Images and Preimages in Random Mappings

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October 4, 1996

Abstract

We present a general theorem which can be used to identify the limiting distribution for a class of combinatorial schemata. For example, many parameters in random mappings can be covered in this way. Especially we can derive the limiting distibution of those points with a given number of total predecessors.

1 Introduction

By a random mapping $\varphi \in \mathcal{F}_n \subseteq \mathcal{F} = \bigcup_{n\geq 0} \mathcal{F}_n$ we mean an arbitrary mapping $\varphi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that every mapping has equal probability n^{-n}. The main purpose of this paper is to obtain limit theorems, when n tends to infinity, for special parameters in random mappings, e.g. for the number of image points. Since every random mapping $\varphi \in \mathcal{F}_n$ has equal probability it suffices to count the number of radom mappings $\varphi \in \mathcal{F}_n$ satisfying a special property, e.g. that the number of image points equals k. By dividing this number by n^n we get the probability of interest. In order to get the limit distribution for $n \to \infty$ it is not necessary to know the exact value. We just have to evaluate these numbers asymptotically. We shalll show that this can be done by a singularity analysis of a proper bivariate generating function.

It should be noted that some of our limit distributions on random mappings are well known (compare with [4, 16]). But our main goal is to provide a general method to derive such limit theorems. In particular, we use bivariate generating functions and singuarity analysis. Especially we are able to characterize the (up to now unknown) limit distribution of the number of those points with a fixed number of total predecessors. It is a Gaussian distribution.

Our basic combinatorial concept is that of labelled combinatorial constructions and the relation to exponential generating functions. A big advantage in such combinatorial constructions is that we can mark a parameter in the constructions which directly leads to bivariate generating function for the number of objects according to their size, and the value of the parameter of interest.

2 Marking in Random Mappings

Every mapping $\varphi \in \mathcal{F}_n$ can be identified with its functional graph G_φ where $V(G_\varphi) = \{1, \ldots, n\}$ and $E(G_\varphi) = \{(i, \varphi(i)) \,|\, 1 \le i \le n\}$. It is obvious that each component of G_φ consists of a cycle (at least of a loop) and every cyclic point is the root of (labelled) tree. (see Figure 1)

FIGURE 1

Hence we can interprete a mapping $\varphi \in \mathcal{F}$ as a set of cycles of trees. Furthermore, since there is no restriction on their structure, the trees (usually known as Cayley trees) can be recursively described as a root followed by a set of trees:

$$\mathcal{F} = \texttt{set}(\texttt{cycle}(\mathcal{T}), \tag{1}$$

$$\mathcal{T} = \circ \cdot \texttt{set}(\mathcal{T}). \tag{2}$$

Both structures, $\mathcal{F}$ and $\mathcal{T}$, fit into the concept of (labelled) combinatorial structures synthetized by Ph. Flajolet [11] (see also [17]). Let us give a short description of such structures.

Let C be a combinatorial stucture of (labelled) elements, let $|c|$ denote the size of $c \in C$ and $c_n = |\{c \in C \,|\, |c| = n\}|$ the number of elements of size n. (Labelled means that there are always $n!$ "isomorphic" elements $c \in C$ of size n which differ by their labels $1, \ldots, n$.) Furthermore we associate to a (labelled) combinatorial structure C the (exponential) generating function

$$\hat{c}(x) = \sum_{c \in C} \frac{x^{|c|}}{(|c|)!} = \sum_{n \ge 0} \frac{c_n}{n!} x^n. \tag{3}$$

The advantage of these generating functions is that there is a correspondence between special combinatorial constructions and special operations with the corresponding generating functions. For example, if the (labelled) combinatorial structure C is the product $C_1 \cdot C_2$ then

$$\hat{c}(x) = \hat{c}_1(x)\hat{c}_2(x). \tag{4}$$

Note that $C_1 \cdot C_2$ is not the set theoretic cartesian product because you have to transform the labelling $\{1, \ldots, k\}$ of $c_1 \in C_1$ and the labelling $\{1, \ldots, n-k\}$ of $c_2 \in C_2$ to a labelling $\{1, \ldots, n\}$ of $c \in C_1 \cdot C_2$. Since there are $k!$ "isomorphic" element in C_1 and $(n-k)!$ "isomorphic" elements in C_2 we have

$$\frac{c_n}{n!} = \sum_{k=0}^{n} \frac{c_{1,k}}{k!} \frac{c_{2,n-k}}{(n-k)!}$$

according to (4). Hence, if $C = \texttt{set}(C_1)$ then we get

$$\hat{c}(x) = \sum_{m\geq 0} \frac{1}{m!}\hat{c}_1^m(x) = e^{\hat{c}_1(x)} \tag{5}$$

and if $C = \texttt{cycle}(C_1)$ then we have

$$\hat{c}(x) = \sum_{m\geq 1} \frac{1}{m}\hat{c}_1^m(x) = \log\frac{1}{1-\hat{c}_1(x)}. \tag{6}$$

Applying this concept to random mappings, the (exponential) generating function

$$\hat{f}(x) = \sum_{n\geq 0} \frac{n^n}{n!}x^n \tag{7}$$

satisfies

$$\hat{f}(x) = \exp\left(\log\frac{1}{1-\hat{t}(x)}\right) = \frac{1}{1-\hat{t}(x)}, \tag{8}$$

where $\hat{t}(x)$, the generating function for Cayley trees, is given by

$$\hat{t}(x) = xe^{\hat{t}(x)}. \tag{9}$$

A big advantage of such combinatorial constructions is that we can formally mark a parameter in the constructions by a symbol like $[u]$. And this marking directly leads to bivariate generating function for the number of objects according to their size, and the value of the parameter of interest.

For example, if we are interested in the number of trees in graphs of random mappings we have to mark the trees in the combinatorial construction

$$\mathcal{F} = \texttt{set}(\texttt{cycle}([u]\mathcal{T})).$$

Formally this leads to

$$\hat{f}(x,u) = \exp\left(\log\frac{1}{1-u\hat{t}(x)}\right) = \frac{1}{1-u\hat{t}(x)}$$

which is exactly the generating function $\hat{f}(x,u) = \sum f_{nk}\frac{x^n}{n!}u^k$ of the numbers f_{nk} of random mappings with k trees in the graph representation. (Note that the number of trees is exactly the number of cyclic points.)

Or if we are interested in the number of components, we have to mark

$$\mathcal{F} = \texttt{set}([u]\texttt{cycle}(\mathcal{T})).$$

and get

$$\hat{f}(x,u) = \exp\left(u\log\frac{1}{1-\hat{t}(x)}\right) = \frac{1}{(1-\hat{t}(x))^u}.$$

Next we will use this marking method to describe special parameters related to image and preimage points.

Points at distance d to a cycle. First we will discuss preimages of cyclic points. For this purpose let Y_φ denote the cyclic points of a random mapping $\varphi \in \mathcal{F}$ and $d \geq 1$. Precisely, we are interested in $\varphi^{-d}(Y_\varphi) \setminus \varphi^{-d+1}(Y_\varphi)$, i.e. non-cyclic points at distance d to the cyclic points.

Lemma 1 *Let*

$$A_0(x,u) = \frac{1}{1-u}$$

and

$$A_{d+1}(x,u) = A_d(x, xe^u)$$

for $d \geq 1$. Then

$$\hat{f}(x,u) = A_d(x, u\hat{t}(x)) \tag{10}$$

is the (exponential) generating function of random mappings where points contained in $\varphi^{-d}(Y_\varphi) \setminus \varphi^{-d+1}(Y_\varphi)$ are marked.

Proof. Let $\hat{t}_d(x,u)$ denote the (exponential) generating function of labelled rooted trees where nodes of distance $d \geq 0$ from the root are marked. Obviously we have

$$\begin{aligned} \hat{t}_0(x,u) &= u\hat{t}(x) \qquad \text{and} \\ \hat{t}_{d+1}(x,u) &= xe^{\hat{t}_d(x,u)} \quad \text{for } d \geq 0 \end{aligned}$$

which directly leads to

$$\hat{f}(x,u) = \frac{1}{1-\hat{t}_d(x,u)} = A_d(x, u\hat{t}(x)).$$

□

Points with in-degree r. Another interesting parameter is the number of points ν with $|\varphi^{-1}(\{\nu\})| = r$, where $r \geq 0$ is a fixed integer.

Lemma 2 *Let $\hat{p}_r(x,u)$ denote the solution of*

$$\hat{p}_r(x,u) = xe^{\hat{p}_r(x,u)} + (u-1)x\frac{\hat{p}_r(x,u)^r}{r!}. \tag{11}$$

Then

$$\hat{f}(x,u) = \frac{1}{1-\left(xe^{\hat{p}_r(x,u)} + (u-1)x\frac{\hat{p}_r(x,u)^{r-1}}{(r-1)!}\right)} \tag{12}$$

is the (exponential) generating function of random mappings where points ν with $|\varphi^{-1}(\{\nu\})| = r$ are marked.

