THE HEIGHT DISTRIBUTION OF LEAVES IN ROOTED TREES

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Abstract. By the use of a double Hankel contour integration the number of trees of size n (in a simply generated family of trees) such that the m-th leaf has height k is asymptotically evaluated. It turns out that the height of the cn-th leaf is asymptotically Maxwell distributed. This generalizes and extends results by W. Gutjahr and G. Ch. Pflug [GP1] and P. Kirschenhofer [Ki1], [Ki3], where completely different methods have been used. In addition the joint height distribution of two different leaves is established. All results can be interpreted in terms of Galton-Watson branching processes.

1. Introduction

Let $\mathcal{F}$ denote a simply generated family of (rooted) trees (introduced by A. Meir and J. W. Moon [MM]). This means that there is a (formal) power series

$$\varphi(t) = \varphi_0 + \varphi_1 t + \varphi_2 t^2 + \cdots \tag{1.1}$$

of non-negative numbers $\varphi_0 > 0$, $\varphi_1 \ge 0$, $\varphi_2 \ge 0, \ldots$, which are used to define a weight $\omega(T)$ for any planted plane (rooted) tree T by

$$\omega(T) = \prod_{i\ge 0} \varphi_i^{D_i(T)}, \tag{1.2}$$

where $D_i(T)$ denotes the number of nodes in T with out-degree i ($D_0(T)$ is the number of leaves, for example) such that $\mathcal{F}$ consists of all trees with weight $\omega(T) > 0$. Let $\mathcal{F}_n$ be the set of trees $T \in \mathcal{F}$ with $|T| = n$ (i.e. T consists of n nodes). Then the generating function

$$y(x) = \sum_{n\ge 0} y_n x^n \tag{1.3}$$

of the numbers

$$y_n = \sum_{|T|=n} \omega(T) = \sum_{T\in\mathcal{F}_n} \omega(T) \tag{1.4}$$

satisfies the (formal) functional equation

$$y = x\varphi(y). \tag{1.5}$$

Hence $\mathcal{F}$, interpreted as a combinatorial structure can be defined recursively by

$$\mathcal{F} = \circ \times \varphi(\mathcal{F}), \tag{1.6}$$

where $\circ$ denotes a node and $\varphi(\mathcal{F})$ a substituted structure (compare with [VF]).

For example, let $\varphi_i = 1$ for all $i \geq 0$, then y_n counts all rooted planted plane trees of size n. From $\varphi(t) = 1/(1-t)$ we get $y = x/(1-y)$ and hence

$$y_n = \frac{1}{n}\binom{2n-2}{n-1}. \tag{1.7}$$

Recursively, all planted plane trees can be described as a node followed by a sequence of trees:

$$\mathcal{F} = \circ + \begin{array}{c}\circ\\ |\\ \mathcal{F}\end{array} + \begin{array}{c}\circ\\ /\ \backslash\\ \mathcal{F}\ \mathcal{F}\end{array} + \cdots = \circ \times \mathcal{F}^*. \tag{1.8}$$

For $\varphi(t) = 1 + t^2$ we get binary trees, for $\varphi(t) = 1 + t + t^2$ Motzkin trees and so on. Gererally if $\varphi_i \in \{0,1\}$ we are always counting special kinds of rooted trees admitting a recursive description. It is easy to extend this concept to count special parameters in these families of trees, the number of leaves for example.

Alternatively if $\varphi(1) = \sum_{i\geq 0} \varphi_i = 1$ then simply generated families of trees can be considered as realizations of a (Galton-Watson) branching process where $\omega(T)$ denotes that probabilty the T occurs (see [AN]). Instead of counting a parameter we directly get the probabilistic distribution of a parameter. However, both concepts are equivalent when we are only interested in discrete parameters in $\mathcal{F}_n$ (e.g. the number of leaves) resp. in the conditional distribution of the parameter of interest in realizations of size n. In fact, if we set $\overline{\varphi}_i = \varphi_i \overline{\tau}^i/\varphi(\overline{\tau})$ for some $\overline{\tau} > 0$ such that $\varphi(\overline{\tau})$ exists, then $\sum_{i\geq 0} \overline{\varphi}_i = 1$ and the conditional distributions are the same (compare with [Ke]). Moreover let τ be the (smallest positive) solution of $t\varphi'(t) = \varphi(t)$ then the branching process corresponding to $\overline{\varphi}_i = \varphi_i \tau^i/\varphi(\tau)$ is critical, i.e. the “offspring expected value” $\sum_{i\geq 0} i\overline{\varphi}_i = 1$. It should be further noted that the “offspring variance”

$$\sigma^2 = \sum_{i\geq 0} (i-1)^2 \overline{\varphi}_i = \sum_{i\geq 0} \frac{(i-1)^2 \varphi_i \tau^i}{\varphi(\tau)} = \frac{\tau^2 \varphi''(\tau)}{\varphi(\tau)} \tag{1.9}$$

of this critical branching process has to play an important role since it appears very often as a constant by analyzing the asymptotic behaviour of simply generated families of trees (see [Al]).

In what follows we will mainly use the combinatorial interpretation but all results will be stated in terms of $\omega(T)$.

The purpose of this paper is to get precise informations of the distribution of the height of leaves. The height of a leaf in a rooted tree is the distance to the root. Since $\mathcal{F}$ consists only of planted plane trees it is possible to number the leaves in a natural way (i.e. from left to right). Let $a_{km}(T) = 1$ if the m-th leaf of T has height k and $a_{km}(T) = 0$ otherwise. Our first result provides an asymptotic formula for

$$a_{kmn} = \sum_{|T|=n} \omega(T) a_{km}(T). \tag{1.10}$$

(Obviously a_{kmn} can be interpreted combinatorically or probabilistically.)

Theorem 1. *Suppose that $\varphi(t)$ has positive radius of convergence $R > 0$ and assume that there exists a (minimal) positive solution $\tau < R$ of $t\varphi'(t) = \varphi(t)$. Let σ be defined by (1.9) and set*

$$d = \gcd\{i \,|\, \varphi_i > 0\}. \tag{1.11}$$

Then we have

$$\begin{aligned} a_{kmn} = & \frac{d\tau\sigma^2\varphi'(\tau)^n k^2}{8\pi\left(m\frac{\varphi(\tau)}{\varphi_0}\left(n - m\frac{\varphi(\tau)}{\varphi_0}\right)\right)^{\frac{3}{2}}} \exp\left(-\frac{\sigma^2 k^2 n}{8m\frac{\varphi(\tau)}{\varphi_0}\left(n - m\frac{\varphi(\tau)}{\varphi_0}\right)}\right) \\ & + \mathcal{O}\left(\varphi'(\tau)^n \frac{k}{n^3}\right) \end{aligned} \tag{1.12}$$

for $n \equiv 1 \mod d$ uniformly for $\varepsilon \le \frac{m}{n} \le \frac{\varphi_0}{\varphi(\tau)} - \varepsilon$ and $k \le C\sqrt{n}$, where $\varepsilon > 0$, $C > 0$ are arbitrary but fixed. (Of course, $a_{kmn} = 0$ for $n \not\equiv 1 \mod d$.)

