NEWMAN'S PHENOMENON FOR GENERALIZED THUE-MORSE SEQUENCES

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Abstract. Let $t_j = (-1)^{s(j)}$ be the Thue-Morse sequence with $s(j)$ denoting the sum of the digits in the binary expansion of j. A well-known result of Newman [10] says that $t_0 + t_3 + t_6 + \cdots + t_{3k} > 0$ for all $k \geq 0$.

In the first part of the paper we show that $t_1 + t_4 + t_7 + \cdots + t_{3k+1} < 0$ and $t_2 + t_5 + t_8 + \cdots + t_{3k+2} \leq 0$ for $k \geq 0$, where equality is characterized by means of an automaton. This sharpens results given by Dumont [4]. In the second part we study more general settings. For $a, g \geq 2$ let $\omega_a = \exp(2\pi \mathrm{i}/a)$ and $t_j^{(a,g)} = \omega_a^{s_g(j)}$, where $s_g(j)$ denotes the sum of digits in the g-ary digit expansion of j. We observe trivial Newman-like phenomena whenever $a|(g-1)$. Furthermore, we show that the case $a = 2$ inherits many Newman-like phenomena for every even $g \geq 2$ and large classes of arithmetic progressions of indices. This, in particular, extends results by Drmota/Skalba [3] to the general g-case.

1. Introduction

Let $t_j = 1, -1, -1, 1, -1, 1, 1, -1, -1, \ldots$ be the Thue-Morse sequence defined by

$$t_j = (-1)^{s(j)} \quad \text{for} \quad j \geq 0, \tag{1.1}$$

where $s(j)$ denotes the sum of the digits in the binary expansion of j. Fix $q \geq 2$ and $i \geq 0$ and consider the subsequence t_{kq+i} with $k \geq 1$. One may ask whether there is a preponderance of the 1's over the -1's in that sequence, or equivalently, of the numbers with even sum of binary digits over the numbers with odd sum of binary digits. In 1969 Newman [10] showed that the 1's prevail in the case of $q = 3$ and $i = 0$. More precisely, by denoting $\tau(n) = \lfloor (n+2)/3 \rfloor$ and

$$S_{q,i}(n) = \sum_{\substack{0 \leq j < n, \\ j \equiv i \pmod q}} t_j, \tag{1.2}$$

Newman's Theorem states that for $n \geq 1$,

$$\frac{3^\alpha}{20} < S_{3,0}(n)\tau(n)^{-\alpha} < 5 \cdot 3^\alpha \qquad \text{with} \qquad \alpha = \log_4 3.$$

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Coquet [1] could give a precise expression for $S_{3,0}(n)$ which involves a continuous 1-periodic fractal function ψ,

$$S_{3,0}(n) = \tau(n)^\alpha \cdot \psi(\log_4 n) - \eta(n)/3,$$

where $\eta(n) \in \{-1, 0, 1\}$. He also displayed the extremal values of $\psi(x)$ on $[0, 1]$ and provided by the way an alternative proof for $S_{3,0}(n) > 0$. It is natural to ask whether there exist similar phenomena for $S_{3,1}(n)$ and $S_{3,2}(n)$. Dumont [4], by using a method of Newman and Slater [11], could prove that $S_{3,1}(n) < 0$ for $n > n_0$. In a short comment he also states that both $S_{3,2}(n) < 0$ and $S_{3,2}(n) > 0$ for infinitely many n. This is not correct since we prove

Theorem 1.1. (1) *$S_{3,1}(n) < 0$ for $n \geq 2$.*
(2) *$S_{3,2}(n) \leq 0$ for $n \geq 3$ with equality if and only if $n = 2^{2k+1}$ for $k \geq 1$ or the binary expansion of n is realized by the automaton given in Figure 1.*

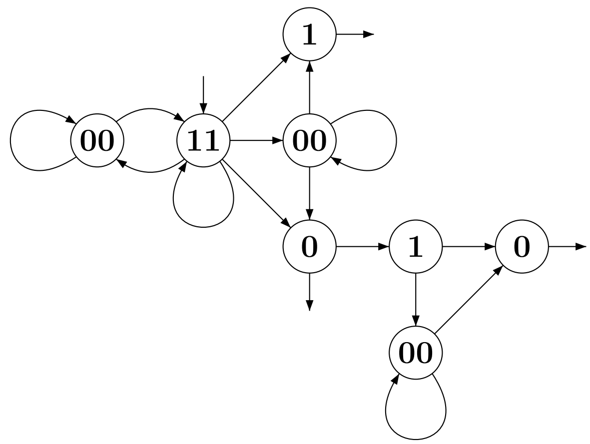

FIGURE 1

The automaton constructs numbers n which can be described in the following. First, a 'head' is constructed by means of alternating **1**...**1**- and **0**...**0**-blocks whereas the length of each block is an even number. After the rightmost **11**-entry of the head a 'tail' is appended which is either of type **0**...**01** (even number of **0**'s), **0**...**0** (odd number of **0**'s) or **0**...**010**...**0** where in the latter case the **0**-blocks have (arbitrary) odd length. So, for instance, for $n = (\mathbf{11110000001100110001 0})_2$ we have $S_{3,2}(n) = 0$.