Proof. According to the recusive structure of Cayley trees $\mathcal{T} = \circ \cdot \mathtt{set}(\mathcal{T})$, the nodes with in-degree r are those followed by r subtrees. Hence the bivariate generating function for trees, with variable u marking nodes with in-degree r satisfies

$$\begin{aligned}\hat{p}_r(x,u) &= x\sum_{m\neq r}\frac{\hat{p}_r(x,u)^m}{m!} + ux\frac{\hat{p}_r(x,u)^r}{r!}\\ &= xe^{\hat{p}_r(x,u)} + (u-1)x\frac{\hat{p}_r(x,u)^r}{r!}\end{aligned}$$

Now a cyclic point in the functional graph of a random mapping has in-degree r if and only if it has in-degree $r-1$ in the corresponding trees. This proves (12). $\square$

Notice that the expression of the bivariate generating function is simpler if we neglect the edges between cyclic points (i.e. cyclic points are marked if they have in-degree $r+1$, and non cyclic points are marked if they have in-degree r). Actually we then consider sequences of Cayley trees instead of random mappings.

Lemma 2bis *The (exponential) generating function of sequences of Cayley trees where marked nodes are those with in-degree r is*

$$\hat{f}(x,u) = \frac{1}{1-\hat{p}_r(x,u)}\,, \tag{13}$$

where $\hat{p}_r(x,u)$ is the same as in Lemma 2.

Points with r-antecedents. Finally we want to count those points where the total number of preimages equals $r \geq 0$.

Lemma 3 *Let $\hat{a}_r(x,u)$ denote the solution of*

$$\hat{a}_r(x,u) = xe^{\hat{a}_r(x,u)} + (u-1)t_r x^r, \tag{14}$$

where $t_r = \frac{r^{r-1}}{r!}$. Then

$$\hat{f}(x,u) = \frac{1}{1-\hat{a}_r(x,u)+(u-1)t_r x^r}\exp\left(\frac{x^r}{r}\sum_{m=0}^{r}\frac{r^{r-m}}{(r-m)!}(u^m-1)\right) \tag{15}$$

is the (exponential) generating function of random mappings where points ν with

$$\left|\bigcup_{d\geq 0}\varphi^{-d}(\{\nu\})\right| = r$$

are marked.

Proof. As in the proof of Lemma 2 we first mark the nodes with total number of preimages r in Cayley trees: a node is marked if and only if it is the root of a tree of total size r (the root is considered to be its own preimage). Hence we get (14), where t_r is the coefficient of x^r in the series expansion

$$\hat{t}(x) = \sum_{n \geq 0} t_n x^n$$

Since $\hat{t}(x)$ satisfies the functional equation (9) we get by Lagrange's inversion theorem

$$t_n = \frac{1}{n}[y^{n-1}]e^{yn} = \frac{n^{n-1}}{n!}.$$

For cyclic points in a random mapping all points in the corresponding component (of the functional graph) are preimages. Hence for a component with m cyclic points, the bivariate generating function, with u marking the number of points having r preimages, is

$$\frac{\left(xe^{\hat{a}_r(x,u)}\right)^m}{m} + \frac{1}{m}(u^m - 1)x^r[v^r]\hat{t}(v)^m,$$

where

$$[v^r]\hat{t}(v)^m = \frac{m}{r}[y^{r-m}]e^{yr} = \frac{mr^{r-m}}{r(r-m)!}$$

is the number of forests composed with m components, and of total size r. This directly gives (15). □

d-th iterate points. It is also interesting to consider $\varphi^d(\{1, \ldots, n\})$, the d-th iterate image points.

Lemma 4 *Set*

$$\begin{aligned} h_0(x) &= 0 \qquad \textit{and} \\ h_{i+1}(x) &= xe^{h_i(x)} \quad \textit{for } i \geq 0. \end{aligned}$$

and let $\hat{i}_d(x, u)$ be the solution of

$$\hat{i}_d(x, u) = xue^{\hat{i}_d(x,u)} - (u-1)h_d(x). \tag{16}$$

Then

$$\hat{f}(x, u) = \frac{1}{1 - \hat{i}_d(x, u) - (u-1)h_d(x)} \tag{17}$$

is the (exponential) generating function of random mappings where points $\nu \in \varphi^d(\{1, \ldots, n\})$ are marked.

Proof. Clearly, $h_d(x)$ is the (exponential) generating function of Cayley trees with height $< d$. In Cayley trees, d-th iterate image points are points at distance $\geq d$ from a leaf. Hence $\hat{i}_d(x,u)$, the bivariate generating function of trees where nodes having a leaf at distance $\geq d$ are marked satisfies (16). For random mappings, since all cyclic points are d-th iterate image points we get

$$\hat{f}(x,u) = \frac{1}{1 - uxe^{\hat{i}_d(x,u)}}$$

which leads to (17). □

Here again, as in the case of points with in-degree r, the expression of the bivariate generating function is simpler if we neglect the edges between cyclic points:

Lemma 4bis *The (exponential) generating function of random mappings where the marked points are those at distance $\geq d$ from a leaf of their own subtree is*

$$\hat{f}(x,u) = \frac{1}{1 - \hat{i}_d(x,u)}\,, \tag{18}$$

where $\hat{i}_d(x,u)$ is the same as in Lemma 4.

Direct d-th iterate points. The most difficult example (from the combinatorial point of view) is the case of d-th iterate image points of non-image points $\varphi^d(\{1,\ldots,n\} \setminus \varphi(\{1,\ldots,n\}))$. In other words we will count those nodes which are connected by a (directed) path of length d to a non-image point. Nevertheless there is a rather easy subcase where edges between cyclic points are neglected, i.e. the problem can be reduced to a problem inside trees. Although this is only a very small change it will turn out that the corresponding limiting distibutions differ. (And it will also be the case for Lemmas 2 and 4 vs. Lemmas 2bis and 4bis.)

Lemma 5 *Set*

$$\begin{aligned} c_0(x,y) &= x \qquad \text{and} \\ c_{i+1}(x,y) &= x\left(e^y - e^{y-c_i(x,y)}\right) \quad \text{for } i \geq 0. \end{aligned}$$

and let $\hat{l}_d(x,u)$ be the solution of

$$\hat{l}_d(x,u) = xe^{\hat{l}_d(x,u)} + (u-1)c_d(x, \hat{l}_d(x,u)). \tag{19}$$

Then

$$\tilde{f}(x,u) = \exp\left(\log \frac{1}{1 - \hat{l}_d(x,u)}\right) = \frac{1}{1 - \hat{l}_d(x,u)}. \tag{20}$$

is the (exponential) generating function of random mappings where marked points are those connected to a leaf by a path of length d which does not contain cyclic edges. (Note that the root of a tree of size 1 is also a leaf.)

Proof. Let $\hat{l}_d(x,u)$ denote the generating function of Cayley trees where nodes having a leaf at distance d are marked. The generating function where leaves are marked is $\hat{l}_0(x,u) = ux + x(e^{l_0(x,u)} - 1)$, and inductively $\hat{l}_d(x,u) = y_d(x,u) + xe^{\hat{l}_d(x,u) - y_{d-1}(x,u)}$, where $y_d(x,u)$ is the generating function for trees with a leaf at distance d to the root: $y_d(x,u) = uc_d(x, \hat{l}_d(x,u))$, and for $i = 1, \ldots, (d-1)$, $y_i(x,u) = c_i(x, \hat{l}_d(x,u))$. Since $xe^{\hat{l}_d(x,u) - y_{d-1}(x,u)} = xe^{\hat{l}_d(x,u)} - c_d(x, \hat{l}_d(x,u))$, represents the trees such that the root is not at distance d to a leaf, $\hat{l}_d(x,u)$ satisfies (19) and hence (20). □

Lemma 5bis *Let $c_i(x,y)$ and $\hat{l}_d(x,u)$ be defined as in Lemma 5 and let $\hat{f}_d(x,u)$ be the (exponential) generating function of random mappings where points $\nu \in \varphi^d(\{1,\ldots,n\} \setminus \varphi(\{1,\ldots,n\}))$ are marked.*

For $d = 0$ and $d = 1$ we have

$$\hat{f}_0(x,u) = \frac{1}{1 - xe^{\hat{l}_0(x,u)}} \quad \text{and} \quad \hat{f}_1(x,u) = \frac{1}{1 - \hat{l}_1(x,u)}. \tag{21}$$

For $d = 2$ set $y_1(x,u) = c_1(x, \hat{l}_2(x,u))$, $y_2(x,u) = uc_2(x, \hat{l}_2(x,u))$, and

$$y_{12}(x,u) = ux\left(e^{\hat{l}_2(x,u)} - e^{\hat{l}_2(x,u)-x} - e^{\hat{l}_2(x,u)-y_1(x,u)} + e^{\hat{l}_2(x,u)-y_1(x,u)-x}\right).$$

Then

$$\hat{f}_2(x,u) = \frac{1}{1 - \hat{l}_2(x,u) - (u-1)\left(y_1(x,u)(1 - y_2(x,u)) - y_{12}(x,u)(1 - \hat{l}_2(x,u))\right)}. \tag{22}$$

Proof. The results for $\hat{f}_0(x,u)$ and $\hat{f}_1(x,u)$ are obvious: for $d = 0$ cyclic points are not leaves of random mappings, and for $d = 1$ edges inside cycles are of no importance. For $d = 2$, the situation gets more delicate: in addition to the interpretations of $y_1(x,u)$ and $y_2(x,u)$ observe that $y_{12}(x,u)$ corresponds to those trees having both a leaf at distance 1 to the root and a leaf at distance 2 to the root. (For the sake of shortness we will use the terms y_1-tree (resp. y_2-tree) for a tree with a leaf at distance 1 (resp. 2) to the root.) Set

$$\begin{aligned} w &= u(y_1 - y_{12}) + (\hat{l}_2 - y_1 - y_2 + y_{12}) \qquad \text{and} \\ s &= uy_{12} + (y_2 - y_{12}). \end{aligned}$$

Then w corresponds to additionally marking y_1-trees which are not y_2-trees. In the same way s corresponds to marking twice y_2-trees which are y_1-trees too. We will show (at the end of the proof) that the generating function

$$\frac{w^m}{m} + \frac{y_2^m}{m} + \frac{1}{m}\sum_{k=1}^{m-1}\sum_{l=1}^{k} A_{mkl}\, s^l (\hat{l}_2 - y_2)^l w^{m-k-l} y_2^{k-l} \tag{23}$$

corresponds to a cycle of m trees where all nodes having a leaf at distance 2 are marked (A_{mkl}, see below, counts the number of cycles of length m containing k y_2-trees, $(k-l)$ of which are followed by an y_2-tree). Since

$$\sum_{m\geq 1}\frac{1}{m}\sum_{k=1}^{m-1}\sum_{l=1}^{k} A_{mkl}\, s^l(\hat{l}_2-y_2)^l w^{m-k-l} y_2^{k-l}$$
$$= \log\frac{1}{1-y_2-\frac{s(\hat{l}_2-y_2)}{1-w}} - \log\frac{1}{1-y_2}$$

we immediately get (22).