Corollary. *Let $0 < c < 1$ and $x > 0$. Suppose that $m\frac{\varphi(\tau)}{\varphi_0} = cn + o(n)$ and $k = x\sqrt{n} + o(\sqrt{n})$ $(n \to \infty)$. Then we have*

$$\sqrt{n}\frac{a_{kmn}}{y_n} = \frac{\sigma^3}{4\sqrt{2\pi}} \frac{x^2}{(c(1-c))^{\frac{3}{2}}} \exp\left(-\frac{\sigma^2}{8}\frac{x^2}{c(1-c)}\right) + o(1) \tag{1.13}$$

for $n \equiv 1 \mod d$ $(n \to \infty)$.

This means that the height of the $c\frac{\varphi_0}{\varphi(\tau)}n$-th leaf is trees of size n is asymptotically Maxwell distributed with mean value

$$\sim \frac{8}{\sqrt{2\pi}\sigma}(c(1-c))^{\frac{1}{2}}\sqrt{n} \tag{1.14}$$

For binary trees this distribution result is due to W. Gutjahr and G. Ch. Pflug [GP1] who used a direct approach with explict formulas and Stirling's asymptotic formula. However, the mean value has been derived by P. Kirschenhofer [Ki1], [Ki2], [Ki3] by a rather involved diagonal method. But we can also use the same methods as in the proof of Theorem 1 to obtain directly

Theorem 2. *Let $\varphi(t)$ be as in Theorem 1. Then we have*

$$\begin{aligned} \sum_{k\ge 0} k a_{kmn} = & \frac{4d\tau\varphi'(\tau)^n}{\pi\sigma^2 n^2}\left(m\frac{\varphi(\tau)}{\varphi_0}\left(n - m\frac{\varphi(\tau)}{\varphi_0}\right)\right)^{\frac{1}{2}} \\ & + \mathcal{O}\left(\frac{\varphi'(\tau)^n}{n^{\frac{3}{2}}}\right) \end{aligned} \tag{1.15}$$

for $n \equiv 1 \mod d$ uniformly for $\varepsilon \le \frac{m}{n} \le \frac{\varphi_0}{\varphi(\tau)} - \varepsilon$, where $\varepsilon > 0$ is arbitrary but fixed.

Hence we get (1.14) again but now uniform for $\varepsilon \le c \le 1 - \varepsilon$ with an error term $\mathcal{O}(1)$.

It should be noted that the bounds $n\frac{\varphi_0}{\varphi(\tau)}$ for the number of leaves and the range $\sqrt{n}$ for the height are natural in the following sense.

First, the number of leaves in trees of size n satisfy a central limit theorem with mean value $\frac{\varphi_0}{\varphi(\tau)}n + \mathcal{O}(1)$ and variance (2.7) (see Section 2).

Second, it has been shown by Ph. Flajolet and A. Odlyzko [FO1] (compare also with [Ko]) that the average height in trees of size n is given by

$$\frac{\sqrt{2\pi}}{\sigma}\sqrt{n} + \mathcal{O}\left(n^{\frac{1}{4}+\varepsilon}\right) \tag{1.16}$$

for $n \equiv 1 \mod d$ (and for any $\varepsilon > 0$). Furthermore the average height is aymptotically theta distributed.

Hence Theorem 1 covers the range of interest. Note that the average height always differs from the average height of the $\frac{\varphi_0}{2\varphi(\tau)}n$-th leaf (which has largest average height) by a factor $\frac{\pi}{2} > 1$. It is not only interesting that this factor is > 1 but it is independent of the simply generated family of trees.

Similarly we can tackle the joint distribution of two leaves. Let $b_{k_1k_2m_1m_2}(T) = 1$ if $m_1 < m_2$ and if the m_i-th leaf has height k_i for $i = 1, 2$ and $b_{k_1k_2m_1m_2}(T) = 0$ otherwise. The next theorem, which generalizes [GP2], gives an asymptotic formula for the numbers

$$b_{k_1k_2m_1m_2n} = \sum_{|T|=n} \omega(T) b_{k_1k_2m_1m_2}(T). \tag{1.17}$$

Theorem 3. *Let $\varphi(t)$ be as in Theorem 1. Then we have*

$$\begin{aligned}
b_{k_1k_2m_1m_2n} &= \\
&= \frac{d\tau\sigma^3\varphi'(\tau)^n}{16\sqrt{2}\pi^{\frac{3}{2}}} \frac{k_1k_2}{\left((m_2-m_1)\frac{\varphi(\tau)}{\varphi_0}\right)^{\frac{1}{2}}\left(m_1\frac{\varphi(\tau)}{\varphi_0}\right)^{\frac{3}{2}}\left(n-m_2\frac{\varphi(\tau)}{\varphi_0}\right)^{\frac{3}{2}}} \cdot \\
&\quad \cdot \exp\left(-\frac{\sigma^2k_1^2}{8m_1\frac{\varphi(\tau)}{\varphi_0}} - \frac{\sigma^2k_2^2}{8\left(n-m_2\frac{\varphi(\tau)}{\varphi_0}\right)}\right) \cdot \\
&\quad \cdot \left(\exp\left(-\frac{\sigma^2(k_1-k_2)^2}{8(m_2-m_1)\frac{\varphi(\tau)}{\varphi_0}}\right) - \exp\left(\frac{\sigma^2(k_1+k_2)^2}{8(m_2-m_1)\frac{\varphi(\tau)}{\varphi_0}}\right)\right) \\
&\quad + \mathcal{O}\left(\varphi'(\tau)^n\frac{k_1+k_2}{n^4}\right)
\end{aligned} \tag{1.18}$$

for $n \equiv 1 \mod d$ uniformly for $\varepsilon \le \frac{m_1}{n} < \frac{m_2}{n} - \varepsilon \le \frac{\varphi_0}{\varphi(\tau)} - 2\varepsilon$ and $k_1, k_2 \le C\sqrt{n}$, where $\varepsilon > 0$, $C > 0$ are arbitrary but fixed.