The discrete function $S_{q,0}(n)$ has also been studied for other fixed values of q (see [1, 2, 3, 7, 8, 9]). Using an asymptotical approach Drmota and Skalba [3] showed that Newman's $q = 3$ can be replaced by an arbitrary multiple of 3, i.e. $q = 3\kappa$ for $\kappa \geq 1$, such that $S_{q,0}(n)$ attains positive values for all but finitely many n. We will generalize this fact in

Theorem 1.2. *Let $\nu \geq 0$ and $\kappa \geq 1$. Then there exists n_0 such that*

(1) *$S_{3\kappa,3\nu}(n) > 0$ for $n > n_0$.*
(2) *$S_{3\kappa,3\nu+1}(n) < 0$ for $n > n_0$.*

A straightforward base g generalization of the Thue-Morse sequence was introduced and investigated by Goldstein, Kelly and Speer [6, Section 5]. Let $a, g \geq 2$ be two fixed positive integers. In analogue to (1.1) define

$$t_k^{(a,g)} = \omega_a^{s_g(k)} \quad \text{for} \quad k \geq 1, \tag{1.3}$$

where $\omega_a = \exp(2\pi \mathrm{i}/a)$ denotes the a-th primitive root of unity (a is sometimes also called the *parity*) and $s_g(k)$ the sum of the digits in the g-ary expansion of k. Similar to (1.2) set

$$S_{q,i}^{(a,g)}(n) = \sum_{\substack{0 \leq j < n, \\ j \equiv i \pmod q}} t_j^{(a,g)}. \tag{1.4}$$

Further let

$$A_{q,i;m}^{(a,g)}(n) = |\{0 \leq j < n : \quad j \equiv i \pmod q), \quad s_g(j) \equiv m \pmod a)\}|,$$

which counts how often ω_a^m shows up on the right hand side of (1.4), i.e.

$$S_{q,i}^{(a,g)}(n) = \sum_{m=0}^{a-1} A_{q,i;m}^{(a,g)}(n)\, \omega_a^m. \tag{1.5}$$

Using this notation Newman's Theorem, for instance, translates into

$$A_{3,0;0}^{(2,2)}(n) > A_{3,0;1}^{(2,2)}(n) \quad \text{for all} \quad n \geq 1.$$

For general triples (a, g, q) we use

Definition 1.3. *The triple (a, g, q) is said to* satisfy an (i, M)-Newman-like phenomenon *if*

$$A_{q,i;M}^{(a,g)}(n) > \max_{\substack{0 \leq m < a \\ m \neq M}} A_{q,i;m}^{(a,g)}(n) \quad \textit{for all but finitely many } n \geq 1.$$

For sake of shortness such occurrences will be referred to as (i, M)-NLP's. The aim of our work is mostly to identify multi-parametric families of NLP's for $a = 2$. Concerning the case $a = g = 2$ infinite lists of triples satisfying $(0, 0)$-NLP's are already well-known:

(i) (Drmota/Skalba [3]): $(2, 2, 3\kappa)$, $(2, 2, 4^\kappa + 1)$ for $\kappa \geq 1$.

(ii) (Leinfellner [9]): $(2, 2, (2^{4\kappa-1} + 1)/3)$ for $\kappa \geq 1$.

As Theorem 1.1 and Theorem 1.2 suggest, there may be $(i, 0)$- and $(i, 1)$-NLP's for more general g. We first show that there exist only trivial (i, M)-NLP's whenever $a|(g - 1)$, thus for $a = 2$, in particular, there are no NLP's if g is odd and $q = \kappa(g + 1)$.

Theorem 1.4. *Let $a|(g - 1)$. Then (a, g, q) satisfies an (i, M)-NLP if and only if $a|q$ and $i \equiv M \pmod a$.*

On the other hand, triples of the form $(2, g, \kappa(g + 1))$ with even $g \geq 4$ are shown to satisfy several $(i, 0)$- and $(i, 1)$-NLP's where i ranges over large intervals depending explicitly on g. Indeed, $I_1 \cup I_2$ make up more than 50% of the positive integers $i \geq 0$.

Theorem 1.5. *Let $g \geq 4$ be even, κ odd and denote*

$$I_1 = \bigcup_{\nu=0}^{\infty} \left[2\nu(g+1), \frac{g}{2} + 2\nu(g+1)\right],$$

$$I_2 = \bigcup_{\nu=0}^{\infty} \left[(2\nu+1)(g+1), \frac{g}{2} + (2\nu+1)(g+1)\right].$$

(1) *If $i \in I_1$ is even or $i \in I_2$ is odd then $(2, g, \kappa(g+1))$ satisfies an $(i,0)$-NLP.*
(2) *If $i \in I_1$ is odd or $i \in I_2$ is even then $(2, g, \kappa(g+1))$ satisfies an $(i,1)$-NLP.*

As an immediate consequence of Theorem 1.5 and Theorem 1.2 we notice that for any i there are infinitely many bases g for which we can observe NLP's.