Therefore it remains to interprete (23). The problem on a cycle is that a y_1-tree forces an additional mark at the next root on the cycle if and only if this next root is not marked, i.e. the corresponding tree is not an y_2-tree. For example, if a cycle of length m contains no y_2-tree then it is immediately clear that $\frac{1}{m}w^m$ is the correct corresponding function, whereas the case of a cycle containing only y_2-trees the generating function of interest is $\frac{1}{m}y_2^m$. For the remaining cases consider a cycle containing exactly k $(0<k<m)$ y_2-trees such that l $(0<l\leq k)$ of these trees are followed by a tree which is not a y_2-tree. Note that

$$A_{mkl} = \frac{m}{l}\binom{m-k-1}{l-1}\binom{k-1}{l-1} \tag{24}$$

is the number of such arrangements on a (labelled) cycle of length m. In any of these cases the corresponding generating function is $\frac{1}{m}y_2^{k-l}s^l w^{m-k-l}(\hat{l}_2-y_2)^l$. This proves (23). $\square$

3 General Theorems

Let $c(x)=\sum c_n x^n$ be the generating function of a combinatorial structure and $c(x,u)=\sum c_{nk}x^n u^k$ the bivariate generating function where a parameter of interest has been marked, i.e. $c(x,1)=c(x)$. We will now be interested in the asymptotic distribution of this parameter in the system of combinatorial objects of size n, when n tends to infinity. For this purpose we introduce a sequence of random variables X_n, $n\geq 1$, defined by

$$\mathbf{Pr}[X_n=k] = \frac{c_{nk}}{c_n} = \frac{[x^n u^k]c(x,u)}{[x^n]c(x,1)},$$

where **Pr** denotes probability. Now the above problem reduces to finding the limiting distribution of X_n.

An important analytic schema, related to combinatorial constructions "sequence" or "set of cycles", is

$$c(x,u) = \frac{1}{1-a(x,u)} \ .$$

The next three theorems study this schema when $a(x,u)$ has an algebraic singularity $\rho(u)$ of squareroot type, such that $a(\rho(1),1)=1$. According to further analytic properties of $a(x,u)$, the limiting distribution of X_n is shown to be either Gaussian, or Rayleigh, or the convolution of Gaussian and Rayleigh ; and in each case the global limit result (convergence of distribution functions) is accompanied by a local limit result (convergence of densities).

Let us first state precisely the general form of the analytic schemas under consideration.

Hypothesis [H]. *Let $c(x,u)=\sum_{n,k} c_{nk}x^n u^k$ be a power series in two variables with non-negative coefficients $c_{nk}\geq 0$, such that $c(x,1)$ has radius of convergence $\rho>0$.*

We suppose that $c(x,u)$ expresses as $c(x,u)=1/d(x,u)$, where $d(x,u)$ has the local representation

$$d(x,u)=g(x,u)+h(x,u)\sqrt{1-\frac{x}{\rho(u)}} \tag{25}$$

for $|u-1|<\varepsilon$ and $|x-\rho(u)|<\varepsilon$, $\arg(x-\rho(u))\neq 0$, where $\varepsilon>0$ is some fixed real number and $g(x,u)$, $h(x,u)$, and $\rho(u)$ are analytic functions.

Furthermore these functions satisfy $g(\rho,1)=0$, $h(\rho,1)>0$, and $\rho(1)=\rho$.

In addition $x=\rho(u)$ is the only singularity on the circle of convergence $|x|=|\rho(u)|$ for $|u-1|<\varepsilon$ and $d(x,u)$, resp. $c(x,u)$, can be analytically continued to a region $|x|<\rho+\delta$, $|u|<1+\delta$, $|u-1|>\frac{\varepsilon}{2}$ for some $\delta>0$.

Under this hypothesis, the limiting distribution of X_n in $c(x,u)$ depends on $\rho'(1)$ and $g_u(\rho(1),1)$, as stated in the following three theorems.

Theorem 1 *Let $c(x,u)$ be a bivariate generating function satisfying* [**H**]. *If $\rho(u)=\rho=const.$ for $|u-1|<\varepsilon$ and $g_u(\rho,1)<0$ then the sequence of random variables X_n defined by*

$$\mathbf{Pr}[X_n=k]=\frac{[x^n u^k]c(x,u)}{[x^n]c(x,1)} \tag{26}$$

has a Rayleigh limiting distribution, i.e.

$$\frac{X_n}{\sqrt{n}}\xrightarrow{d}\mathcal{R}(\lambda), \tag{27}$$

where $\lambda=\frac{h(\rho,1)^2}{2g_u(\rho,1)^2}$ and $\mathcal{R}(\lambda)$ has density $\lambda x\exp\left(-\frac{\lambda}{2}x^2\right)$ for $x\geq 0$. Expected value and variance are given by

$$\mathbf{E}X_n=\sqrt{\frac{\pi}{2\lambda}}\sqrt{n}+\mathcal{O}(1) \quad \text{and} \quad \mathbf{V}X_n=\left(2-\frac{\pi}{2}\right)\frac{n}{\lambda}+\mathcal{O}(\sqrt{n}). \tag{28}$$

Moreover we have the following local law

$$\mathbf{Pr}[X_n = k] = \frac{\lambda k}{n} \exp\left(-\frac{\lambda k^2}{2n}\right) + \mathcal{O}((k+1)n^{-\frac{3}{2}}) + \mathcal{O}(n^{-1}) \tag{29}$$

uniformly for all $k \geq 0$.

Theorem 2 *Let $c(x,u)$ be a bivariate generating function satisfying* $[\mathbf{H}]$. *If $\rho'(1) \neq 0$ and $\alpha = \frac{\partial}{\partial u} g(\rho(u),u)|_{u=1} = 0$ then X_n has a Gaussian limiting distribution, i.e.*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1), \tag{30}$$

where $\mu = -\rho'(1)/\rho$ and $\sigma^2 = \mu^2 + \mu - \rho''(1)/\rho$. Expected value and variance are given by

$$\mathbf{E}X_n = \mu n + \mathcal{O}(1) \quad \textit{and} \quad \mathbf{V}X_n = \sigma^2 n + \mathcal{O}(\sqrt{n}). \tag{31}$$

Furthermore there is local law of the form

$$\mathbf{Pr}[X_n = k] = \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{(k-\mu n)^2}{2\sigma^2 n}\right) + \mathcal{O}(n^{-\frac{3}{4}}) \tag{32}$$

uniformly for all $k \geq 0$.

Theorem 3 *Let $c(x,u)$ be a bivariate generating function satisfying* $[\mathbf{H}]$. *If $\rho'(1) \neq 0$ and $\alpha = \frac{\partial}{\partial u} g(\rho(u),u)|_{u=1} < 0$ then the limiting distribution of X_n is the convolution of a Gaussian and a Rayleigh distribution, i.e.*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1) * \mathcal{R}(\lambda), \tag{33}$$

where $\lambda = \frac{h(\rho,1)^2\sigma^2}{2\alpha^2}$ and μ and σ^2 are defined as in Theorem 2. Expected value and variance are given by

$$\mathbf{E}X_n = \mu n - \frac{\sqrt{\pi}\alpha}{h(\rho,1)}\sqrt{n} + \mathcal{O}(1) \quad \textit{and} \quad \mathbf{V}X_n = \left(\sigma^2 + \frac{(4-\pi)\alpha^2}{h(\rho,1)^2}\right) n + \mathcal{O}(\sqrt{n}) \tag{34}$$

and there is local law of the form

$$\begin{aligned} \mathbf{Pr}[X_n = k] &= \frac{\lambda}{1+\lambda}\frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{(k-\mu n)^2}{\sigma^2 n}\right) \\ &+ \frac{\lambda}{(1+\lambda)^{\frac{3}{2}}}\frac{k-\mu n}{\sigma^2 n} \exp\left(-\frac{\lambda}{1+\lambda}\frac{(k-\mu n)^2}{\sigma^2 n}\right) \Phi\left(\frac{k-\mu n}{\sqrt{(1+\lambda)\sigma^2 n}}\right) \\ &+ \mathcal{O}(n^{-\frac{3}{4}}) \end{aligned} \tag{35}$$

uniformly for all $k \geq 0$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt. \tag{36}$$

(If $\alpha > 0$ then the corresponding Rayleigh distribution is supported on the negative real axis and a similar local law holds.)