Corollary. *Let $0 < c_1 < c_2 < 1$ and $x_1, x_2 > 0$. Suppose that $m_i\frac{\varphi(\tau)}{\varphi_0} = c_in + o(n)$ and $k_i = x_i\sqrt{n} + o(\sqrt{n})$ $(n \to \infty)$ for $i = 1, 2$. Then we have*

$$\begin{aligned}
n\frac{b_{k_1k_2m_1m_2n}}{y_n} &= \frac{\sigma^4}{16\pi}\frac{x_1x_2}{(c_2-c_1)^{\frac{1}{2}}(c_1(1-c_2))^{\frac{3}{2}}} \cdot \\
&\quad \cdot \exp\left(-\frac{\sigma^2x_1^2}{8c_1} - \frac{\sigma^2x_2^2}{8(1-c_2)}\right) \cdot \\
&\quad \cdot \left(\exp\left(-\frac{\sigma^2(x_2-x_1)^2}{8(c_2-c_1)}\right) - \exp\left(-\frac{\sigma^2(x_2+x_1)^2}{8(c_2-c_1)}\right)\right) \\
&\quad + o(1)
\end{aligned} \tag{1.19}$$

for $n \equiv 1 \mod d$ *(*$n \to \infty$*).*

It should be noted that the methods used here can be extended to discuss the case of three, four, ... leaves but the expressions used there get more and more involved. Nevertheless there is some hope to prove a simiar theorem to [GP3] where it is shown that in the case of binary trees the normalized height contour converges to Brownian excursion.

2. Proof of Theorem 1

Set $y_m(T) = 1$ if T has exactly m leaves and $y_m(T) = 0$ otherwise and let

$$y(x,u) = \sum_{n,m} y_{mn} x^n u^m \tag{2.1}$$

be the generating function of the numbers

$$y_{mn} = \sum_{|T|=n} \omega(T) y_m(T). \tag{2.2}$$

Then it is easy to verify that $y(x,z)$ satisfies the functional equation

$$y(x,z) = \varphi_0 x(u-1) + x\varphi(y(x,u)), \tag{2.3}$$

where it is quite useful to apply the combinatorial interpretation. Recall (1.6), saying that simply generated families of trees can be described recursively by

$$\mathcal{F} = \varphi_0 \cdot \circ + \varphi_1 \cdot \begin{array}{c}\circ\\ |\\ \mathcal{F}\end{array} + \varphi_2 \cdot \begin{array}{c}\circ\\ /\ \backslash\\ \mathcal{F}\ \mathcal{F}\end{array} + \cdots. \tag{2.4}$$

In order to "count" the leaves it is only necessary to mark those nodes in this construction correspondion to leaves:

$$\mathcal{F} = \varphi_0 \cdot \bullet + \varphi_1 \cdot \begin{array}{c}\circ\\ |\\ \mathcal{F}\end{array} + \varphi_2 \cdot \begin{array}{c}\circ\\ /\ \backslash\\ \mathcal{F}\ \mathcal{F}\end{array} + \cdots. \tag{2.5}$$

Now (2.5) directly translates to (2.3).

A direct application of [Dr, Theorem 3] gives that y_{mn} satisfy a central limit theorem with mean value

$$E_n = \frac{1}{y_n} \sum_{m\ge 0} m y_{mn} = \frac{\varphi_0}{\varphi(\tau)} n + \mathcal{O}(1) \tag{2.6}$$

and variance

$$V_n = \frac{1}{y_n} \sum_{m\ge 0} (m - E_n)^2 y_{mn} = \left(\frac{\varphi_0}{\varphi(\tau)} - \frac{\varphi_0^2}{\varphi(\tau)^2} - \frac{\varphi_0^2 \varphi'(\tau)}{\varphi(\tau)^3 \varphi''(\tau)} \right) n + \mathcal{O}(1). \tag{2.7}$$

This property justifies the restriction $m < \frac{\varphi_0}{\varphi(\tau)} n$ in Theorem 1.

Now let

$$A_k(x,u) = \sum_{m,n} a_{kmn} x^n u^m \tag{2.8}$$

be the generating function of those leaves with height k.

Proposition 1. *Let $y(x,z)$ be defined in (2.1). Then we have*

$$A_k(x,u) = \varphi_0 u x^{k+1} \left(\frac{\varphi(y(x,u)) - \varphi(y(x,1))}{y(x,u) - y(x,1)} \right)^k. \tag{2.9}$$

Proof. We will proceed by induction. Trivially

$$A_0(x,u) = \varphi_0 x u. \tag{2.10}$$

Now let $A_k(x,u)$ be given. Then we have (recall the combinatorial interpreation (2.4))

$$\begin{aligned} A_{k+1}(x,u) &= x\varphi_1 A_k(x,u) + x\varphi_2 \left(A_k(x,u) y(x,1) + y(x,u) A_k(x,u) \right) \\ &\quad + x\varphi_3 (A_k(x,u) y(x,1)^2 + y(x,u) A_k(x,u) y(x,1) \\ &\qquad + y(x,u)^2 A_k(x,u)) + \cdots \\ &= x A_k(x,u) \sum_{i \ge 1} \varphi_i \sum_{j=0}^{i-1} y(x,1)^j y(x,u)^{i-1-j} \\ &= x A_k(x,u) \sum_{i \ge 1} \varphi_i \frac{y(x,u)^i - y(x,1)^i}{y(x,u) - y(x,1)} \\ &= x A_k(x,u) \frac{\varphi(y(x,u)) - \varphi(y(x,1))}{y(x,u) - y(x,1)} \end{aligned} \tag{2.11}$$

which proves (2.9). □

In order to use $A_k(x,u)$ to evaluate a_{kmn} we have to know some information about $y(x,u)$.

Proposition 2. *Suppose that $\varphi(t)$ has positive radius of convergence $R > 0$, let $d = \gcd\{i \mid \varphi_i > 0\} = 1$, and assume that there exists a (minimal real) positive solution $\tau < R$ of $t\varphi'(t) = \varphi(t)$. Then there exists some real $\varepsilon > 0$ such that there is a unique solution $x = f(u)$ ($y = y(f(u),u)$) of*

$$\begin{aligned} y &= \varphi_0 x(u-1) + x\varphi(y) \\ 1 &= x\varphi'(y) \end{aligned} \tag{2.12}$$

for $|u-1| < \varepsilon$ satisfying $f(1) = \frac{1}{\varphi'(\tau)}$. Furthermore $y(x,u)$ is only singular at $x = f(u)$ for $|x| \le x_0 + \varepsilon$, $\arg\left(1 - \frac{x}{f(u)}\right) \ne \pi$ and has the local representation

$$y(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{f(u)}} \tag{2.13}$$

for $|x - f(u)| < \varepsilon$, $\arg\left(1 - \frac{x}{f(u)}\right) \ne \pi$, where $g(x,u)$ and $h(x,u)$ are analytic in this range and the latter satisfies

$$h(f(u),u) = \sqrt{\frac{2y(f(u),u)}{f(u)\varphi''(y(f(u),u))}}. \tag{2.14}$$

If $d > 1$ there are similar expansions around $f(u)\exp\left(j\frac{2\pi i}{d}\right)$ $(0 \le j < d)$.