Corollary 1.6. *Let $i \geq 0$ be even (resp. odd). Then for all even $g \geq 2$ the triple $(2, g, \kappa(g+1))$ satisfies an $(i,0)$-NLP (resp. $(i,1)$-NLP).*

Finally we show that there are only few primes q where an NLP occurs. This a direct generalization of [3, Theorem 2]. Let p be an odd prime and $g \geq 2$ an even integer. Set $s = \mathrm{ord}_p(g)$ the multiplicative order of g in the multiplicative group modulo p. Then $s|(p-1)$ and $t = (p-1)/s$ is called the co-order of g. Furthermore let $\mathbb{P}_t$ denote the set of odd primes for which g has co-order t.

Theorem 1.7. *Let $g \geq 2$ be an even integer. Then every prime $p \in \mathbb{P}_t$ such that $(2, g, p)$ satisfies an $(0,0)$-NLP is bounded by*

$$p \leq Ct^2(\log t)^2,$$

where $C > 0$ only depends on g.

Furthermore,

$$\#\{p \leq x : \ (2, g, p) \text{ satisfies and } (0,0)\text{-NLP}\} = o\left(\frac{x}{\log x}\right),$$

that is, almost no primes satisfy a $(0,0)$-NLP.

2. Possible extensions

Drmota and Skalba [3] observed that while considering $q = (g^a - 1)/(g-1)$ the *parity* a can not be too large in order to obtain $(0,0)$-NLP's. More precisely, they proved that $(a, 2, 2^a - 1)$ satisfies a $(0,0)$-NLP if and only if $2 \leq a \leq 6$. Numerical simulations motivate several conjectures (see below) that we want to deal with in a forthcoming paper. Conjecture 1 gives evidence that NLP's aren't rare at all, while Conjecture 2 is a weak analogon of Theorem 1.5 for the case $a = 3$. Concerning Conjecture 3, there are expected to be infinitely many parities a and for each of them again an infinite number of bases g such that there hold $(0,0)$-NLP's. This casts a more positive light compared to the result of Drmota/Skalba.

- **Conjecture 1:**
  For all $0 \leq i \leq q-1$ there exists a $M = M(i)$ such that $(3, 2, 7\kappa)$ satisfies an (i, M)-NLP.
- **Conjecture 2:**
  Let $g \geq 3$ and $(g-1, 3) = (\kappa, 3) = 1$. Then the triple $(3, g, \kappa(g^2+g+1))$ satisfies a $(0,0)$-NLP, a $(1,1)$-NLP and a $(2,2)$-NLP.
- **Conjecture 3:**
  (1) Let $a \equiv 0 \pmod 2$ and $g = (\nu + \frac{1}{2})a + 1$ with $\nu \geq 0$. Then $(a, g, \kappa(g^a - 1)/(g-1))$ satisfies a $(0,0)$-NLP.
  (2) Let $a \equiv 0 \pmod 3$ and $g = (\nu + \frac{2}{3})a + 1$ with $\nu \geq 0$. Then $(a, g, \kappa(g^a - 1)/(g-1))$ satisfies a $(0,0)$-NLP.

3. Proof of Theorem 1.1

For the following basic properties of $S_{q,i}(n)$ we refer to [3]. A general exposition will be given later in Section 5.1. To begin with, since (see relation (8) and the proof of Lemma 5 in [3])

$$S_{q,i}(2^k) = \frac{1}{q} \sum_{i=0}^{q-1} \zeta_q^{-li} \prod_{j=0}^{k-1} \left(1 - \zeta_q^{l2^j}\right)$$

we have

$$\begin{aligned} S_{3,1}(2^k) = S_{3,2}(2^k) &= -1 \cdot \frac{\sqrt{3}^k}{3}, && \text{if } k \text{ is even} \geq 2, \\ S_{3,1}(2^k) &= \sqrt{3} \cdot \frac{\sqrt{3}^k}{3}, && \text{if } k \text{ is odd}, \\ S_{3,2}(2^k) = S_{3,1}(2^0) = S_{3,2}(2^0) &= 0, && \text{if } k \text{ is odd}. \end{aligned}$$

Moreover, since for all $n' < 2^k$ it holds (see relation (9) in [3])

$$S_{q,i}(2^k + n') = S_{q,i}(2^k) - S_{q,i-2^k}(n'),$$

all expansion of $S_{q,i}(n)$ into values of powers of 2 can be seen as paths in the graph of Figure 2.

