim g(\zeta_n)[x^n]f(x) \qquad (n \to \infty)$$

and consequently $f(x)$ dominates $g(x)$. □

First we state two typical cases where $g(x)$ is dominated in $g(x)F(uw(x))$ and where we can classify the kind of limiting distributions.

Rule 1 *Suppose that $F(z)$ is acceptable with finite radius of convergence R. Let $w(r) = R$ and assume that $w(x)$ and $g(x)$ are regular at $x = r$ (i.e. the radii of convergence of $w(x)$ and $g(x)$ are greater than r). Then $g(x)$ is dominated in $y(x,u) = g(x)F(uw(x))$.*

If in addition $F(z)$ is an alglog-function or admissible then the limiting distribution is Gaussian with mean value $\sim \mu_w(R)^{-1}n$ and variance $\sim \sigma_w^2(R)\mu_w(R)^{-3}n$.

Proof. First we recall that usually (in the remaining cases we have similar formulas, see [6])

$$[x^n]w(x)^k = \frac{w(\zeta_{\frac{n}{k_0}})^k \zeta_{\frac{n}{k_0}}^{-n}}{\sqrt{2n\pi\sigma_w^2(\zeta_{\frac{n}{k_0}})\mu_w(\zeta_{\frac{n}{k_0}})^{-3}}} \left(\exp\left(-\frac{(k-k_0)^2}{2n\sigma_w^2(\zeta_{\frac{n}{k_0}})\mu_w(\zeta_{\frac{n}{k_0}})^{-3}}\right) + \mathcal{O}(n^{-\frac{1}{2}})\right) \tag{49}$$

as $n \to \infty$ uniformly for $k \geq 0$ and $\mu_w(R) - \varepsilon \leq \frac{n}{k_0} \leq \mu_w(R) + \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Furthermore, the same asymptotic formula holds (despite of the factor $g(\zeta_{\frac{n}{k_0}})$) if we calculate $[x^n]g(x)w(x)^k$. This can be shown by using exactly the same saddle points method as for the proof of (49).

Now suppose that $F(z)$ is an alglog-function. Then use (49) with $k_0 = n\mu_w(R)^{-1}$ and the asymptotic expansion of $f_k = [z^k]F(z)$ to determine an asymptotic expansion for $[x^n u^k]F(uw(x)) = f_k[x^n]w(x)^k$. Since the factor f_k has minor influence on the distribution behaviour it follows that the limiting distribution corresponding to $\overline{y}(x,u) = F(uw(x))$ is Gaussian.

If $F(z)$ is admissible then let R_k denote the saddle point defined by $\mu_F(R_k) = k$. For sufficiently large n let k_0 be determined by $w(\zeta_{\frac{n}{k_0}}) = R_{k_0}$. Hence

$$\begin{aligned}[x^n u^k]F(uw(x)) &= \frac{F(R_{k_0})}{2\pi\sqrt{n\sigma_F^2(R_{k_0})\sigma_w^2(\zeta_{\frac{n}{k_0}})\mu_w(\zeta_{\frac{n}{k_0}})^{-3}}} \cdot \\ &\cdot \left(\exp\left(-\frac{(k-k_0)^2}{2\sigma_F^2(R_{k_0})}\right) + \exp\left(-\frac{(k-k_0)^2}{2n\sigma_w^2(\zeta_{\frac{n}{k_0}})\mu_w(\zeta_{\frac{n}{k_0}})^{-3}}\right) + o(1)\right)\end{aligned}$$

Now $\lim\limits_{x\to R}(R-x)\mu_F(x) = \infty$ and $\mu_F(x) = o(\sigma_F^2(x))$ (as $x \to R$) imply that

$$n\sigma_w^2(\zeta_{\frac{n}{k_0}})\mu_w(\zeta_{\frac{n}{k_0}})^{-3} = o(\sigma_F^2(R_{k_0}))$$

as $n \to \infty$. Hence we obtain a Gaussian limit law.