Proof. It is well known (see [MM]) that $x_0 = \frac{1}{\varphi'(\tau)}$ is the only singularity of $y(x) = y(x,1)$ on the circle $|x| = x_0$ and that $y(x,1)$ can be analytically continued to $|x| < x_0 + \varepsilon$, $\arg(x_0 - x) \ne \pi$ (for some $\varepsilon > 0$). Namely $y(x) = y(x,1)$ is the solution of the functional equation

$$F(y,x,u) = \varphi_0 x(u-1) + x\varphi(y) - y = 0 \tag{2.15}$$

for $u \equiv 1$. $x_0 = \frac{1}{\varphi'(\tau)}$ is the smallest positive solution of $F_y(y(x),x,1) = x\varphi'(y(x)) - 1 = 0$. This means that x_0 is the first point where the implicit funtion theorem cannot be applied to continue the solution $y(x)$ of $F(y,x,1) = 0$. It turns out that x_0 is the radius of convergence of $y(x)$, that $x = x_0$ is the only singularity on the circle $x = x_0$ and that $y(x) = y(x,1)$ has a local expansion of the kind (2.13), where $u \equiv 1$. The analytic continuation property follows from this expansion locally around $x = x_0$ and from the fact that $d = 1$ implies

$$|x\varphi'(y(x))| < x_0\varphi'(y(x_0)) = 1 \tag{2.16}$$

for $|x| \le x_0$ and $x \ne x_0$.

Now vary u locally around 1 and consider the solution $y(x,u)$ of $F(y,x,u) = 0$. By continuity and the implicit function theorem this solution always exists for $|x| < x_0 + \varepsilon$, $|x - x_0| > \varepsilon$. The existence of a solution $0 < \tau < R$ of $t\varphi'(t) = \varphi(t)$ shows that $\varphi''(t) \not\equiv 0$. Hence $\varphi''(\tau) > 0$. Thus the implicit function theorem ensures that $f(u)$ is the unique solution of (2.12) if u varies locally around 1. In order to obtain some information about the local bahaviour of $y(x,u)$ around $x = f(u)$ we will use Weierstrass' preparation theorem [KK] for $F(y,x,u) = \varphi_0 x(u-1) + x\varphi(y) - y$ at $y = \tau$, $x = x_0$, $u = 1$. Since $F_y(\tau,x_0,1) = 0$ and $F_{yy}(\tau,x_0,1) = x_0\varphi''(\tau) \ne 0$ there exist analytic functions $H(y,x,u)$, $h_1(x,u)$, $h_2(x,u)$ with $H(\tau,x_0,1) \ne 0$, $h_1(x_0,1) = h_2(x_0,1) = 0$, and

$$F(y,x,u) = H(y,x,u)\left((y-\tau)^2 + h_1(x,u)(y-\tau) + h_2(x,u)\right) \tag{2.17}$$

locally around $y = \tau$, $x = x_0$, $u = 1$. Hence $y(x,u)$ satisfies

$$(y(x,u)-\tau)^2 + h_1(x,u)(y(x,u)-\tau) + h_2(x,u) = 0 \tag{2.18}$$

or can be expressed explicitely by

$$y(x,u) = -\frac{h_1(x,u)}{2} \pm \sqrt{\frac{h_1(x,u)^2}{4} - h_2(x,u)} \tag{2.19}$$

Now $F_x(\tau,x_0,1) = x_0\varphi'(\tau) \ne 0$ implies $(h_2)_x(x_0,1) \ne 0$ which gives

$$\frac{\partial}{\partial x}\left(\frac{h_1(x,u)^2}{4} - h_2(x,u)\right)_{x=x_0,u=1} \ne 0 \tag{2.20}$$

since $h_1(x_0,1) = 0$. Hence by Weierstrass' preparation theorem there exist analytic functions $h_3(x,u)$, $h_4(u)$ with $h_3(x_0,1) \ne 0$, $h_4(1) = 0$, and

$$h_2(x,u) - \frac{h_1(x,u)^2}{4} = h_3(x,u)((x-x_0) + h_4(u)) \tag{2.21}$$

locally around $x = x_0$, $u = 1$. Now set

$$g(x,u) = \tau - \frac{h_1(x,u)}{2} \tag{2.22}$$

and

$$h(x,u) = \sqrt{h_3(x,u)(x_0 - h_4(u))} \tag{2.23}$$

which gives

$$y(x,u) = g(x,u) \pm h(x,u)\sqrt{1 - \frac{x}{x_0 - h_4(u)}}. \tag{2.24}$$

But $x_0 - h_4(u) = f(u)$ since there are no other singularities of $y(x,u)$ around $x = x_0$, $u = 1$ except $x = f(u)$. Hence (2.13) follows from (2.24). Furthermore by (2.13) $y(x,u)$ can be analytically continued to $|x - x_0| < \varepsilon$, $\arg(1 - x/f(u)) \neq \pi$. Finally the representation (2.14) for $h(f(u),u)$ can be obtained by using Taylor's theorem for $F(y,x,u)$.

The case $d > 1$ can be managed similarly. The only difference is that $x\varphi'(y(x,1)) = 1$ if only if $x = x_0 \exp\left(j\frac{2\pi i}{d}\right)$ $(0 \le j < d)$ for $|x| \le x_0$ so that we get d singularities and hence similar expansions around these singularities. $\square$

Furthermore we will use the following integral:

Proposition 3. *Let γ be a Hankel contour starting from $+e^{2\pi i}\infty$, passing around 0 and terminating at $+\infty$.*

Picture 1

Then we have for $\lambda, \mu > 0$

$$\frac{1}{2\pi i}\int_\gamma e^{-\lambda\sqrt{-t}-\mu t}\,dt = \frac{\lambda\mu^{-\frac{3}{2}}}{2\sqrt{\pi}}\exp\left(-\frac{\lambda^2}{4\mu}\right). \tag{2.25}$$

Proof. (2.25) directly follows from the substitutions $t = v^2$, $v = \frac{x}{\sqrt{\mu}} + i\frac{\lambda}{2\mu}$ and from

$$\int_{-\infty+ic}^{\infty+ic} e^{-x^2}\,dx = \sqrt{\pi}. \tag{2.26}$$

$\square$

Now we will prove the asymptotic expansion (1.12) of Theorem 1. For simplicity we will only discuss the case $d = 1$. For $d > 1$ similar considerations around the points $x = x_0\exp\left(j\frac{2\pi i}{d}\right)$ $(0 \le j < d)$, $u = 1$ must be used.