Remark. It should be noticed, that condition $g(\rho, 1) = 0$ in the theorems is not a real restriction. In fact, it turns out that the case $g(\rho, 1) = 0$ is the most difficult one, and the limiting distribution for other cases can also be found.

If $g(\rho, 1) > 0$ then $c(x, u)$ has a local representation of the form

$$c(x,u) = \frac{1}{g(x,u) + h(x,u)\sqrt{1 - x/\rho(u)}} = G(x,u) - H(x,u)\sqrt{1 - \frac{x}{\rho(u)}}. \tag{37}$$

On the other hand, if $g(\rho, 1) < 0$, the algebraic singularity is not the dominating one. Here $d(\overline{\rho}, 1) = 0$ for $\overline{\rho} < \rho$ and (usually) $d_x(\overline{\rho}, 1) \neq 0$. Hence by the Weierstrass preparation theorem $d(x, u)$ has a local representation of the form $d(x, u) = D(x, u)(1 - x/\overline{\rho}(u))$, where $D(x, u)$ and $\overline{\rho}(u)$ are analytic functions satisfying $D(\overline{\rho}, 1) \neq 0$, $\overline{\rho}(1) = \overline{\rho}$, and $\overline{\rho}'(1) \neq 0$. Thus

$$c(x,u) = \frac{1/D(x,u)}{1 - \frac{x}{\overline{\rho}(u)}}. \tag{38}$$

In both cases, (37) and (38), we can apply Bender's theorem [5] (compare also with [6] and [7]) to get asymptotic normality if $\rho'(1) < 0$ (Evaluating the expected value shows that $\rho'(1)$ cannot be positive).

When $\rho(u) = \text{const.}$ (see also [14] for this case), the limiting distribution is Gaussian for $g(\rho, 1) > 0$, and discrete for $g(\rho, 1) < 0$ (for example there is a derivated geometric law for the schema $c(x, u) = (1 - ua(x))^{-1}$).

Finally we want to remark that the assumption $\rho(u) = \text{const}$ in Theorem 1 can be weakened to $\rho'(1) = 0$. However, the proof would be a little bit more complicated. Furthermore no example is known, where $\rho'(1) = 0$ but $\rho(u) \neq$ const..

Before proving Theorems 1,2,3 (see section 5) we want to motivate why such theorems have some importance in relation to random mappings.

4 Applications to Random Mappings

In this section we apply our theorems to obtain the limiting distributions for various parameters of random mappings. It should be noted that some of the obtained results are known but our intention is to provide all the results by applying only one general principle. The underlying point is that the combinatorial specification of random mappings out of Cayley trees, together with the analytic form of the Cayley trees series, imply that all bivariate generating functions $\hat{f}(x,u)$ constructed in section 2 satisfy hypothesis [**H**].

4.1 Analytic frame

The basic property is that solutions of functional equations usually have algebraic singularities of squareroot type.

Proposition 1 *Let $F(a,x,u)$ be a power series on three variables with non-negative coefficients and $F(0,0,0)=0$. Suppose that the system of equations*

$$\begin{aligned} a_0 &= F(a_0,x_0,1), &(39)\\ 1 &= F_a(a_0,x_0,1) &(40)\end{aligned}$$

has positive solutions $a_0>0$, $x_0>0$ (which are supposed to be minimal) such that $(a_0,x_0,1)$ is contained in the region of convergence of $F(a,x,u)$ and that

$$F_x(a_0,x_0,1)\neq 0 \quad \text{and} \quad F_{aa}(a_0,x_0,1)\neq 0. \tag{41}$$

Then there exists a unique analytic solution $a=a(x,u)=\sum_{nk} a_{nk}x^n u^k$ of

$$a=F(a,x,u) \tag{42}$$

with non-negative coefficients $a_{nk}\geq 0$ and $a_{00}=0$ such that $a(x,u)$ has the local representation

$$a(x,u)=g(x,u)-h(x,u)\sqrt{1-\frac{x}{\rho(u)}} \tag{43}$$

for $|u-1|<\varepsilon$ and $|x-\rho(u)|<\varepsilon$, $\arg(x-\rho(u))\neq 0$, where $g(x,u)$, $h(x,u)$, and $\rho(u)$ are analytic functions which satisfy

$$g(x_0,1)=a_0,\quad h(x_0,1)=\sqrt{\frac{2x_0F_x(a_0,x_0,1)}{F_{aa}(a_0,x_0,1)}},\quad \textit{and}\quad \rho(1)=x_0 \tag{44}$$

and $\varepsilon>0$ is some fixed real number. Furthermore, if there are n_1,n_2,n_3 and $k_1<k_2<k_3$ such that $a_{n_1k_1}a_{n_2k_2}a_{n_3k_3}>0$ and $\gcd(k_3-k_1,k_2-k_1)=1$ and if

$$\gcd\left\{n-l\,:\,\sum_k a_{nk}>0\right\}=1,$$

where

$$l=\min\left\{m\,:\,\sum_k a_{mk}>0\right\},$$

then $x = \rho(u)$ is the only singularity on the circle of convergence $|x| = |\rho(u)|$ for $|u-1| < \varepsilon$ and there exists some $\delta > 0$ such that $a(x,u)$ can be analytically continued to the region $|x| < x_0 + \delta$, $|u| < 1 + \delta$, $|u-1| > \frac{\varepsilon}{2}$.

The *Proof* of Proposition 1 is a combination of the implicit function theorem and the Weierstrass preparation theorem (cf. [7, 8]).

Now it is easy to see the connection to random mappings. In any of the above combinatorial constructions the solution $a(x,u)$ (satisfying $a(\frac{1}{e},1) = 1$) of a functional equation of the type (42) is used to construct a final generating function which is more or less of the form

$$c(x,u) = \frac{1}{1-a(x,u)}.$$

Hence we can directly apply our theorems to obtain the kind of asymptotic distribution.

4.2 Distribution of parameters

The examples mentioned in section 2 cover the three types of limiting distributions, Gaussian, Rayleigh, or a convolution of both. It should be noted that in applications 2,4 and 5, small structural modifications (neglecting cyclic edges) lead to different limit laws.

For the sake of shortness we will only mention the weak convergence law. However, in all the cases the local law and the asymptotic expansions for mean and variance hold, too.

Application 1 ([16]) *Let X_n denote the number of non cyclic points at a fixed distance $d > 0$ to a cycle, in random mappings of size n. Then*

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{R}(1). \tag{45}$$

Proof. From Proposition 1 it follows that $\hat{t}(x)$ has a local representation of the kind

$$\hat{t}(x) = a(x) - b(x)\sqrt{1-ex},$$

where $a(x)$ and $b(x)$ are analytic functions around $x_0 = \frac{1}{e}$ with $a(\frac{1}{e}) = 1$ and $b(\frac{1}{e}) = \sqrt{2}$. Furthermore we can use the Taylor series expansion of

$$A_d(x,u)^{-1} = \sum_{l,k \geq 0} c_{lk}(u-1)^l (x - \frac{1}{e})^k,$$

where $c_{00} = 0$ and $c_{10} = -1$ to see that $A_d(x, u\hat{t}(x,u))$ has a representation of the kind

$$A_d(x, u\hat{t}(x,u))^{-1} = g(x,u) + h(x,u)\sqrt{1-ex},$$

where $g(\frac{1}{e},1) = c_{00} = 0$ and $h(\frac{1}{e},1) = -c_{10}b(\frac{1}{e}) = \sqrt{2}$. Hence we can apply Theorem 1. □

Application 2 ([4]) *Let $r \geq 0$ be a fixed integer and let X_n denote the number of points ν with $|\varphi^{-1}(\{\nu\})| = r$ in mappings $\varphi \in \mathcal{F}_n$. Then*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1), \tag{46}$$

where $\mu = \frac{1}{er!}$ and $\sigma^2 = \mu + (1+(r-1)^2)\mu^2$.