In any of these cases the factor $g(\zeta_{\frac{n}{k_0}})$ has no influence on the distribution behaviour. Thus $g(x)$ is dominated. $\square$

Rule 2 *Suppose that $w(x)$ is acceptable with finite radius of convergence r such that $w(r) = R$ is finite. Furthermore assume that $F(y)$ is regular at $y = w(r)$ and $g(x)$*

is regular at $x = r$. Then $g(x)$ is dominated in $y(x,u) = g(x)F(uw(x))$ and has a discrete limiting distribution with

$$\lim_{n\to\infty} \mathbf{Pr}[X_n = k] = \frac{kf_k R^{k-1}}{F'(R)}. \tag{50}$$

Proof. Since $w(r) = R$ is finite $w(x)$ consists of no admissible term with radius of convergence r. Hence we can represent $w(x)$ as

$$w(x) = R - C(1-x/r)^{\alpha'} L\left(\frac{1}{1-x/r}\right)(1 + o(1-x/r))$$

For some $\alpha' > 0$ which is not an integer. This leads to

$$[x^n]g(x)F(w(x)) \sim Cg(r)F'(R)\frac{n^{-\alpha'-1}}{\Gamma(-\alpha')}L(n) \tag{51}$$

and for any fixed $k \geq 1$ to

$$[x^n]g(x)w(x)^k \sim kCg(r)R^{k-1}\frac{n^{-\alpha'-1}}{\Gamma(-\alpha')}L(n). \tag{52}$$

(51) and (52) prove Rule 2. □

These two cases are surely the simplest ones since $g(x)$ is regular at the singular curve of $F(uw(x))$ and since there is only one singularity. In the following we will try to formulate a "general criterion to decide whether $g(x)$ is dominated or not. For example, if $g(x)$ and $w(x)$ are alglog-function and $F(w(x))$ has an essential singularity at $x = r$ then $g(x)$ is usually dominated In order to be more exact we can "scale" singulaties by the notion of dominance introduced in Definition 2 The same "scaling" can be used for entire functions. Therefore we can expect the following Rule.

Rule 3 *Suppose that $g(x)$, $w(x)$, and $F(w(x))$ are acceptable If $F(w(x))$ dominates $g(x)$ then $g(x)$ is (usually) dominated in $g(x)F(uw(x))$.*

Heuristically we can justify Rule 3 by the following observation. First note that the fact that $g(x)$ is dominated in $g(x)F(uw(x))$ is more or less equivalent to the statement that $w(x)^k$ dominates $g(x)$ for $k \sim \mathbf{E}X_n$ and this can be checked by $\mu_g = o(\sqrt{k\sigma_w^2})$. Next, if $\mu_W^2\sigma_F^2 = \mathcal{O}(\mu_F\sigma_w^2)$ then

$$\sigma_{F\circ w}^2 = \mu_F\sigma_w^2 + \mu_W^2\sigma_F^2 = \mathcal{O}(\mu_F\sigma_w^2)$$

and for the saddle point of interest $x = \zeta_n$ we have $\mu_F \sim \mathbf{E}X_n$. Hence, if $F(w(x))$ dominates $g(x)$ then we can expect that $g(x)$ is dominated in $g(x)F(uw(x))$. In fact, we do not know any example disproving Rule 3.

What remains is to discuss the case $y(x,u) = F(uw(x))$. In section 2.2 we have formulated (quite powerful) theorems related to this field. Those cases which are not covered by these theorems must be treated separately. Usually expected value and variance cause no difficulty. In order to get the limiting distribution you have to get an asymptotic expansion for $[x^n]w(x)^k$ in the range of interest. Section 3 provides some methods to do so (see also [6, 13]).

4.2 g is dominating in $y(x,u)$

Alternatively to the previous section, where the factor $g(x)$ has actually no influence on the asymptotic limit distribution of $y(x,u) = g(x)F(uw(x))$, this section is devoted to the case where $g(x)$ is of dominant importance. This means that the saddle point ζ_n of $g(x)$ given by $\zeta_n g'(\zeta_n)/g(\zeta_n) = n$ can be used instead of the exact saddle points in the evaluation of the mean, variance and probability. Hence we get

$$\mathbf{E}X_n \sim \frac{w(\zeta_n)F'(w(\zeta_n))}{F(w(\zeta_n))}, \tag{53}$$

$$\mathbf{V}X_n \sim \frac{w(\zeta_n)^2F''(w(\zeta_n))}{F(w(\zeta_n))} - \left(\frac{w(\zeta_n)F'(w(\zeta_n))}{F(w(\zeta_n))}\right)^2 + \frac{w(\zeta_n)F'(w(\zeta_n))}{F(w(\zeta_n))}, \tag{54}$$

and

$$\frac{y_{nk}}{y_n} \sim \frac{f_k w(\zeta_n)^k}{F(w(\zeta_n))} \tag{55}$$

in the range of interest, where $f_k = [z^k]F(z)$. For short we will say that $g(x)$ is dominating. As a first rule we have:

Rule 4 *Suppose that $g(x)$ is acceptable and has finite radius of convergence r. If $w(x)$ and $F(w(x))$ are regular at $x = r$ (i.e. their radii of convergence are larger than r) then $g(x)$ is dominating in $g(x)F(uw(x))$ and has a discrete limiting distribution with*

$$\lim_{n\to\infty} \mathbf{Pr}[X_n = k] = \frac{f_k w(r)^k}{F(w(r))}. \tag{56}$$

Proof. If $g(x)$ is acceptable and has finite radius of convergence then we can apply Lemma 2 to $g(x)F(w(x))$ and to $g(x)w(x)^k$ for any fixed $k \ge 0$ which directly gives (56). □

In general we can only expect a Rule of the following kind which is precisely the converse statement to Rule 3.

Rule 5 *Suppose that $g(x)$, $w(x)$, and $F(w(x))$ are acceptable If $g(x)$ dominates $F(w(x))$ then $g(x)$ is (usually) dominating in $g(x)F(uw(x))$.*

Again we will give an heuristic argument for Rule 5. Mostly it suffices to know the behaviour of $[x^n]g(x)w(x)^k$ if $k = \mathcal{O}(\mathbf{E}X_n)$. If $g(x)$ dominates $F(w(x))$ then

$$\frac{xF'(w(x))w'(x)}{F(w(x))} = \frac{w(x)F'(w(x))}{F(w(x))}\frac{xw'(x)}{w(x)} = o\left(\sqrt{\sigma_g^2(x)}\right). \tag{57}$$

Hence if we assume something like (53) then (57) essentially says that $g(x)$ dominates $w(x)^k$ for $k = \mathcal{O}(\mathbf{E}X_n)$. Therefore we can expect that $g(x)$ is dominating in this case.

One very interesting thing in the dominating case is that there are only a few kinds of limiting distributions which can be classified in the following way.

Theorem 11 *Suppose that $g(x)$ is acceptable with radius of convergence $r \in (0,\infty]$ and suppose that $g(x)$ is dominating in $y(x,u) = g(x)F(uw(x))$, i.e. (53), (54), and (55) hold.*

a) Assume that $\lim_{x\to r-} w(x) = w(r)$ exists and that $F(z)$ is regular at $w(r)$. Then X_n (related to $y(x,u)$) has a discrete limiting distribution given by

$$\lim_{n\to\infty} \mathbf{Pr}[X_n = k] = \frac{f_k w(r)^k}{F(w(r))}.$$

b) Assume that $\lim_{x\to r-} w(x) = \infty$ and that $F(z)$ is entire and admissible. Then X_n is asymptotically normally distributed with mean value (53) and variance (54).

c) Assume that $\lim_{x\to r-} w(x) = w(r)$ exists and that $F(z)$ is singular at $z = w(r)$. If $F(z)$ is admissible then X_n is asymptotically normally distributed with mean value (53) and variance (54).

d) Assume that $\lim_{x\to r-} w(x) = w(r)$ exists and that $F(z)$ is singular at $z = w(r)$. If $F(z)$ an alglog-function having a dominating term with $\alpha > 0$ then X_n is asymptotically Gamma distributed with parameter α and $\mathbf{E}X_n \sim \alpha/\log \frac{w(r)}{w(\zeta_n)}$.

Remark. Note that the radius of convergence r of $g(x)$ can only be infinite in part b). Hence, if $g(x)$ is entire and dominating then there is a Gaussian limiting distribution. □

Proof. a) This part is almost trivial. From $\lim_{n\to\infty} \zeta_n = r$ it follows that

$$\lim_{n\to\infty} \mathbf{Pr}[X_n = k] = \lim_{n\to\infty} \frac{y_{nk}}{y_n} = \lim_{n\to\infty} \frac{f_k w(\zeta_n)^k}{F(w(\zeta_n))} = \frac{f_k w(r)^k}{F(w(r).)}$$

b) Now assume that $\lim_{x\to r-} w(x) = \infty$ and let $F(z)$ be entire and admissible. By [17] we have

$$f_{k_0+l} = \frac{F(R_{k_0})}{R_{k_0}^{k_0+l}\sqrt{2\pi\sigma^2(R_{k_0})}} \left(\exp\left(-\frac{l^2}{2\sigma^2(R_{k_0})} \right) + o(1) \right), \quad (k_0 \to \infty)$$

uniformly for all l such that $k_0 + l$ is a positive integer, where R_{k_0} is given by $k_0 = R_{k_0} F'(R_{k_0})/F(R_{k_0})$. Note that k_0 need not be an integer. We will use this formula for