Basically we will use Cauchy's formula

$$a_{kmn} = \frac{1}{(2\pi i)^2}\int_{\Gamma_1}\int_{\Gamma_2}\frac{A_k(x,u)}{x^{n+1}u^{m+1}}\,dx\,du, \tag{2.27}$$

where the paths of integration $\Gamma_1 = \Gamma_{11}\cup\Gamma_{12}\cup\Gamma_{13}\cup\Gamma_{14}$ and $\Gamma_2 = \Gamma_{21}\cup\Gamma_{22}\cup\Gamma_{23}\cup\Gamma_{24}$ are of the following kind ($x_0 = \frac{1}{f'(1)}$):

$$\begin{aligned}
\Gamma_{11} &= \left\{ x = x_0\left(1+\frac{t}{n}\right) \,\middle|\, \Re t < 0, |t| = 1\right\} \\
\Gamma_{12} &= \left\{ x = x_0\left(1+\frac{t}{n}\right) \,\middle|\, 0 \le \Re t \le \log^2 n, \Im t = 1\right\} \\
\Gamma_{13} &= \overline{\Gamma_{12}} \\
\Gamma_{14} &= \left\{ x \,\middle|\, |x| = \left|1+\frac{\log^2 n + i}{n}\right|, \arg\left(1+\frac{\log^2 n+i}{n}\right) \le |\arg x| \le \pi\right\}
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}
\Gamma_{21} &= \left\{ u = 1+\frac{s}{m} \,\middle|\, \Re s < -R(t), |s+R(t)| = 1\right\} \\
\Gamma_{22} &= \left\{ u = 1+\frac{s}{m} \,\middle|\, -R(t) \le \Re s \le \log^2 m, \Im s = 1\right\} \\
\Gamma_{23} &= \overline{\Gamma_{22}} \\
\Gamma_{24} &= \left\{ u \,\middle|\, |u| = \left|1+\frac{\log^2 m + i}{m}\right|, \arg\left(1+\frac{\log^2 m+i}{m}\right) \le |\arg u| \le \pi\right\},
\end{aligned} \tag{2.29}$$

where $R(t) = \max\left\{0, \frac{\varphi(\tau)}{\varphi_0}\frac{m}{n}\Re t\right\}$. Our aim is to show that

$$\begin{aligned}
a_{kmn} &= \varphi_0 \frac{x_0^{1-n}}{nm}\frac{1}{2\pi i}\int_\gamma e^{-\lambda_1\sqrt{-t}-\mu_1 t}\,dt \frac{1}{2\pi i}\int_\gamma e^{-\lambda_2\sqrt{-v}-\mu_2 v}\,dv \\
&\quad + \mathcal{O}\left(\frac{x_0^{-n}}{mn}\left(\frac{k}{n}+\frac{k}{m}\right)\right),
\end{aligned} \tag{2.30}$$

where

$$\begin{aligned}
\lambda_1 &= \frac{\sigma k}{\sqrt{2n}}, & \mu_1 &= 1 - \frac{m}{n}\frac{\varphi(\tau)}{\varphi_0}, \\
\lambda_2 &= \frac{\sigma k}{\sqrt{2m}}\sqrt{\frac{\varphi_0}{\varphi(\tau)}}, \quad \text{and} & \mu_2 &= 1.
\end{aligned} \tag{2.31}$$

Trivially, (2.30) and (2.25) yield (1.12) for $d = 1$.

First we will approximate $A_k(x,u)x^{-n}u^{-m}$ for $x \in \Gamma_{11}\cup\Gamma_{12}\cup\Gamma_{13}$ and $u \in \Gamma_{21}\cup\Gamma_{22}\cup\Gamma_{23}$. From $f'(1) = -x_0\frac{\varphi_0}{\varphi(\tau)}$ we get

$$\begin{aligned}
y(x,u) - \tau &= -\sqrt{\frac{2\varphi(\tau)}{\varphi''(\tau)}}\sqrt{1-\frac{x}{f(u)}}\left(1+\mathcal{O}\left(\frac{|s|}{m}\right)\right) \\
&\quad + \mathcal{O}\left(\left|1-\frac{x}{f(u)}\right|\right) + \mathcal{O}\left(\frac{|s|}{m}+\frac{|t|}{n}\right) \\
&= -\sqrt{\frac{2\varphi(\tau)}{\varphi''(\tau)}}\sqrt{-\frac{t}{n}-\frac{\varphi_0}{\varphi(\tau)}\frac{s}{m}} + \mathcal{O}\left(\frac{|s|}{m}+\frac{|t|}{n}\right)
\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}
&\left(x\frac{\varphi(y(x,u))-\varphi(y(x,1))}{y(x,u)-y(x,1)}\right)^k = \\
&= \left(1-\frac{\sigma}{\sqrt{2}}\left(\sqrt{-\frac{t}{n}}+\sqrt{-\frac{t}{n}-\frac{\varphi_0}{\varphi(\tau)}\frac{s}{m}}\right)+\mathcal{O}\left(\frac{|s|}{m}+\frac{|t|}{n}\right)\right)^k \\
&= \exp\left(-\frac{\sigma}{\sqrt{2}}\left(\sqrt{-\frac{k^2}{n}t}+\sqrt{-\frac{k^2}{n}t-\frac{\varphi_0}{\varphi(\tau)}\frac{k^2}{m}s}\right)\right)\cdot \\
&\qquad \cdot\left(1+\mathcal{O}\left(\frac{k|t|}{n}+\frac{k|s|}{m}\right)\right),
\end{aligned} \tag{2.33}$$

where we have used

$$\frac{\varphi(\tau+t_1)-\varphi(\tau+t_2)}{(\tau+t_1)-(\tau+t_2)} = \varphi'(\tau)+\frac{\varphi''(\tau)}{2}(t_1+t_2)+\mathcal{O}(|t_1|^2+|t_2|^2) \tag{2.34}$$

for sufficiently small t_1, t_2. Using the substitutions $x = x_0(1+t/n)$, $u = 1+s/m$ and setting