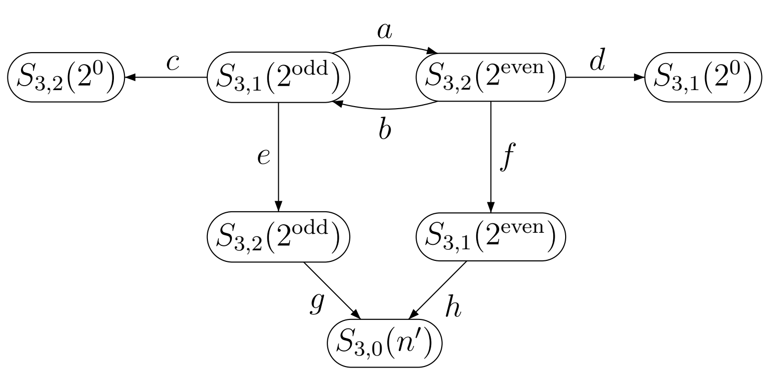

Figure 2

To start with, observe that by Newman's Theorem

$$S_{3,0}(2^k - n') \le S_{3,0}(2^k) - S_{3,0}(n') < \frac{2}{3}\sqrt{3}^k \qquad \text{for all} \quad n' < 2^k. \tag{3.1}$$

Proof of Theorem 1.1. First we consider the case of $S_{3,1}(n)$. Of course, if $s_2(n) = 1$ then $S_{3,1}(n) < 0$. Let now $s_2(n) > 1$ and $n = 2^k + \dots$ with k even. Then

$$S_{3,1}(n) = S_{3,1}(2^k) - S_{3,0}(n') = (-1) \cdot \frac{\sqrt{3}^k}{3} - S_{3,0}(n') < 0 \tag{3.2}$$

by Newman's Theorem. Now, let k be odd. Denote

$$\begin{aligned} \mathcal{A}_1 &= \{(ab)^m,\ (ab)^m a,\ (ab)^m af,\ (ab)^m afh,\ (ab)^m ad\},\\ \mathcal{A}_2 &= \{(ab)^m c,\ (ab)^m e,\ (ab)^m eg\}. \end{aligned}$$

Let $n \in \mathcal{A}_1$. Then by (3.2),

$$S_{3,1}(n) \le \frac{\sqrt{3}^k}{3}\left(-\sqrt{3} - \frac{-1}{\sqrt{3}}\right) < 0.$$

On the other hand, if $n \in \mathcal{A}_2$ then by (3.1),

$$S_{3,1}(n) \le \frac{\sqrt{3}^k}{3}(-\sqrt{3} - 0) + S_{3,0}(n') < -\frac{\sqrt{3}}{3}\sqrt{3}^k + \frac{2}{3}\sqrt{3}^{k-2} < 0.$$

Consider now $S_{3,2}(n)$. If $s_2(n) = 1$ then $S_{3,2}(n) \le 0$ with equality if and only if k is odd. Suppose $s_2(n) > 1$ and k odd. Then by Newman's Theorem

$$S_{3,2}(n) = S_{3,2}(2^k) - S_{3,0}(n') < 0.$$

Let now k be even and put

$$\begin{aligned} \mathcal{B}_1 &= \{(ba)^m,\ (ba)^m d,\ (ba)^m f,\ (ba)^m fh\},\\ \mathcal{B}_2 &= \{(ba)^m beg\},\\ \mathcal{B}_3 &= \{(ba)^m b,\ (ba)^m bc,\ (ba)^m be\}. \end{aligned}$$

First note that the edge b gives maximal contribution (namely 0) to the final sum, if the corresponding 1's in the binary expansion of n are adjacent. So, for $n \in \mathcal{B}_1$ and by (3.1) it holds

$$S_{3,2}(n) < \frac{\sqrt{3}^k}{3}\left(0 - \sqrt{3}^{-l} + \sqrt{3}^{-l-2}\right) + \frac{2}{3}\sqrt{3}^{k-l-2} = 0.$$

If $n \in \mathcal{B}_2$ then

$$S_{3,2}(n) \le \frac{\sqrt{3}^k}{3}(0+0) - S_{3,0}(n') < 0.$$

Finally, if $n \in \mathcal{B}_1$ then $S_{3,2}(n) \le 0$ where equality holds if and only if the 1's corresponding to the adjacent expansion terms $S_{3,1}(2^{\text{odd}})$ and $S_{3,2}(2^{\text{even}})$ are adjacent and there is at most one digit 1 at some lower odd position 2^k or at the 2^0-position. The automaton can now be easily constructed. $\square$

4. Proof of Theorem 1.4

Let

$$d(n)=\begin{cases}(n-i)/q, & q|n-i\\ [(n-i)/q]+1, & \text{otherwise.}\end{cases}$$

Since $s_g(n)\equiv n \pmod{g-1}$ and $a|(g-1)$ we have $s_g(n)\equiv n \pmod a$. Thus, if $a\not|q$ then by (1.4) and (1.5),

$$S_{q,i}^{(a,g)}(n)=\sum_{\substack{0\le j<n,\\ j\equiv i \pmod q}}\omega_a^j=\omega_a^i\sum_{k=0}^{d(n)}\omega_a^{kq}=\omega_a^i\frac{\omega_a^{q(d(n)+1)}-1}{\omega_a^q-1}.$$