$$k_0 = \frac{F'(w(\zeta_n))w(\zeta_n)}{F(w(\zeta_n))} \sim \mathbf{E}X_n,$$

i.e. $R_{k_0} = w(\zeta_n)$. Consequently

$$\mathbf{Pr}[X_n = k_0 + l] \sim \frac{f_{k_0+l}w(\zeta_n)^{k_0+l}}{F(w(\zeta_n))} \sim \frac{1}{\sqrt{2\pi\sigma^2(R_{k_0})}} \exp\left(-\frac{l^2}{2\sigma^2(R_{k_0})}\right)$$

uniformly for $l = \mathcal{O}(\sigma(R_{k_0}))$ leading to asymptotic normality with mean value (53) and variance (54).

c) If $w(r)$ exists and if $F(z)$ is singular at $z = w(r)$ and again admissible then we are in a similar situation as above and by the same proof we get asymptotic normality .

d) Finally assume that $F(z)$ has a singularity of the kind (16). By [11] we get

$$f_k \sim w(r)^{-k} \frac{k^{\alpha-1}}{\Gamma(\alpha)} L(k). \tag{58}$$

Hence

$$\begin{aligned}
\mathbf{Pr}[X_n = k] \quad &\sim \quad \frac{\left(1 - \frac{w(r)}{w(\zeta_n)}\right)^{\alpha}}{\Gamma(\alpha)} \frac{L(k)}{L\left(\frac{1}{1-w(\zeta_n)/w(r)}\right)} k^{\alpha-1} \left(\frac{w(\zeta_n)}{w(r)}\right)^k \\
&\sim \quad \frac{\log^{\alpha} \frac{w(\zeta_n)}{w(r)}}{\Gamma(\alpha)} k^{\alpha-1} \exp\left(k \log \frac{w(\zeta_n)}{w(r)}\right)
\end{aligned}$$

in the range of interest $k \gg\ll \left(\log \frac{w(r)}{w(\zeta_n)}\right)^{-1}$. This proves Theorem 11. □

Remark. The restriction $\alpha > 0$ in Theorem 11-d is not essential. The case of alglog-functions $F(z)$ with $F(w(r)) < \infty$ can always be treated as above but the limiting distribution cannot be classified in such a simple way. Here we have

$$\mathbf{Pr}[X_n = k] \sim \frac{k^{\alpha-1}L(k)}{\Gamma(\alpha)F(w(r))} \left(\frac{w(\zeta_n)}{w(r)}\right)^k.$$

Thus everything depends on the behaviour of $w(\zeta_n)$ and on α.

4.3 Examples and Extensions

We discuss some typical schemas $y(x,u) = g(x)F(uw(x))$ occuring in combinatorial structures related to the "sequence-of" and "cycle-of" constructions. Most of the time g is either dominated or dominating in $y(x,u)$, and we refer to sections 4.1 and 4.2. But there are also some interesting cases where g is neither dominated nor dominating in $y(x,u)$. We investigate some of these cases : we get hypergeometric limiting distributions for some g and $F(w)$ having the same finite radius of convergence, and Gaussian limiting distributions for some entire functions g and $F(w)$.

4.3.1 F is exponential

We consider the product schema $y(x,u) = g(x)\exp(uw(x))$.

- a) Assume first that g has a finite radius of convergence, and w is an entire function. In this case g is dominating in $y(x,u)$, and the limiting distribution, dictated by F is Poisson by Theorem 11-a.

As an illustration, the number of cycles with fixed length l in permutations, with bivariate generating function

$$y(x,u) = \frac{\exp(-\frac{x^l}{l})}{1-x}\ \exp(u\frac{x^l}{l})$$

is Poisson distributed (with parameter $1/l$).

Note that by Theorem 11-a, the number of cycles with fixed length l in random mappings, with bivariate generating function

$$y(x,u) = \frac{\exp(-\frac{\hat{a}(x)^l}{l})}{1-\hat{a}(x)}\ \exp(u\frac{\hat{a}(x)^l}{l})$$

has the same Poisson limiting distribution.

- b) Consider the case where both g and w are entire functions. If g is dominated in $y(x,u)$, the limiting distribution is Gaussian, dictated by $\exp(uw(x))$([4]). If g is dominating in $y(x,u)$ then the limiting distribution is Gaussian, too, by Theorem 11-b.

- c) We now investigate some cases when g and w have the same finite radius of convergence.