$$\begin{aligned}
C_1 &= \{t\,|\,|t|=1, \Re t<0\}\cup\{t\,|\,0\le \Re t\le \log^2 n, \Im t = \pm 1\}, \\
C_2 &= \{s\,|\,|s+R(t)|=1, \Re s<-R(t)\}\cup\{s\,|\,-R(t)\le \Re s\le \log^2 m, \Im s=\pm 1\}
\end{aligned} \tag{2.35}$$

we immediately get

$$\begin{aligned}
a_{kmn} &= \varphi_0\frac{x_0^{1-n}}{nm}\frac{1}{(2\pi i)^2}\int_{C_1}\int_{C_2} e^{-s-t}e^{-\sqrt{-\lambda_1^2 t}-\sqrt{-\lambda_1^2 t-\lambda_2^2 s}}\,ds\,dt \\
&\quad +\mathcal{O}\left(\frac{x_0^n}{mn}\left(\frac{k}{n}+\frac{k}{m}\right)\right),
\end{aligned} \tag{2.36}$$

where we have applied

$$\begin{aligned}
x^{-n} &= x_0^{-n}e^{-t}\left(1+\mathcal{O}\left(\frac{|t|^2}{n}\right)\right), \\
u^{-m} &= e^{-s}\left(1+\mathcal{O}\left(\frac{|s|^2}{n}\right)\right),
\end{aligned} \tag{2.37}$$

$$\begin{aligned}
&\int_{C_1}\int_{C_2}\left|e^{-s-t}e^{-\sqrt{-\lambda_1^2 t}-\sqrt{-\lambda_1^2 t-\lambda_2^2 s}}\right|\,|t^L||s^M|\,|ds|\,|dt| \\
&\quad\le \int_{C_1}\int_{C_2}\left|e^{-s-t}\right|\,|t^L||s^M|\,|ds|\,|dt| = \mathcal{O}_{L,M}(1)
\end{aligned} \tag{2.38}$$

for $L, M\ge 0$, and

$$\begin{aligned}
\sup_{x\in\Gamma_1, u\in\Gamma_{24}} |A(x,u)x^{-n}u^{-m}| &= \mathcal{O}(x_0^{-n}m^{-L}) \\
\sup_{x\in\Gamma_{14}, u\in\Gamma_2} |A(x,u)x^{-n}u^{-m}| &= \mathcal{O}(x_0^{-n}n^{-L})
\end{aligned} \tag{2.39}$$

for any fixed $L > 0$. Now we use the substitution

$$v = s + \frac{\varphi(\tau)}{\varphi_0}\frac{m}{n}\,t. \tag{2.40}$$

to get (2.30) where we have to bear in mind that

$$\int\limits_{\gamma\cap\{t \mid \Re t \geq K\}} e^{-\lambda\sqrt{-t}-\mu t}\,dt = \mathcal{O}\left(\mu^{-1}e^{-K\mu}\right) \tag{2.41}$$

uniformly for $K > 0$.

Now (1.13) of the Corollary follows trivially by using the well known asymptotic expansion

$$y_n = \frac{d\tau}{\sqrt{2\pi}\sigma}x_0^{-n}n^{-\frac{3}{2}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \tag{2.42}$$

for $n \equiv 1 \mod d$. (We can use (2.13) and [FO2] to obtain (2.42), too.)

Proof of Theorem 2

In order to prove Theorem 2 we again use Cauchy's formula and the same path of integration (2.28), (2.29) applied to

$$\begin{aligned}\sum_{n,m}\left(\sum_{k\geq 0} k a_{kmn}\right)x^n u^m &= \frac{\partial}{\partial z}\left[\sum_{k\geq 0} A_k(x,u)z^k\right]_{z=1} \\ &= \frac{u}{\varphi_0}\frac{(\varphi(y(x,u)) - \varphi(y(x,1)))\,(y(x,u) - y(x,1))}{(u-1)^2} \\ &= \frac{u}{\varphi_0 x}\left(\frac{y(x,u)-y(x,1)}{u-1}\right)^2 - u\frac{y(x,u)-y(x,1)}{u-1}.\end{aligned} \tag{3.1}$$

It turns out that the first term is dominating. So it suffices to discuss this one. The approximation ($x = x_0(1+t/n)$, $u = 1 + s/m$)

$$\begin{aligned}y(x,u) - y(x,1) &= \sqrt{\frac{2\varphi(\tau)}{\varphi''(\tau)}}\left(\sqrt{-\frac{t}{n}} - \sqrt{-\frac{t}{n} - \frac{\varphi_0}{\varphi(\tau)}\frac{s}{m}}\right) + \mathcal{O}\left(\frac{|t|}{n} + \frac{|s|}{m}\right) \\ &= \sqrt{\frac{2\varphi_0^2}{\varphi(\tau)\varphi''(\tau)}}\frac{\frac{s}{m}}{\sqrt{-\frac{t}{n}} + \sqrt{-\frac{t}{n} - \frac{\varphi_0}{\varphi(\tau)}\frac{s}{m}}} \\ &\quad + \mathcal{O}\left(\frac{|t|}{n} + \frac{|s|}{m}\right)\end{aligned} \tag{3.2}$$

and the substitutions (2.36) and $w = t\left(1 - \frac{m}{n}\frac{\varphi(\tau)}{\varphi_0}\right)$ directly lead to the integral

$$\begin{aligned}&\frac{1}{(2\pi i)^2}\int_{\Gamma_1}\int_{\Gamma_2}\frac{u}{\varphi_0 x}\left(\frac{y(x,u)-y(x,1)}{u-1}\right)^2 dx\,du = \\ &= \frac{2\tau}{\sigma^2}\frac{x_0^{-n}}{m\frac{\varphi(\tau)}{\varphi_0}\left(n - m\frac{\varphi(\tau)}{\varphi_0}\right)}\frac{1}{(2\pi i)^2}\int_\gamma\int_\gamma e^{-v-w}\frac{dv\,dw}{(\alpha\sqrt{-w}+\beta\sqrt{-v})^2} \\ &+ \mathcal{O}\left(\frac{x_0^{-n}}{n^{\frac{3}{2}}}\right),\end{aligned} \tag{3.3}$$

where

$$\alpha = \left(n - m\frac{\varphi(\tau)}{\varphi_0}\right)^{-\frac{1}{2}} \quad \text{and} \quad \beta = \left(m\frac{\varphi(\tau)}{\varphi_0}\right)^{-\frac{1}{2}}. \tag{3.4}$$

Hence Theorem 2 follows follows from the following formula.