For $n=(ka-1)q+i$ with $k\ge 1$ holds $d(n)\equiv -1 \pmod a$ and $S_{q,i}^{(a,g)}(n)=0$. Hence no NLP occurs. On the other hand, in the case $a|q$ the statement of the theorem is obviously true since $S_{q,i}^{(a,g)}(n)=\omega_a^i\,(d(n)+1)$. $\square$

5. Proof of Theorem 1.2 and Theorem 1.5

5.1. **Preliminaries.** The strategy for studying the discrete function $S_{q,i}^{(a,g)}(n)$ for large n consists in expanding the function in a Fourier series and looking at the behaviour of the asymptotically dominating term $\bar{S}_{q,i}^{(a,g)}(n)$. The growth of this term is basically determined by the absolute maximal eigenvalue $\lambda_{\max}$ of the matrix

$$\mathbf{M}(\omega_a)=\prod_{m=0}^{s-1}(\mathbf{I}+\omega_a\mathbf{T}^{g^m}+\omega_a^2\mathbf{T}^{2g^m}+\cdots+\omega_a^{g-1}\mathbf{T}^{(g-1)g^m}),$$

where $s=\mathrm{ord}_q(g)$ and $\mathbf{T}$ denotes the matrix which 'shifts' the canonical basis of $\mathbb{C}^q$ via $\mathbf{Te_i}=\mathbf{e_{i+1}}$. This is a straightforward generalization of the case $g=2$ treated in detail in [3] and [6].

Moreover, the function $S_{q,i}^{(a,g)}(n)$ can be made explicit by considering a simple generating relation. To begin with, observe that for $1\le\varepsilon\le g-1$ it holds

$$\begin{aligned}\sum_{n<\varepsilon g^k}y^{s_g(n)}z^n &= \left(1+yz^{g^k}+\cdots+y^{\varepsilon-1}z^{(\varepsilon-1)g^k}\right)\sum_{n<g^k}y^{s_g(n)}z^n\\ &= \tfrac{1-y^\varepsilon z^{\varepsilon g^k}}{1-yz^{g^k}}\cdot\prod_{j=0}^{k-1}(1+yz^{g^j}+\cdots+y^{g-1}z^{(g-1)g^j}).\end{aligned}\tag{5.1}$$

Let

$$S_{q,i}^{(a,g)}(y,n)=\sum_{\substack{0\le j<n,\\ j\equiv i \pmod q}}y^{s_g(j)}\tag{5.2}$$

and $\zeta_q=\exp(2\pi\mathrm{i}/q)$. By employing two different ways of counting y-powers we get

$$\sum_{i=0}^{q-1}\zeta_q^{li}S_{q,i}^{(a,g)}(y,\varepsilon g^k)=\sum_{n<\varepsilon g^k}y^{s_g(n)}\zeta_q^{ln}$$

and by (5.1),

$$S_{q,i}^{(a,g)}(y,\varepsilon g^k) = \frac{1}{q}\sum_{l=0}^{q-1} \zeta_q^{-li} \frac{1-y^\varepsilon \zeta_q^{\varepsilon l g^k}}{1-y\zeta_q^{lg^k}} \cdot \prod_{j=0}^{k-1} \frac{1-y^g\zeta_q^{lg^{j+1}}}{1-y\zeta_q^{lg^j}}. \tag{5.3}$$

Thus, in principle, it is possible to evaluate $S_{q,i}^{(a,g)}(y,n)$ at multiples of g-powers. For general $n=\varepsilon g^k + n'$ with $n' < g^k$ definition (5.2) provides a simple recursive relation, namely

$$S_{q,i}^{(a,g)}(y,\varepsilon g^k + n') = S_{q,i}^{(a,g)}(y,\varepsilon g^k) + y^\varepsilon S_{q,i-\varepsilon g^k}^{(a,g)}(y,n'), \tag{5.4}$$

which enables to split off higher multiples of g-powers. For $1 \le l \le q-1$ let

$$\eta_l^\varepsilon(k) = \frac{1-\omega_a^\varepsilon \zeta_q^{\varepsilon l g^k}}{1-\omega_a \zeta_q^{lg^k}} \qquad \text{and} \qquad \lambda_l(k) = \prod_{j=0}^{k-1} \frac{1-\omega_a^g \zeta_q^{lg^{j+1}}}{1-\omega_a\zeta_q^{lg^j}} \tag{5.5}$$

denote the factors appearing in (5.3). Since $\lambda_l(k_1 s + k_2) = \lambda_l(s)^{k_1} \cdot \lambda_l(k_2)$ and $\eta_l^\varepsilon(k_1 s + k_2) = \eta_l^\varepsilon(0)^{k_1} \cdot \eta_l^\varepsilon(k_2)$ we see that

$$S_{q,i}^{(a,g)}(\omega_a,\varepsilon g^k) = \frac{1}{q}\sum_{l=0}^{q-1} \zeta_q^{-li} \left(\eta_l^\varepsilon(0)\lambda_l(s)\right)^{k_1} \eta_l^\varepsilon(k_2)\lambda_l(k_2). \tag{5.6}$$