Function g is alg-log: Let us assume that

$$g(x) = \frac{1}{(1-\frac{x}{x_0})^A}\ \log^B \frac{1}{1-\frac{x}{x_0}}, \tag{59}$$

then the distribution of interest is Gaussian for different forms of $w(x)$.

- If $w(x)$ is a logarithmic function

$$w(x) = a\log\frac{1}{1-\frac{x}{x_0}} + C + \mathcal{O}(|\log^{-2}(1-x/x_0)|), \tag{60}$$

  for $A \neq 0$ and $a = 1$, g is dominating in $y(x,u)$, since the heuristic for Rule 5 applies, thus by Theorem 11-b, the limiting distribution is Gaussian with mean value asymptotic to $\log n$.

  For $A = 0$ or $a \neq 1$, $g(x)$ and $F(w(x))$ are incomparable, but the limiting distribution is still Gaussian with mean and variance asymptotic to $a\log n$. This result can be obtained by an application of Corollary 2 :

$$\mathbf{Pr}[X_n = k] \sim \frac{1}{k!}\frac{a^k\log^k(n)}{n^a},$$

for $k = c \log n + \lambda\sqrt{c \log n}$, $\lambda \in R$; the limiting distribution is Poisson with infinite mean, hence the Gaussian result.

For example, if $\hat{a}(x)$ is the exponential generating function of Cayley trees, for $(1-\hat{a}(x))^{-1} \exp(u \log(1-\hat{a}(x))^{-1})$, or $\log(1-\hat{a}(x))^{-1} \exp(u \log(1-\hat{a}(x))^{-1})$, the limiting distribution is Gaussian with mean value asymptotic to $\frac{1}{2} \log n$.

- If $w(x)$ is an algebraic function of the form $x/(1-x)^a$, $a \in \mathbf{R} \backslash \{0, -1, -2, \ldots\}$, then g is dominated in $y(x,u)$, since $F(w(x))$ dominates $g(x)$ (and for $k \to \infty$, $w^k(x)$ dominates $g(x)$ by Lemma 3). Hence the limiting distribution is dictated by $\exp(uw(x))$: it is Gaussian with mean value asymptotic to $n^{1/(a+1)}$.

  For example, consider the exponential generating function for sets of *cycles* or *sequences* of points : $y(x) = \exp\left(\log \frac{1}{1-x} - x + \frac{x}{1-x}\right)$. If we mark the *sequences*, the bivariate generating function is $y(x,u) = \frac{e^{-x}}{1-x} \exp(u \frac{x}{1-x})$, and the limiting distribution is Gaussian with mean value asymptotic to $\sqrt{n}$.

Function g is exponential.

- If $w(x)$ is a logarithmic function, then g is dominating in $y(x,u)$ (Rule 5 applies), and by Theorem 11-b) the limiting distribution is Gaussian with mean value asymptotic to $w(\zeta_n)$.

  For example if $g(x) = \exp(\frac{x}{1-x})$ the mean value is asymptotically $\frac{1}{2} \log n$ (this is the case if we mark the *cycles* in the exponential generating function for sets of *cycles* or *sequences* of points), whereas if $g(x) = \exp(\exp(\frac{x}{1-x}))$ the mean value is asymptotically $\log(\log n)$.

- When $w(x)$ is an algebraic function of the form $x/(1-x)^a$, different cases may occur.

  If $g(x)$ dominates $F(w(x))$ (e.g. $g(x) = \exp\exp(\frac{x}{1-x})$, or $g(x) = \exp \frac{x}{(1-x)^A}$ with $A > a$), then Rule 5 applies: g is dominating in $y(x,u)$ and the limiting distribution is Gaussian with mean value asymptotic to $w(\zeta_n)$, where ζ_n is the saddle point of g.

  On the opposite, if $F(w(x))$ dominates $g(x)$ (i.e. $g(x) = \exp \frac{x}{(1-x)^A}$ with $a > A$), then the limiting distribution is Gaussian, dictated by $\exp(uw(x))$. (Again you can check Rule 3 by observing that $w^k(x)$ dominates $g(x)$ for $k = \mathbf{E}X_n$ by Lemma 3.)

4.3.2 F is alg-log

We essentially consider the product schema $y(x,u) = g(x) 1/(1-uw(x))$.

- a) If g is an entire function, it is dominated in $y(x,u)$, and the limiting distribution is dictated by F. It might be discrete, or Gaussian, or hypergeometric (see [10]).