Proposition 4. *Let γ be a Hankel contour as in Proposition 3. Then we have for $\alpha, \beta > 0$*

$$\frac{1}{(2\pi i)^2}\int_\gamma\int_\gamma e^{-v-w}\frac{dv\,dw}{(\alpha\sqrt{-w}+\beta\sqrt{-v})^2} = \frac{2}{\pi}\frac{\alpha\beta}{(\alpha^2+\beta^2)^2}. \tag{3.5}$$

Proof. We will use the following two substitutions. First we set $v = x^2$, $w = y^2$, where x and y tend from $-\infty$ to ∞ such that $\Im x > 0$ and $\Im y > 0$. Then we use $x = \frac{r}{2}\left(z + \frac{1}{z}\right)$ and $y = \frac{r}{2i}\left(z - \frac{1}{z}\right)$, where $0 < r < \infty$ and z varies around a positively oriented closed curve C including 0 and the arc $\{w = e^{it} \mid \frac{\pi}{2} < t < \pi\}$ but excluding the arc $\{w = e^{it} \mid \frac{3\pi}{2} < t < 2\pi\}$. (It should be noted that the range for r, z for a curve C indepenent of r in the second substitution is only valid in the limit case where the paths of integration for x, y pass the point $x = 0$ resp. $y = 0$.) Hence we get

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_\gamma\int_\gamma e^{-v-w}\frac{dv\,dw}{(\alpha\sqrt{-w}+\beta\sqrt{-v})^2}\\
&= \frac{1}{\pi^2}\int\limits_{\substack{-\infty\\ \Im x>0}}^{\infty}\int\limits_{\substack{-\infty\\ \Im y>0}}^{\infty} e^{-x^2-y^2}\frac{xy}{(\alpha x+\alpha y)^2}\,dx\,dy\\
&= \frac{-1}{\pi^2}\int_0^\infty e^{-r^2} r\,dr\int_C \frac{z^4-1}{(\alpha(z^2+1)-i\beta(z^2-1))^2}\,\frac{dz}{z}\\
&= \frac{-1}{\pi^2}\,\frac{1}{2}\,2\pi i\frac{2\alpha\beta i}{(\alpha^2+\beta^2)^2},
\end{aligned} \tag{3.6}$$

where we had to calculate the residues of

$$F(z) = \frac{z^4-1}{z\left(\alpha(z^2+1)-i\beta(z^2-1)\right)^2} \tag{3.7}$$

at $z = z_1 = 0$

$$\mathrm{Res}(F(z), z = z_1) = \frac{-1}{(\alpha+i\beta)^2} \tag{3.8}$$

and at $z = z_2 = i\sqrt{\frac{\alpha+i\beta}{\alpha-i\beta}}$

$$\mathrm{Res}(F(z), z = z_2) = \frac{1}{2}\frac{1}{(\alpha+i\beta)^2} + \frac{1}{2}\frac{1}{(\alpha-i\beta)^2}. \tag{3.9}$$

This proves Proposition 4. $\square$

Proof of Theorem 3

The proof of Theorem 3 is rather similar to that of Theorem 1 but a little bit more involved.

In order to simplify our notation set

$$\begin{aligned}\phi_1(x,u,v) &= x\sum_{i\ge 1}\varphi_i \sum_{j_1+j_2=i-1} y(x,u)^{j_1}y(x,v)^{j_2} \\ &= x\frac{\varphi(y(x,u))-\varphi(y(x,v))}{y(x,u)-y(x,v)}\end{aligned} \tag{4.1}$$

and

$$\phi_2(x,u,v) = x\sum_{i\ge 2}\varphi_i \sum_{j_1+j_2+j_3=i-2} y(x,u)^{j_1}y(x,v)^{j_2}y(x,1)^{j_3}, \tag{4.2}$$

where $y(x,u)$ is defined in (2.1). First we will prove an explicit formula for the generating function

$$B_{k_1k_2}(x,u_1,u_2) = \sum_{m_1,m_2,n} b_{k_1k_2m_1m_2n}x^n u_1^{m_1}u_2^{m_2}. \tag{4.3}$$

Proposition 5. *Let $\phi_1(x,u,v)$ and $\phi_2(x,u,v)$ be defined in (4.1) and (4.2). Then we have*

$$\begin{aligned}B_{k_1k_2}(x,u_1,u_2) = \varphi_0^2x^2u_1u_2\phi_2(x,u_1u_2,u_2)\phi_1(x,u_1u_2,u_2)^{k_1-1}\cdot \\ \cdot\, \phi_1(x,u_2,1)^{k_2-1}\frac{1-q^{\min\{k_1,k_2\}}}{1-q},\end{aligned} \tag{4.4}$$

where

$$q = \frac{\phi_1(x,u_1u_2,1)}{\phi_1(x,u_1u_2,u_2)\phi_1(x,u_2,1)}. \tag{4.5}$$

Proof. First let

$$\overline{A}_k(x,u,z) = \sum_{n,m,l}\overline{a}_{kmln}x^nu^mz^l \tag{4.6}$$

be the generating function of the numbers

$$\overline{a}_{kmln} = \sum_{|T|=n}\omega(T)\overline{a}_{kml}(T), \tag{4.7}$$

where $\overline{a}_{kml}(T)=1$ if T has l leaves and the m-th leaf has height k and $\overline{a}_{kml}(T)=0$ otherwise. As in Proposition 1 we get

$$\overline{A}_k(x,u,z) = \varphi_0xu\phi_1(x,uz,z)^k. \tag{4.8}$$

(Trivially $A_k(x,u) = \overline{A}_k(x,u,z)$.)

Next we use the combinatorial interpretation to obain

$$\begin{aligned}
&B_{k_1+1,k_2+1}(x,u_1,u_2) = \\
&\quad = x\overline{A}_{k_1}(x,u_1,u_2)\overline{A}_{k_2}(x,u_2,1)\cdot \\
&\qquad \cdot \sum_{i\ge 2}\varphi_i \sum_{j_1+j_2+j_3=i-2} y(x,u_1u_2)^{j_1}y(x,u_2)^{j_2}y(x,1)^{j_3} \\
&\qquad + xB_{k_1k_2}(x,u_1,u_2)\sum_{i\ge 1}\varphi_i \sum_{j_1+j_2=i-1} y(x,u_1u_2)^{j_1}y(x,1)^{j_2} \\
&\quad = x\phi_1(x,u_1u_2,u_2)^{k_1}\phi_1(x,u_2,1)^{k_2}\phi_2(x,u_1u_2,u_2) \\
&\qquad + xB_{k_1k_2}(x,u_1,u_2)\phi_1(x,u_1u_2,1),
\end{aligned} \tag{4.9}$$

where the first terms corresponds to the case where the two leaves of interest are contained in different subtrees leading away from the root and the second term to the case where both leaves are contained in the same subtree. Using the initial conditions $B_{k_10}(x,u_1,u_2) = B_{0k_2}(x,u_1,u_2) = 0$ we directly get