Thus the growth of $|S_{q,i}^{(a,g)}(\omega_a,\varepsilon g^k)|$ is asymptotically determined by $\Lambda_l = |\eta_l^\varepsilon(0)\lambda_l(s)|$. More precisely, let

$$L_{\max} = \left\{ l : \quad |\eta_l^\varepsilon(0)\lambda_l(s)| \ge |\eta_{\hat{l}}^\varepsilon(0)\lambda_{\hat{l}}(s)| \text{ for all } 0 \le \hat{l} \le q-1 \right\}$$

and set $\Lambda = |\eta_l^\varepsilon(0)\lambda_l(s)|$ for $l \in L_{\max}$. Then for $k = k_1 s + k_2$ we have

$$\begin{aligned} \bar{S}_{q,i}^{(a,g)}(\omega_a,\varepsilon g^k) &= \frac{1}{q}\sum_{l\in L_{\max}} \zeta_q^{-li}\eta_l^\varepsilon(k)\lambda_l(k) \\ &= \frac{\Lambda^{k_1}}{q}\sum_{l\in L_{\max}} \zeta_q^{-li} \exp(\mathrm{i}k_1\theta_0)\eta_l^\varepsilon(k_2)\lambda_l(k_2), \end{aligned} \tag{5.7}$$

where $\theta_0 = \arg(\eta_l^\varepsilon(0)\lambda_l(s))$. Note that in the case $g=2$ (treated in [3]) we have $\eta_l^\varepsilon(k) \equiv 1$ and thus the calculation of $L_{\max}$ is just right the calculation of the maximal $|\lambda_l(s)|$. For the case $a=2$, $g>2$ determining $L_{\max}$ is a more difficult task since for $\kappa > 1$ we have

$$\max_l |\eta_l^\varepsilon(0)| \cdot \max_l |\lambda_l(s)| > \Lambda,$$

i.e. we cannot independently maximize $|\eta_l^\varepsilon(0)|$ and $|\lambda_l(s)|$. We deal with this additional difficulty in Lemma 5.2.

5.2. **Outline of proof.** From now on let $a = 2$, $g \equiv 0 \pmod 2$ and $q = \kappa(g+1)$ with $(\kappa, 2) = 1$. Recall that the case $g \equiv 1 \pmod 2$ is totally characterized for all q in Theorem 1.4. Our investigation on the fractal behaviour of $S_{q,i}^{(2,g)}(-1, n)$ now splits up into several steps. First we determine $L_{\max}$ (Lemma 5.2 and Lemma 5.3) and get an explicit expression for $\bar{S}_{q,i}^{(2,g)}(-1, \varepsilon g^k)$ (Lemma 5.4). Then, starting from a sufficiently large $n = \varepsilon_1 g^k + \varepsilon_2 g^{k-1} + \dots$, we use the recursive relation (5.4) to 'expand' the function to values of the function at points of lower g-order. We obtain a finite tail which can be estimated by a geometric series with small modulus (Corollary 5.5). A sufficient criterion is then given which implies $(i, 0)$- and $(i, 1)$-NLP's depending on the parity of i (Lemma 5.6). Finally by distinguishing several cases on the leading coefficient ε_1 and using the criterion of Lemma 5.6 we obtain the results of Theorem 1.5. The case $g = 2$ of Theorem 1.2 will be treated separately.

5.3. **Determination of $L_{\max}$.** For convenience put

$$\varphi_g = \frac{\pi}{2(g+1)}, \quad l_1 = \kappa g/2 \quad \text{and} \quad l_2 = \kappa(g/2+1).$$

To begin with, we calculate the values of $\lambda_l(k)$ and $\eta_l^\varepsilon(k)$ for $l = l_1$ and $l = l_2$. For later reference we include the following useful identity

$$\frac{1 - z^\alpha}{1 - z} = z^{\alpha/2 - 1/2} \frac{\sin(\alpha \arg z/2)}{\sin(\arg z/2)} = z^{\alpha/2-1/2}\, U_{\alpha-1}(\cos(\arg z/2)), \tag{5.8}$$

where $U_{\alpha-1}(x)$ is the Chebyshev polynomial of the second kind of degree $\alpha - 1$.

Lemma 5.1. *It holds*

$$\lambda_l(k) = \begin{cases} (\cot \varphi_g)^k, & k \text{ even, } l \in \{l_1, l_2\} \\ -\mathrm{i}\zeta_{g+1}^{1/2} (\cot \varphi_g)^k, & k \text{ odd, } l = l_1 \\ \mathrm{i}\zeta_{g+1}^{-1/2} (\cot \varphi_g)^k, & k \text{ odd, } l = l_2 \end{cases}$$

$$\eta_l^\varepsilon(k) = \begin{cases} \exp(-\mathrm{i}\theta)\, U_{\varepsilon-1}(\cos \varphi_g), & l = l_1 \\ \exp(\mathrm{i}\theta)\, U_{\varepsilon-1}(\cos \varphi_g), & l = l_2 \end{cases}$$

where $\theta = (\varepsilon - 1) \cdot (-1)^k \varphi_g$.