Proof. Let

$$F(p,x,u) = xe^p + (u-1)x\frac{p^r}{r!}.$$

Then another application of Proposition 1 provides a local representation of

$$\hat{p}_r(x,u) = a(x,u) - b(x,u)\sqrt{1 - \frac{x}{\rho(u)}},$$

where $\hat{p}_r(x,1) = \hat{t}(x)$ and consequently $\rho(1) = \frac{1}{e}$, $a(\frac{1}{e},1) = 1$, and $b(\frac{1}{e},1) = \sqrt{2}$. Hence we obain

$$\begin{aligned}
\hat{f}(x,u) &= \frac{1}{1 - \left(\hat{p}_r(x,u) - (u-1)x\frac{\hat{p}_r(x,u)^r}{r!} + (u-1)x\frac{\hat{p}_r(x,u)^{r-1}}{(r-1)!}\right)} \\
&= \frac{1}{g(x,u) + h(x,u)\sqrt{1 - x/\rho(u)}},
\end{aligned}$$

in which $g(\frac{1}{e},1) = 0$, $h(\frac{1}{e},1) = \sqrt{2}$, and

$$\alpha = \frac{\partial}{\partial u} g(\rho(u),u)|_{u=1} = -\frac{\partial}{\partial u} a(\rho(u),u)|_{u=1} + \frac{1}{er!} - \frac{1}{e(r-1)!}.$$

Since $p(u) = a(\rho(u),u) = \hat{p}_r(\rho(u),u)$ satisfies the system of equations

$$\begin{aligned}
p(u) &= F(p(u),\rho(u),u) \\
1 &= F_p(p(u),\rho(u),u)
\end{aligned}$$

implicit differentiation gives

$$\begin{aligned}
\rho'(1) &= -\frac{F_u(1,\frac{1}{e},1)}{F_x(1,\frac{1}{e},1)} = -\frac{1}{er!}, \\
\rho''(1) &= \frac{1}{F_{pp}F_x^3}(F_x^2(F_{pu}^2 - F_{pp}F_{uu}) + F_u^2(F_{px}^2 - F_{pp}F_{xx}) \\
&\qquad -2F_xF_u(F_{px}F_{pu} - F_{pp}F_{xu})) \\
&= -\frac{(r-1)^2}{e^3(r!)^2}, \\
p'(1) &= \frac{F_{px}F_u - F_{pu}F_x}{F_{pp}F_x} = \frac{1}{er!} - \frac{1}{e(r-1)!}.
\end{aligned}$$

Thus we can apply Theorem 2. □

Application 2bis *Let $r \geq 0$ be a fixed integer and let X_n denote the number of nodes with in-degree r in a sequence of Cayley trees of total size n. Then*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1) * \mathcal{R}(\lambda),$$

where μ and σ^2 are the same as in Application 2, and $\lambda = \frac{\sigma^2 (er!)^2}{(1-r)^2}$. In the special case $r = 1$, the limiting distribution is only Gaussian since $\lambda^{-1} = 0$.

Proof. In this case $\alpha = p'(1)$ is not equal to 0, except for $r = 1$. Hence the convolution results by Theorem 3. □

Application 3 *Let $r \geq 0$ be a fixed integer and let X_n denote the number of points ν with*

$$\left| \bigcup_{d \geq 0} \varphi^{-d}(\{\nu\}) \right| = r$$

in mappings $\varphi \in \mathcal{F}_n$. Then

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1), \tag{47}$$

where $\mu = \frac{r^{r-1}}{r!} e^{-r}$ and $\sigma^2 = \mu - 2r\mu^2$.

Proof. First notice that the analytic factor $\exp(\cdots)$ in (15) has no influence on the parameters of interest α, $\rho'(1)$, and $\rho''(1)$. Therefore we can neglect it. Hence we can proceed as in the proof on Application 2. Here we have

$$\begin{aligned} \rho'(1) &= -t_r e^{-r-1}, \\ \rho''(1) &= (1+2r) t_r^2 e^{-2r-1}, \\ a'(1) &= t_r e^{-r}. \end{aligned}$$

Consequently $\alpha = 0$ and we obtain a Gaussian limiting distribution. □

Application 4 *Let $d \geq 0$ be a fixed integer and let X_n denote the number of points $\nu \in \varphi^d(\{1, \ldots, n\})$ mappings $\varphi \in \mathcal{F}_n$. Then*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1), \tag{48}$$

where $\mu = h_d(\frac{1}{e})$ and $\sigma^2 = \frac{2}{e} h_d'(\frac{1}{e})(1-\mu) - \mu$.

Proof. The proof is almost the same as the proof of Application 2. □

We want to mention that the mean value was already determined in [13].

Application 4bis *Let $d \geq 0$ be a fixed integer and let X_n denote the number of points at distance $\geq d$ from a leaf of their own subtree, in random mappings of size n. Then*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1) * \mathcal{R}(\lambda), \tag{49}$$

where μ and σ^2 are the same as in Application 4, and $\lambda = \frac{\sigma^2}{h_d^2(\frac{1}{e})}$.

Proof. In this case $\alpha = h_d(\frac{1}{e}) \neq 0$. Hence the convolution result. □

Application 5 *Let $d \geq 0$ be fixed and X_n denote the number of nodes which are connected to a leaf by a path of length d containing no cyclic edge, in random mappings of size n. Then*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1) * \mathcal{R}(\lambda), \tag{50}$$

where $\mu = c_d = c_d(\frac{1}{e}, 1)$, $\sigma^2 = c_d(1 - \frac{2}{e}c_{d,x}) + 2c_d c_{d,y} - c_{d,y}^2$, and $\lambda^{-1} = (c_d - c_{d,y})^2/\sigma^2$ ($c_{d,x} = \frac{\partial}{\partial x}c_d(\frac{1}{e}, 1)$, $c_{d,y} = \frac{\partial}{\partial y}c_d(\frac{1}{e}, 1)$). Since $\lambda^{-1} = 0$ for $d = 1$ the limiting distribution is only Gaussian in this special case.

Proof. The proof runs along the same lines as the preceding ones. You only have to apply Theorem 3 since $\alpha = c_d - c_{d,y} \neq 0$ for $d \neq 1$. □

Application 5bis *Let $d \in \{0, 1, 2\}$ be fixed and let X_n be the number of points $x \in \varphi^d(\{1, \ldots, n\} \setminus \varphi(\{1, \ldots, n\}))$ in mappings $\varphi \in \mathcal{F}_n$. Then*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1), \tag{51}$$

where μ and σ^2 are as in Application 5, i.e.

$$c_d(e^{-1}, 1) = \begin{cases} e^{-1} & \text{for } d = 0, \\ 1 - e^{-e^{-1}} & \text{for } d = 1 \\ 1 - e^{-(1-e^{-1/e})} & \text{for } d = 2, \end{cases}$$

and

$$\sigma^2 = \begin{cases} e^{-1} - 2e^{-2} & \text{for } d = 0, \\ e^{-e^{-1}}\left(1 - 2e^{-1}\right)\left(1 - e^{-e^{-1}}\right) & \text{for } d = 1, \\ 2e^{-1+e^{-1/e}-e^{-1}} - e^{-1+e^{-1/e}} & \text{for } d = 2. \\ \quad -2e^{-2+e^{-1/e}-e^{-1}} - e^{-2+2e^{-1/e}-2e^{-1}} & \\ \quad +2e^{-3+2e^{-1/e}-e^{-1}} & \end{cases}$$

Proof. Especially in the case $d = 2$ you have to calculate α very carefully but in all the cases $\alpha = 0$.□

Note: in the case $d > 2$, the combinatorial description is much more involved. Nevertheless it may be conjectured that the limiting distribution is still Gaussian.

5 Proof of the Theorems

The proofs of Theorems 1,2,3 proceed in the following way. First we derive asymptotic expansions for mean value and variance, then we prove a weak limit theorem using characteristic functions and finally we establish the corresponding local limit theorem. This procedure seems to be redundant, and in fact it is. But our aim is not only to prove special theorems, but to provide an example for a general method to analyze the asymptotic distribution of a parameter in combinatorial constructions.

5.1 Preliminaries

We first list some useful formulae related to Gaussian and Rayleigh distributions.

Lemma 6 *Let γ be a Hankel contour starting from $+e^{2\pi i}\infty$, passing around 0 and tending to $+\infty$. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{-z}}{\sqrt{-z}-is}\,dz = \frac{1}{\sqrt{\pi}}\varphi_{\mathcal{R}}(\sqrt{2}s), \tag{52}$$

where

$$\begin{aligned} \varphi_{\mathcal{R}}(t) &= \int_0^\infty e^{itx} x e^{-x^2/2}\,dx \\ &= 1 + ite^{-t^2/2}\left(\sqrt{\frac{\pi}{2}} - i\int_0^t e^{u^2/2}\,du\right) \end{aligned}$$

denotes the characteristic function of the Rayleigh distribution.

Proof. It suffices to compare the Taylor expansion around $s = 0$. By the Hankel intregral representation of $\Gamma(s)^{-1}$ we get

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{-z}}{\sqrt{-z}-is} = \frac{1}{2\pi i}\int_\gamma \sum_{n\ge 0}(is)^n(-z)^{-\frac{n+1}{2}}e^{-z}\,dz = \sum_{n\ge 0}\frac{(is)^n}{\Gamma\left(\frac{n+1}{2}\right)}.$$

On the other hand we have

$$\begin{aligned} \varphi_{\mathcal{R}}(t) &= \int_0^\infty \sum_{n\ge 0}(it)^n x^{n+1}e^{-x^2/2}\,dx \\ &= \sum_{n\ge 0}(it)^n 2^{\frac{n}{2}}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma(n+1)} \\ &= \sqrt{\pi}\sum_{n\ge 0}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}\left(\frac{it}{\sqrt{2}}\right)^n, \end{aligned}$$

where we have used the duplication formula for the Γ-function. $\square$

Lemma 7 *Let γ be as in Lemma 6. Then*

$$\frac{1}{2\pi i}\int_\gamma e^{-s\sqrt{-z}-z}\,dz = \frac{s}{2\sqrt{\pi}}e^{-s^2/4} \tag{53}$$

and

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{-z}}{\sqrt{-z}}\,dz = \frac{1}{\sqrt{\pi}}. \tag{54}$$