- b) If g is exponential with the same radius of convergence as $(1-w(x))^{-1}$, then g is dominating in $g(x)/(1-uw(x))^{\alpha}$, $\alpha > 0$, since Rule 5 applies. By Theorem 11-d, the distribution is asymptotically Gamma with parameter α.

- c) We now concentrate on the case when g is an alg-log function of the form (59) with the same radius of convergence as $F(w(x)) = 1/(1-w(x))$.

The limiting distribution in the simple schema $F(uw(x)) = (1-uw(x))^{-1}$ is shown in [10] to be depending on $w(r) = \lim_{x\to r-} w(x)$, where r is the radius of convergence of $w(x)$: It is Gaussian if $w(r) > 1$, and for $w(x)$ with an algebraic singularity, it is discrete if $w(r) < 1$, and hypergeometric if $w(r) = 1$. These 3 cases are also to be considered for the product schema $g(x)F(uw(x))$.

case $\lim_{x\to r-} w(x) < 1$**.** Assume that $w(x)$ has a local expansion of the form

$$w(x) = \tau - c(1-x/r)^d + \sum_{p\geq 2} c_p(1-x/r)^{pd} \tag{61}$$

with $0 < d < 1$, $c > 0$ and $r = x_0$. Then $F(w(x)) = 1/(1-w(x))$ is analytic at τ, and $g(x)$ dominates $F(w(x))$ by Rule 5. Thus g is dominating in $y(x,u)$ and by Theorem 11-a, the limiting distribution is geometric with asymptotic mean value $\frac{\tau F'(w(\tau))}{F(w(\tau))}$.

case $\lim_{x\to r-} w(x) > 1$**.** Then $w(x)$ is analytic at x_0 such that $w(x_0) = 1$.

- If $g(x)$ has an algebraic factor ($A \neq 0$), the mean value and standard deviation of X_n are of order n. It is easy to show that $g(x)$ dominates $F(w(x))$ (and for $k = \lambda n$, $g(x)$ also dominates $w(x)^k$). Thus g is dominating in $y(x,u)$ and by Theorem 11-d, the limiting distribution is Gamma with parameter 1.

- If $g(x)$ is only logarithmic ($A = 0$), $F(w(x))$ dominates $g(x)$. Here the mean value and variance of X_n are of order n (and for $k = \mathbf{E}X_n + \lambda\sqrt{n}$, $w(x)^k$ dominates $g(x)$). Hence g is dominated in $y(x,u)$, and the limiting distribution is Gaussian, dictated by F.

case $\lim_{x\to r-} w(x) = 1$**.** Here again assume that $w(x)$ has a local expansion of the form (61) with $r = x_0$. It can be easily shown that the mean value and standard deviation of X_n in $y(x,u)$ are of order n^d.

- If $g(x)$ is only logarithmic ($A = 0$), $F(w(x))$ dominates $g(x)$. And for $k = \lambda n^d$, $w(x)^k$ dominates $g(x)$. Hence g is dominated in $y(x,u)$, and the limiting distribution is hypergeometric, dictated by F. For $d = 1/2$, the distribution is Rayleigh.

- If $g(x)$ has an algebraic factor ($A \neq 0$), then g is neither dominated nor dominating in $y(x,u)$. But the limiting distribution of X_n is still hypergeometric (see [10]).

  We shall give some examples of such cases, related to random mappings. Let $\hat{a}(x)$ be the exponential generating function for Cayley trees.

  In $y(x,u) = (1 - \hat{a}(x))^{-1}(1 - u\hat{a}(x))^{-1}$, the distribution is asymptotically normal (see [10]).

  In $y(x,u) = (1-\hat{a}(x))^{-2}(1-u\hat{a}(x))^{-1}$, $g(x)$ is neither dominated nor dominating. It is shown in [14] that the random variable X_n, which represents the cycle length of a random point in a random mapping of size n, has the asymptotic probability $\sqrt{\frac{2\pi}{n}}(1 - \Phi(\frac{k}{\sqrt{n}}))$, where $\Phi(x) = \frac{1}{2i\pi}\int_{-\infty}^{x} e^{-t^2/2}dt$.

  In $y(x,u) = (1-\hat{a}(x))^{-1}(1-u\hat{a}(x))^{-2}$, $g(x)$ is again neither dominated nor dominating. The limiting distribution is Rayleigh. It is the distribution of the ρ-length of a random point in a random mapping (see [14]).

  In $y(x,u) = (1-\hat{a}(x))^{-1}(1-u\hat{a}(x))^{-3}$, the limiting distribution is Maxwell (see [10]).

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