Proof. Using (5.5) and the fact that $\zeta_q^{lg^{j+2}} = \zeta_q^{lg^j}$ for $l \in \{l_1, l_2\}$ we see that the calculation of $\lambda_l(k)$ reduces to the computation of ζ_q^l and ζ_q^{lg} for $l \in \{l_1, l_2\}$. Moreover, it is easy to verify that $\zeta_q^{l_1} = \zeta_q^{l_2 g}$ and $\zeta_q^{l_2} = \zeta_q^{l_1 g}$ which together with identity (5.8) gives the expressions for $\lambda_l(k)$ and $\eta_l^\varepsilon(k)$. ☐

Note that the eigenvalue $\lambda_{l_1}(s) = \lambda_{l_2}(s) = (\cot \varphi_g)^s > 0$ is an increasing function of g with $\sqrt[s]{\lambda_{l_1}(s)} = \sqrt{3}, 3.077\dots, 4.381\dots, 5.671\dots$ for $g = 2, 4, 6$, respectively.

We include a technical lemma which handles the general multiplier $\eta_l^\varepsilon(0)$ which modifies the eigenvalue $\lambda_l(s)$ via relation (5.6).

Lemma 5.2. *Let* $1 \le \varepsilon \le g-1$, $z = \exp(\mathrm{i}\varphi)$ *and*

$$f_1(\varphi) = \left|\frac{1-z^g}{1+z}\right|, \qquad f_2(\varphi) = \left|\frac{1-z^{g^2}}{1+z^g}\right|.$$

If $f_1(\varphi) > \cot\varphi_g$ *then*

$$f_1(\varphi)f_2(\varphi)\left|\frac{1-(-z)^\varepsilon}{1+z}\right| < (\cot\varphi_g)^2 \cdot \frac{\sin(\varepsilon\varphi_g)}{\sin\varphi_g}.$$

Proof. For $g = 2$ the statement of the lemma is equivalent to the first step of the proof in Lemma 4 in [3]. Assume now $g \ge 4$ and put $J = [\varphi_1, \varphi_2] = [\pi - 2\varphi_g, \pi + 2\varphi_g]$. We split the proof up into several steps.

(1) First we claim that

$$f_1(\varphi) \ge \cot\varphi_g \quad \text{if and only if} \quad \varphi \in J;$$

equality holds if and only if $\varphi = \varphi_1$ or $\varphi = \varphi_2$. To begin with, by using (5.8) we easily note that for $\varphi_1 < \varphi < \varphi_2$ it holds

$$f_1(\varphi) = \left|\frac{\sin(g\varphi/2)}{\cos(\varphi/2)}\right| > \cot\varphi_g.$$

Viceversa, observe that $f_1(\varphi)$ is an oscillating function in φ which is symmetric with respect to $\varphi = \pi$. Moreover, note that its envelope $\mathrm{env}_1(\varphi) = |\cos(\varphi/2)|^{-1}$ is strictly increasing on $[0, \pi]$. Now, put $J' = [\varphi', \pi]$, where $\varphi' = (1-2/g)\pi$ denotes the largest zero of $f_1(\varphi)$ less than $\varphi = \pi$. Then for $g \ge 4$ it holds

$$\max_{\varphi\in[0,\pi]\setminus J'} f_1(\varphi) < |\cos(\varphi'/2)|^{-1} = (\sin(\pi/g))^{-1} < \cot\varphi_g.$$

Furthermore, $f_1(\varphi)$ is strictly increasing on $[\varphi', \varphi_1]$ with $f_1(\varphi_1) = \cot\varphi_g$. This completes the proof of the first step.

(2) By the first step, the investigation can now be focused on the interval J. Let $\mathrm{env}_2(\varphi) = |\cos(g\varphi/2)|^{-1}$ be the envelope of $f_2(\varphi)$. We claim that

$$f_3(\varphi) = f_1(\varphi)\cdot \mathrm{env}_2(\varphi)\cdot \left|\frac{1-(-z)^\varepsilon}{1+z}\right|$$

is strictly decreasing on $[\varphi_1, \pi]$. In equivalent terms, we have to show that

$$\begin{aligned} f_3(\pi - 2\varphi) &= \frac{\sin(\varepsilon\varphi)}{\sin^2\varphi}\cdot\tan(g\varphi) \\ &= \frac{\sin(\varepsilon\varphi)}{\sin\varphi\sqrt{\cos(g\varphi)}}\cdot\frac{\sin(g\varphi)}{\sin\varphi\sqrt{\cos(g\varphi)}} \end{aligned}$$

is strictly increasing on $[0, \varphi_g]$. But this is clear due to the fact that for all $1 \le \varepsilon \le g$ the function

$$\frac{\sin(\varepsilon\varphi)}{\sin\varphi\sqrt{\cos(g\varphi)}}$$

is strictly increasing on $[0,\varphi_g]$. This completes the proof of the second step.