Proof. (53) and (54) follow immediately from the substitution $z = w^2$.□

Lemma 8 *Let γ be as in Lemma 6 and α, β real constants. Then*

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\int\limits_{\gamma} \frac{e^{i\alpha w - w^2/2 - z}}{\sqrt{-z} - i\beta w}\,dz\,dw \tag{55}$$

$$= \frac{e^{-\alpha^2/2}}{\sqrt{2}(\frac{1}{2}+\beta^2)} - \frac{\sqrt{\pi}\alpha\beta}{(\frac{1}{2}+\beta^2)^{\frac{3}{2}}}\exp\left(-\frac{\alpha^2}{4(\frac{1}{2}+\beta^2)}\right)\Phi\left(-\frac{\alpha\beta}{\sqrt{\frac{1}{2}+\beta^2}}\right).$$

Proof. Since both sides of (56) can be interpreted as analytic functions in α, β around the real axis it suffices to prove (56) for the case $\alpha\beta > 0$. In this case we can use the substitutions $z = \frac{u^2}{2}$ and $w = v + i\alpha$ to obtain

$$\frac{e^{-\alpha^2/2}}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}(u^2+v^2)}}{\frac{1}{\sqrt{2}}u + \beta v + i\alpha\beta}\,u\,du\,dv.$$

Then we can apply the polar substitution $u = r\cos\varphi$, $v = r\sin\varphi$ to get

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}(u^2+v^2)}}{\frac{1}{\sqrt{2}}u + \beta v + i\alpha\beta}\,u\,du\,dv$$

$$= \int\limits_{0}^{\infty}\int\limits_{0}^{2\pi}\frac{r^2 e^{\frac{1}{2}r^2}\cos\varphi}{r\left(\frac{1}{\sqrt{2}}\cos\varphi + \beta\sin\varphi\right) + i\alpha\beta}\,d\varphi\,dr$$

$$= \int\limits_{0}^{\infty} r^2 e^{-\frac{1}{2}r^2}\int\limits_{|z|=1}\frac{\frac{1}{2}(z+z^{-1})}{r\left(\frac{1}{2\sqrt{2}}(z+z^{-1}) + \frac{\beta}{2i}(z-z^{-1})\right) + i\alpha\beta}\,\frac{dz}{iz}\,dr$$

$$= \int\limits_{0}^{\infty} r^2 e^{-\frac{1}{2}r^2}\frac{\sqrt{2}\pi}{r\left(\frac{1}{2}+\beta^2\right)}\left(1 - \frac{\alpha\beta}{\sqrt{\alpha^2\beta^2 + r^2\left(\frac{1}{2}+\beta^2\right)}}\right)dr$$

$$= \frac{\sqrt{2}\pi}{\frac{1}{2}+\beta^2} - \frac{2\pi^{\frac{3}{2}}\alpha\beta}{\left(\frac{1}{2}+\beta^2\right)^{\frac{3}{2}}}\exp\left(\frac{\alpha^2\beta^2}{2\left(\frac{1}{2}+\beta^2\right)}\right)\Phi\left(-\frac{\alpha\beta}{\sqrt{\frac{1}{2}+\beta^2}}\right),$$

where the integral $\int_0^{2\pi} \ldots d\varphi$ is solved by using the substitution $z = e^{i\varphi}$ and the residue theorem:

$$\int\limits_{|z|=1} \frac{z^2+1}{r(\frac{i}{\sqrt{2}}+\beta)z^2 - 2\alpha\beta z + r(\frac{i}{\sqrt{2}}-\beta)} \frac{dz}{z}$$

$$= 2\pi i \left(\frac{-\frac{i}{\sqrt{2}} - \beta}{r(\frac{1}{2}+\beta^2)} + \frac{\frac{i}{\sqrt{2}}\alpha\beta + \beta\sqrt{\alpha^2\beta^2 + r^2(\frac{1}{2}+\beta^2)}}{r(\frac{1}{2}+\beta^2)\sqrt{\alpha^2\beta^2+r^2(\frac{1}{2}+\beta^2)}} \right)$$

$$= \frac{\sqrt{2}\pi}{r(\frac{1}{2}+\beta^2)} \left(1 - \frac{\alpha\beta}{\sqrt{\alpha^2\beta^2 + r^2(\frac{1}{2}+\beta^2)}} \right).$$

The residues have to be calculated for

$$z_1 = 0 \quad \text{and for} \quad z_2 = \frac{\alpha\beta - \sqrt{\alpha^2\beta^2 + r^2(\frac{1}{2}+\beta^2)}}{r(\frac{1}{2}+\beta^2)}.$$

This completes the proof of Lemma 8.□

5.2 Proof of Theorem 1

We first derive asymptotic expansions for mean value and variance. Since

$$c(x,u) = \frac{1}{g(x,u) + h(x,u)\sqrt{1 - x/\rho(u)}} \tag{56}$$

we get

$$\begin{aligned} [x^n]c(x,1) &= [x^n]\frac{1}{h(x,1)}\left(1 - \frac{x}{\rho}\right)^{-\frac{1}{2}} + \frac{\rho g_x(\rho,1)}{h(\rho,1)} + \mathcal{O}(\sqrt{1-x/\rho}) \\ &= \frac{\rho^{-n} n^{-\frac{1}{2}}}{h(\rho,1)\sqrt{\pi}}(1 + \mathcal{O}(n^{-1})) \end{aligned}$$

and

$$\begin{aligned} [x^n]c_u(x,1) &= [x^n]\left(\frac{-g_u(x,1)}{h(x,1)^2}\left(1-\frac{x}{\rho}\right)^{-1} + \frac{-h_u(x,1)}{h(x,1)^2}\left(1-\frac{x}{\rho}\right)^{-\frac{1}{2}} \right) \\ &= \frac{-g_u(\rho,1)\rho^{-n}}{h(\rho,1)^2}(1 + \mathcal{O}(n^{-\frac{1}{2}})). \end{aligned}$$

Hence

$$\mathbf{E}X_n = \frac{[x^n]c_u(x,1)}{[x^n]c(x,1)} = \frac{-g_u(\rho,1)}{h(\rho,1)}\sqrt{\pi n} + \mathcal{O}(1).$$

Similarly we get

$$[x^n]c_{uu}(x,1) = \frac{4g_u(\rho,1)^2}{\sqrt{\pi}h(\rho,1)^3}\rho^{-n}n^{\frac{1}{2}}(1+\mathcal{O}(n^{-\frac{1}{2}}))$$

and

$$\mathbf{V}X_n = \frac{[x^n]c_{uu}(x,1)}{[x^n]c(x,1)} + \mathbf{E}X_n - (\mathbf{E}X_n)^2 = (4-\pi)\frac{g_u(\rho,1)^2}{h(\rho,1)^2}n + \mathcal{O}(n^{\frac{1}{2}}).$$

Next we will determine the characteristic function of $X_n/\sqrt{n}$. Since

$$\mathbf{E}e^{itX_n/\sqrt{n}} = \frac{[x^n]c(x,e^{it/\sqrt{n}})}{[x^n]c(x,1)} \tag{57}$$

we have to expand $[x^n]c(x,u)$ for $u = e^{it/\sqrt{n}} = 1 + i\frac{t}{\sqrt{n}} + \mathcal{O}(n^{-1})$. For this purpose we will use Cauchy's formula

$$[x^n]c(x,u) = \frac{1}{2\pi i}\int_\Gamma c(z,u)\,\frac{dz}{z^{n+1}} \tag{58}$$

for the following path of integration $\Gamma = \Gamma_1 \cup \Gamma_2$:

$$\begin{aligned} \Gamma_1 &= \left\{ z = \rho\left(1+\frac{s}{n}\right) \;:\; s \in \gamma' \right\}, & (59)\\ \Gamma_2 &= \left\{ z = Re^{i\vartheta} \;:\; R = \rho\left|1+\frac{\log^2 n + i}{n}\right|, \arg\left(1+\frac{\log^2 n + i}{n}\right) \le |\vartheta| \le \pi \right\}, \end{aligned}$$

where $\gamma' = \{s : |s| = 1, \Re s \le 0\} \cup \{s : 0 < \Re s < \log^2 n, \Im s = \pm 1\}$ is the major part of a Hankel contour γ.

First let us concentrate on the path Γ_1. By using the substitution $z = \rho\left(1+\frac{s}{n}\right)$ we get

$$\begin{aligned} \frac{1}{2\pi i}\int\limits_{\Gamma_1} c(z,u)\,\frac{dz}{z^{n+1}} &= \frac{\rho^{-n}}{2\pi i}\int\limits_{\gamma'} \frac{e^{-s}(1+\mathcal{O}(s^2n^{-1}))}{g_u(\rho,1)\frac{it}{\sqrt{n}} + h(\rho,1)\sqrt{\frac{-s}{n}} + \mathcal{O}\left(\frac{s}{n}\right)}\,\frac{ds}{n} \\ &= \frac{\rho^{-n}n^{-\frac{1}{2}}}{h(\rho,1)}\int\limits_{\gamma'} \frac{e^{-s}}{\sqrt{-s} + i\frac{g_u(\rho,1)}{h(\rho,1)}t}\,ds + \mathcal{O}(\rho^{-n}n^{-1}). \quad (60) \end{aligned}$$

Since

$$\int\limits_{\gamma\setminus\gamma'} \frac{e^{-s}}{\sqrt{-s}+iCt}\,ds = \mathcal{O}\left(e^{-\log^2 n}\right)$$

we immediately get by (60) and Lemma 6

$$\frac{1}{2\pi i}\int\limits_{\Gamma_1} c(z,u)\,\frac{dz}{z^{n+1}} = \frac{\rho^{-n}n^{-\frac{1}{2}}}{\sqrt{\pi}h(\rho,1)}\varphi_{\mathcal{R}}\left(\frac{-\sqrt{2}g_u(\rho,1)}{h(\rho,1)}t\right) + \mathcal{O}(\rho^{-n}n^{-1}). \tag{61}$$