$$\begin{aligned}
&B_{k_1k_2}(x,u_1,u_2) = \varphi_0^2x^2u_1u_2\phi_2(x,u_1u_2,u_2)\phi_1(x,u_1u_2,u_2)^{k_1-1}\cdot \\
&\quad \cdot \phi_1(x,u_2,1)^{k_2-1}\left(1+q+q^2+\cdots+q^{\min\{k_1,k_2\}-1}\right),
\end{aligned} \tag{4.10}$$

which proves (4.4). $\square$

In addition to (2.25) of Proposition 3 we will need

Proposition 6. *Let γ be a Hankel contour as in Proposition 3. Then we have for $\lambda,\mu>0$*

$$\frac{1}{2\pi i}\int_\gamma e^{-\lambda\sqrt{-t}-\mu t}(-t)^{-\frac12}\,dt = \frac{1}{\sqrt{\mu\pi}}\exp\left(-\frac{\lambda^2}{4\mu}\right). \tag{4.11}$$

Proof. We either can differentiate (4.11) with respect to λ or can proceed as in the proof of Proposition 3. $\square$

Next set

$$x = x_0\left(1+\frac{t}{n}\right) \tag{4.12}$$

and

$$u_i = 1+\frac{s_i}{m_i} \tag{4.13}$$

for $i = 1, 2$. As above we get

$$
\begin{aligned}
\phi_1(x, u_1u_2, u_2) &= 1 - \frac{\sigma}{\sqrt{2}}\left(\sqrt{-\frac{t}{n} - \frac{\varphi_0}{\varphi(\tau)}\frac{s_2}{m_2}} + \sqrt{-\frac{t}{n} - \frac{\varphi_0}{\varphi(\tau)}\left(\frac{s_1}{m_1} + \frac{s_2}{m_2}\right)}\right)\\
&\quad + \mathcal{O}\left(\frac{|t|}{n} + \frac{|s_1|}{m_1} + \frac{|s_2|}{m_2}\right),\\
\phi_1(x, u_2, 1) &= 1 - \frac{\sigma}{\sqrt{2}}\left(\sqrt{-\frac{t}{n}} + \sqrt{-\frac{t}{n} - \frac{\varphi_0}{\varphi(\tau)}\frac{s_2}{m_2}}\right) + \mathcal{O}\left(\frac{|t|}{n} + \frac{|s_2|}{m_2}\right),\\
\phi_1(x, u_1u_2, 1) &= 1 - \frac{\sigma}{\sqrt{2}}\left(\sqrt{-\frac{t}{n}} + \sqrt{-\frac{t}{n} - \frac{\varphi_0}{\varphi(\tau)}\left(\frac{s_1}{m_1} + \frac{s_2}{m_2}\right)}\right)\\
&\quad + \mathcal{O}\left(\frac{|t|}{n} + \frac{|s_1|}{m_1} + \frac{|s_2|}{m_2}\right),\\
q &= 1 + \sqrt{2}\sigma\sqrt{-\frac{t}{n} - \frac{\varphi_0}{\varphi(\tau)}\frac{s_2}{m_2}} + \mathcal{O}\left(\frac{|t|}{n} + \frac{|s_1|}{m_1} + \frac{|s_2|}{m_2}\right),
\end{aligned}
\tag{4.14}
$$

and

$$
\phi_2(x, u_1u_2, u_2) = x_0\frac{\varphi''(\tau)}{2} + \mathcal{O}\left(\sqrt{\frac{|t|}{n} + \frac{|s_1|}{m_1} + \frac{|s_2|}{m_2}}\right). \tag{4.15}
$$

Now we use a triple Hankel contour integration similar to (2.28), (2.29), and the substitutions

$$
v = s_2 + \frac{m_2}{n}\frac{\varphi(\tau)}{\varphi_0}t, \qquad w = s_1 + \frac{m_1}{m_2}s_2 + \frac{m_1}{n}\frac{\varphi(\tau)}{\varphi_0}t \tag{4.16}
$$

to obtain a representation of the kind

$$
\begin{aligned}
&b_{k_1k_2m_1m_2n} =\\
&= \frac{1}{(2\pi i)^3}\int_{\Gamma_1}\int_{\Gamma_2}\int_{\Gamma_3}\frac{B_{k_1k_2}(x, u_1, u_2)}{x^{n+1}u_1^{m_1+1}u_2^{m_2+1}}\,dx\,du_1\,du_2\\
&= C\frac{1}{(2\pi i)^3}\int_{\gamma} e^{-\lambda_1\sqrt{-t}-\mu_1 t}\,dt\int_{\gamma} e^{-\lambda_2\sqrt{-w}-\mu_2 w}\,dw\cdot\\
&\quad\cdot\left\{-\int_{\gamma} e^{-\lambda_{31}\sqrt{-v}-\mu_3 v}(-v)^{-\frac{1}{2}}\,dv + \int_{\gamma} e^{-\lambda_{32}\sqrt{-v}-\mu_3 v}(-v)^{-\frac{1}{2}}\,dv\right\}\\
&+ \mathcal{O}\left(\frac{x_0^n}{nm_1m_2}\left(\frac{k_1+k_2}{n} + \frac{k_1}{m_1} + \frac{k_2}{m_2}\right)\right),
\end{aligned}
\tag{4.17}
$$

where

$$
C = \sqrt{\frac{\varphi(\tau)}{\varphi_0}}\frac{\varphi_0^2\varphi''(\tau)x_0^{3-n}}{2\sqrt{2}\sigma n m_1\sqrt{m_2}},
$$

$$\begin{aligned}
\lambda_1 &= \frac{\sigma k_2}{\sqrt{2n}}, & \mu_1 &= 1 - \frac{m_2}{n}\frac{\varphi(\tau)}{\varphi_0}, \\
\lambda_2 &= \frac{\sigma k_1}{\sqrt{2m_1}}\sqrt{\frac{\varphi(\tau)}{\varphi_0}}, & \mu_2 &= 1, \\
\lambda_{31} &= \frac{\sigma(k_1+k_2)}{\sqrt{2m_2}}\sqrt{\frac{\varphi(\tau)}{\varphi_0}}, & \mu_3 &= 1 - \frac{m_1}{m_2}, \quad \text{and} \\
\lambda_{32} &= \frac{\sigma(k_1+k_2-2\min\{k_1,k_2\})}{\sqrt{2m_2}}\sqrt{\frac{\varphi(\tau)}{\varphi_0}}.
\end{aligned} \tag{4.18}$$

Hence everything follows by applying (2.25) and (4.11) to (4.17).

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