Now we make use of the fact that there are $\varepsilon_1 > 0$, $\delta_1 > 0$ such that

$$\max_{|x|=x_1, |\arg x| \ge \vartheta_1} |c(x,u)| = |c(x_1 e^{i\vartheta_1}, u)|$$

for $1 \le x_1 \le 1+\delta_1$ and $|u-1| < \varepsilon_1$. You only have to observe that $c_{nk} \ge 0$, that $x = \rho$ is the only singularity on the circle of convergence, and that $c(x,u)$ has the local representation (56). Hence it follows from

$$\left| c\left(\rho\left(1+\frac{\log^2 n + i}{n}\right), e^{it/\sqrt{n}}\right)\right| = \mathcal{O}\left(n^{\frac{1}{2}} \log n\right)$$

that

$$\int_{\Gamma_2} c(z,u)\,\frac{dz}{z^{n+1}} = \mathcal{O}\left(n^{\frac{1}{2}} \log n\, e^{-\log^2 n}\right). \tag{62}$$

Consequently, by (57), (58), (61), and (62)

$$\mathbf{E} e^{itX_n/\sqrt{n}} = \varphi_{\mathcal{R}}\left(\frac{-\sqrt{2} g_u(\rho,1)}{h(\rho,1)} t\right) + \mathcal{O}(n^{-\frac{1}{2}}).$$

Thus we have proved a weak limit theorem.

In order to prove the local limit theorem we again use Cauchy's formula

$$[x^n u^k] c(x,u) = \frac{1}{(2\pi i)^2} \int_\Gamma \int_\Delta c(z,u)\,\frac{du}{u^{k+1}}\,\frac{dz}{z^{n+1}},$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ is as above (see 59) and Δ will be properly chosen.

If $z = \rho\left(1+\frac{s}{n}\right) \in \Gamma_1$ then the mapping $u \mapsto c(z,u)$ has a polar singularity at $u_0 = 1 + \frac{t_0}{\sqrt{n}}$, where

$$t_0 = -\frac{h(\rho,1)}{g_u(\rho,1)}\sqrt{-s} + \mathcal{O}\left(\frac{s}{n}\right)$$

with residue

$$\frac{1}{g_u(\rho,1)}\left(1+\mathcal{O}\left(\frac{s}{n}\right)\right).$$

Hence we can transform Δ in a way that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\Delta c(z,u)\,\frac{du}{u^{k+1}} \\
&= -\frac{u_0^{-k-1}}{g_u(\rho,1)}\left(1+\mathcal{O}\left(\frac{s}{n}\right)\right) + \frac{1}{2\pi i}\int_{|u|=1+\varepsilon_2} c(z,u)\,\frac{du}{u^{k+1}} \\
&= \frac{-1}{g_u(\rho,1)} \exp\left(\frac{k}{\sqrt{n}}\frac{h(\rho,1)}{g_u(\rho,1)}\sqrt{-s}\right)\left(1+\mathcal{O}\left(\frac{(k+1)s}{n}\right)\right) + \mathcal{O}\left((1+\varepsilon_2)^{-k}\right).
\end{aligned}$$

Consequently

$$
\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\Gamma_1}\int_{\Delta} c(z,u)\,\frac{du}{u^{k+1}}\,\frac{dz}{z^{n+1}} \\
&= \frac{-1}{g_u(\rho,1)}\,\frac{\rho^{-n}}{2\pi i}\int_{\gamma'}\exp\left(-s+\frac{k}{\sqrt{n}}\frac{h(\rho,1)}{g_u(\rho,1)}\sqrt{-s}\right)\left(1+\mathcal{O}\left(\frac{(k+1)s^2}{n}\right)\right)\,\frac{ds}{n} \\
&\quad +\mathcal{O}\left(\rho^{-n}(1+\varepsilon_2)^{-k}\right) \\
&= \frac{k}{n^{\frac{3}{2}}}\frac{h(\rho,1)}{g_u(\rho,1)^2}\frac{\rho^{-n}}{2\sqrt{\pi}}\exp\left(-\frac{k^2}{4n}\frac{h(\rho,1)^2}{g_u(\rho,1)^2}\right)+\mathcal{O}\left(\rho^{-n}\frac{k+1}{n^2}\right) \\
&\quad +\mathcal{O}\left(\rho^{-n}\frac{(1+\varepsilon_2)^{-k}}{n}\right).
\end{aligned}
$$

By elementary considerations we obtain

$$
\max_{z\in\Gamma_2,|u|=1} c(z,u)=\mathcal{O}\left(n^{\frac{1}{2}}\log n\right).
$$

Hence, by choosing $\Delta=\{u\,:\,|u|=1\}$ for $z\in\Gamma_2$ we can estimate the remaining integral by

$$
\frac{1}{(2\pi i)^2}\int_{\Gamma_2}\int_{\Delta} c(z,u)\,\frac{du}{u^{k+1}}\,\frac{dz}{z^{n+1}}=\mathcal{O}\left(n^{\frac{1}{2}}\log n\, e^{-\log^2 n}\right)
$$

and finally have proved the the local limit theorem.

5.3 Proof of Theorem 2

As above we have

$$
[x^n]c(x,1)=\frac{\rho^{-n}n^{-\frac{1}{2}}}{h(\rho,1)\sqrt{\pi}}(1+\mathcal{O}(n^{-1}))
$$

and from

$$
\begin{aligned}
c_u(x,1) &= \frac{-\rho'(1)}{2\rho h(\rho,1)}\left(1-\frac{x}{\rho}\right)^{-\frac{3}{2}}-\frac{\alpha}{h(\rho,1)^2}\left(1-\frac{x}{\rho}\right)^{-\frac{1}{2}} \\
&+ \mathcal{O}\left((1-x/\rho)^{\frac{1}{2}}\right)
\end{aligned}
$$

we immediately get ($\alpha=0$)

$$
\begin{aligned}
\mathbf{E}X_n &= \frac{[x^n]c_u(x,1)}{[x^n]c(x,1)} \\
&= -\frac{\rho'(1)}{\rho}\,n+\mathcal{O}(1) \\
&= \mu\, n+\mathcal{O}(1)
\end{aligned}
$$

and from a little bit more refined analysis

$$\mathbf{V}X_n = \sigma^2 n + \mathcal{O}(\sqrt{n}).$$

Since

$$\varphi_{(X_n-\mu n)/\sqrt{\sigma^2 n}}(t) = e^{-it\sqrt{n}\mu/\sigma}\varphi_{X_n}\left(\frac{t}{\sqrt{\sigma^2 n}}\right) = e^{-it\sqrt{n}\mu/\sigma}\frac{[x^n]c(x,e^{it/\sqrt{\sigma^2 n}})}{[x^n]c(x,1)}$$

we have to determine

$$[x^n]c(x,u) = \frac{1}{2\pi i}\int_\Gamma c(z,u)\,\frac{dz}{z^{n+1}}$$

for $u = e^{it/\sqrt{\sigma^2 n}} = 1 + i\frac{t}{\sqrt{\sigma^2 n}} - \frac{t^2}{2\sigma^2 n} + \mathcal{O}(n^{-2})$, in which we use the following path of integration $\Gamma = \Gamma_1 \cup \Gamma_2$:

$$\begin{aligned}
\Gamma_1 &= \left\{z = \rho(u)\left(1+\frac{s}{n}\right)\,:\, s \in \gamma'\right\}, \\
\Gamma_2 &= \left\{z = Re^{i(\vartheta - \arg(u))}\,:\, R = |\rho(u)|\left|1+\frac{\log^2 n + i}{n}\right|,\right. \\
&\qquad\qquad \left.\arg\left(1+\frac{\log^2 n + i}{n}\right) \le |\vartheta| \le \pi\right\}.
\end{aligned} \tag{63}$$

From

$$\rho(u)^{-n} = \rho^{-n}\exp\left(\sqrt{n}\frac{\mu}{\sigma} - \frac{t^2}{2}\right)\left(1+\mathcal{O}(n^{-\frac{1}{2}})\right)$$

and from

$$g(z,u) + h(z,u)\sqrt{1-z/\rho(u)} = h(\rho,1)\sqrt{\frac{-s}{n}} + \mathcal{O}\left(\frac{s}{n}\right)$$

for $x \in \Gamma_1$ we directly get by applying Lemma 7

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_1} c(z,u)\,\frac{dz}{z^{n+1}} \\
&= \frac{\rho(u)^{-n}}{h(\rho,1)\sqrt{n}}\frac{1}{2\pi i}\int_{\gamma'}\frac{e^{-s}}{\sqrt{-s}}\left(1+\mathcal{O}\left(\frac{s^2}{n}+\left|\frac{s}{n}\right|\right)\right)\,ds \\
&= \exp\left(\sqrt{n}\frac{\mu}{\sigma} - \frac{t^2}{2}\right)\frac{\rho^{-n}n^{-\frac{1}{2}}}{h(\rho,1)\sqrt{\pi}}\left(1+\mathcal{O}(n^{-\frac{1}{2}})\right).
\end{aligned}$$

It remains to estimate the integral on Γ_2. But this can be done as in the proof of Theorem 1 since

$$\max_{z\in\Gamma_2}|c(z,u)| = \mathcal{O}(n^{\frac{1}{2}}\log n).$$

In order to prove the local law we again use Cauchy's formula

$$[x^n u^k]c(x,u) = \frac{1}{(2\pi i)^2} \int_\Delta \int_\Gamma c(z,u)\, \frac{dz}{z^{n+1}}\, \frac{du}{u^{k+1}},$$

where $\Delta = \{u \,:\, |u| = 1\}$. For $u \in \Delta_1 = \{u = e^{it} \,:\, |t| \le n^{-5/12}\}$ let $\Gamma = \Gamma_1 \cup \Gamma_2$ as in the proof of the weak limit theorem, for $u \in \Delta_2 = \{u = e^{it} \,:\, n^{-5/12} < |t| < \varepsilon\}$ (for some sufficiently small $\varepsilon > 0$) let $\Gamma = \{z \,:\, |z| = \frac{1}{2}(\rho + |\rho(e^{it})|)\}$, and for $u \in \Delta_3 = \{u = e^{it} \,:\, \varepsilon \le |t| \le \pi\}$ let $\Gamma = \{z \,:\, |z| = \rho(1+\delta)\}$ for some sufficiently small $\delta > 0$.

First, let $u = e^{it} \in \Gamma_1$, i.e. $|t| \le n^{-5/12}$, and $z \in \Gamma_1$. By direct approximation we have

$$g(z,u) + h(z,u)\sqrt{1 - z/\rho(u)} = n^{-\frac{1}{2}} h(\rho,1)\sqrt{-s}(1 + \mathcal{O}(n^{-\frac{1}{3}}))$$

(Note that $\alpha = 0$ and that $|s| \ge 1$.) and

$$z^{-n}u^{-k} = \rho^{-n} e^{-it(k-\mu n) - \frac{1}{2}t^2\sigma^2 n - s}(1 + \mathcal{O}(n^{-\frac{1}{4}})).$$

Hence we obtain by using the methods of [12] and a saddle point like integration (compare with [9])

$$\begin{aligned}
&\frac{1}{(2\pi i)^2} \int_{\Delta_1} \int_{\Gamma_1} c(z,u)\, \frac{dz}{z^{n+1}}\, \frac{du}{u^{k+1}} \\
&= \frac{\rho^{-n}}{\sqrt{n}h(\rho,1)} \frac{1}{(2\pi i)^2} \int_{-n^{-5/12}}^{n^{-5/12}} \int_{\gamma'} \frac{e^{-it(k-\mu n) - \frac{1}{2}t^2\sigma^2 n - s}}{\sqrt{-s}} (1 + \mathcal{O}(n^{-\frac{1}{4}}))\, ds\, dt \\
&= \frac{\rho^{-n}}{\sqrt{n}h(\rho,1)} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{2\pi\sigma^2 n}} \left(\exp\left(-\frac{(k-\mu n)^2}{2\sigma^2 n} \right) + \mathcal{O}(n^{-\frac{1}{4}}) \right).
\end{aligned}$$

Therefore the proof is finished if the remaining integrals are sufficiently small. As above we have

$$\frac{1}{(2\pi i)^2} \int_{\Delta_1} \int_{\Gamma_2} c(z,u)\, \frac{dz}{z^{n+1}}\, \frac{du}{u^{k+1}} = \mathcal{O}\left(\rho^{-n} n^{\frac{1}{12}} \log n\, e^{-\log^2 n} \right).$$

Next we get

$$\frac{1}{(2\pi i)^2} \int_{\Delta_2} \int_{\Gamma} c(z,u)\, \frac{dz}{z^{n+1}}\, \frac{du}{u^{k+1}} = \mathcal{O}\left(\rho^{-n} e^{-cn^{1/6}} \right)$$

since $|\rho(u)| \ge \rho(1 + c_1 n^{-5/6})$ for $u \in \Delta_2$ (and some sufficiently small constants $c, c_1 > 0$). Finally, since $c(z,u)$ is bounded for $u \in \Delta_3$ and $z \in \Gamma$ we obtain

$$\frac{1}{(2\pi i)^2} \int_{\Delta_3} \int_{\Gamma} c(z,u)\, \frac{dz}{z^{n+1}}\, \frac{du}{u^{k+1}} = \mathcal{O}\left(\rho^{-n}(1+\delta)^{-n} \right)$$

which completes the proof of the local theorem.

5.4 Proof of Theorem 3

As in the proof of Theorem 2 we get

$$\mathbf{E}X_n = \mu n - \sqrt{\pi}\frac{\alpha}{h(\rho,1)}\sqrt{n} + \mathcal{O}(1)$$

and

$$\mathbf{V}X_n = \left(\sigma^2 + (4-\pi)\frac{\alpha^2}{h(\rho,1)^2}\right) n + \mathcal{O}(\sqrt{n}).$$

Now, if we use the same normalization and the same path of integration (63) as in Theorem 2, we obtain for $u = e^{it/\sqrt{\sigma^2 n}}$ by applying Lemma 6

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_1} c(z,u)\,\frac{dz}{z^{n+1}} \\
&= \frac{\rho(u)^{-n}}{h(\rho,1)\sqrt{n}}\frac{1}{2\pi i}\int_{\gamma'} \frac{e^{-s}}{\sqrt{-s}+i\alpha t/(h(\rho,1)\sigma)}\left(1+\mathcal{O}\left(\frac{s^2}{n}+\left|\frac{s}{n}\right|\right)\right) ds \\
&= \frac{\rho^{-n}n^{-\frac{1}{2}}}{h(\rho,1)\sqrt{\pi}}\exp\left(\sqrt{n}\frac{\mu}{\sigma}-\frac{t^2}{2}\right)\varphi_{\mathcal{R}}\left(\frac{-\sqrt{2}\alpha}{h(\rho,1),\sigma}\right)\left(1+\mathcal{O}(n^{-\frac{1}{2}})\right).
\end{aligned}$$

The remaining integral on Γ_2 can be estimated as above. Hence we have proved the weak convergence property.

In order to prove the local law we will proceed as in the proof of Theorem 2. As above we can concentrate on the path of integration $\Delta_1 \times \Gamma_1$. The remaining integrals are negligible. Direct approximation yields

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\Delta_1}\int_{\Gamma_1} c(z,u)\,\frac{dz}{z^{n+1}}\,\frac{du}{u^{k+1}} \\
&= \frac{\rho^{-n}}{\sqrt{n}h(\rho,1)}\frac{1}{(2\pi i)^2}\int_{-n^{-5/12}}^{n^{-5/12}}\int_{\gamma'} \frac{e^{-it(k-\mu n)-\frac{1}{2}t^2\sigma^2 n - s}}{\sqrt{-s}+i\frac{\alpha}{h(\rho,1)}\sqrt{n}t}(1+\mathcal{O}(n^{-\frac{1}{4}}))\,ds\,dt.
\end{aligned}$$

Hence an application of Lemma 8 and easy tail estimates complete the proof of Theorem 3.

6 Conclusions

The main purpose of this paper is to provide general techniques to obtain the limiting distribution of parameters in combinatorial constructions. It is the second paper of a (planed) series of papers ([10]) devoted to this topic. Theorems 1–3 should be considered as examples of analytic theorems providing an link between combinatorial constructions and its asymtotic distributions. (They seem to be proper theorems to discuss random mappings.) The authors are convinced that the methods presented in the preceding proofs can be used in many other (different) problems. The basic ideas are singularity analysis (introduced by Flajolet and Odlyzko [12]) and saddle point approximation.

Random mappings are widely and intensively discussed in literature, e.g. in Kolchin's book [15] a probabilistic approach via branchning processes is presented whereas Aldous and Pitman [3] use a completely different probabilistic concept related to Aldous' continuum random trees [1, 2]. Our concept of generating functions goes back to Arney and Bender [4] and to Flajolet and Odlyzko [13]. They could identify many limiting distribution and provided asymptotic expansions for mean and variance. (One gap could be filled by Application 3.) It should be mentioned, too, that Arney and Bender [4] discussed a little bit more general case, namely that the number of immediate predecessors of a point $|\varphi^{-1}(\{\nu\})|$ is not arbitrary but must be contained in a subset D of non-negative integers, i.e. the corresponding tree function $\hat{t}_D(x)$ satisfies the functional equation

$$\hat{t}_D(x) = x \sum_{n \in D} \frac{\hat{t}_D(x)^n}{n!} = x\phi(\hat{t}_D(x)).$$

and the corresponding generating function for those mappings

$$\hat{f}_D(x) = \frac{1}{1 - x\phi'(\hat{t}_D(x))}.$$

From Proposition 1 it follows that if D contains a number ≥ 2 then it has a squareroot singularity

$$\hat{t}_D(x) = g_D(x) - h_D(x)\sqrt{1 - x/\rho_D},$$

around $x = \rho_D = t_0/\phi(t_0)$, where $t_0 > 0$ satisfies $t_0\phi'(t_0) = \phi(t_0)$. Hence $\rho_D\phi'(t_0) = 1$ and we are in a similar situation as in the classical case. Especially we can adapt all the combinatorial constructions to this more general case and obtain analog results by applying our theorems.

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Keywords: random mappings, combinatorial constructions, limiting distributions

AMS–classification: 05 A 16, 60 F

Short title: Random Mappings