(3) Let $J''=[\varphi'',\pi]$ where $\varphi''=\pi(1-1/g+2/g^2)$ denotes the smallest zero of $f_2(\varphi)$ larger that φ_1. By the second step we have $f_3(\varphi)\le f_3(\varphi'')$ on $[\varphi'',\pi]$. Since

$$f_1(\varphi)f_2(\varphi)\left|\frac{1-(-z)^{\varepsilon}}{1+z}\right|$$

is strictly decreasing on $[\varphi_1,\varphi'']$, it remains to show that

$$f_1(\varphi)f_2(\varphi)\left|\frac{1-(-\exp(\mathrm{i}\varphi_1))^{\varepsilon}}{1+\exp(\mathrm{i}\varphi_1)}\right|=(\cot\varphi_g)^2\cdot\frac{\sin(\varepsilon\varphi_g)}{\sin\varphi_g}>f_3(\varphi''). \tag{5.9}$$

We calculate

$$f_3(\varphi'')=\frac{\cos(\pi/g^2)}{\sin(\pi/g)\cdot\sin^2(\pi(g/2-1)/g^2)}\cdot\sin(\varepsilon\pi(g/2-1)/g^2).$$

Of course,

$$\sin(\varepsilon\pi(g/2-1)/g^2)<\sin(\varepsilon\varphi_g)$$

for $g\ge 2$. Secondly, for $g\ge 6$ we also have

$$\frac{\cos(\pi/g^2)}{\sin(\pi/g)\cdot\sin^2(\pi(g/2-1)/g^2)}<\frac{(\cot\varphi_g)^2}{\sin\varphi_g},$$

which gives (5.9) for $g\ge 6$. For the single case $g=4$, relation (5.9) can be verified by hand. This finishes the proof of the lemma.

□

The following lemma shows that the indices l_1 and l_2 indeed maximize the quantity $|\eta_l^{\varepsilon}(0)|\lambda_l(s)$. The proof uses a set splitting argument as seen in [3, Lemma 4] extended to the general g-case.

Lemma 5.3. *It holds*

$$L_{\max}=\{l_1,l_2\}\,.$$

Proof. Consider

$$\lambda_l(s)=\prod_{j=0}^{s-1}\delta_l(j)\quad\text{with}\quad\delta_l(j)=\frac{1-\zeta_q^{lg^{j+1}}}{1+\zeta_q^{lg^j}}$$

and partition all indices $j\in\{0,1,\dots,s-1\}=M$ into four disjunct sets M_0, M_1, M_2 and M_3 where

$$\begin{aligned}
M_0&=\{j\text{ with }|\delta_l(j)|=\cot\varphi_g\},\\
M_1&=\{j\text{ with }|\delta_l(j)|>\cot\varphi_g\},\\
M_2&=\{j+1\pmod{s}\text{ with }j\in M_1\}\quad\text{and}\\
M_3&=M\setminus(M_0\cup M_1\cup M_2\cup M_3)\,.
\end{aligned}$$

It is clear that either $M_0 = \{\}$ or $M_0 = M$. If $M_0 = \{\}$ then by Lemma 5.2,

$$\begin{aligned} |\eta_l^\varepsilon(0)\lambda_l(s)| &= \left| \frac{1-\left(-\zeta_q^l\right)^\varepsilon}{1+\zeta_q^l} \right| \cdot \prod_{j\in M_1} |\delta_l(j)\delta_l(j+1)| \cdot \prod_{j\in M_3} |\delta_l(j)| \\ &< \frac{\sin(\varepsilon\varphi_g)}{\sin\varphi_g} \cdot (\cot\varphi_g)^{2|M_1|} \cdot (\cot\varphi_g)^{|M_3|} = \frac{\sin(\varepsilon\varphi_g)}{\sin\varphi_g} \cdot (\cot\varphi_g)^s. \end{aligned}$$

The case $M_0 = M$ appears if and only if $l = l_1 = \kappa g/2$ or $l = l_2 = \kappa(g/2+1)$ where

$$|\eta_l^\varepsilon(0)\lambda_l(s)| = \frac{\sin(\varepsilon\varphi_g)}{\sin\varphi_g}(\cot\varphi_g)^s.$$

This completes the proof. □

5.4. **Calculation of the leading term.** By using the formula (5.7) it is now straightforward to calculate the leading term $\bar{S}_{q,i}^{(2,g)}(-1,\varepsilon g^k)$. In what follows let

$$\begin{aligned} \psi_0(g,i,\varepsilon) &:= \sin\left(\varphi_g(2\varepsilon-2i-1)\right) + \sin\left(\varphi_g(2i+1)\right), \\ \psi_1(g,i,\varepsilon) &:= -\cos\left(\varphi_g(2\varepsilon+2i+1)\right) + \cos\left(\varphi_g(2i+1)\right). \end{aligned}$$

Lemma 5.4. *If k is even then*

$$\begin{aligned} \bar{S}_{q,i}^{(2,g)}(-1,\varepsilon g^k) &= \frac{(-1)^i}{q} \cdot \frac{(\cot\varphi_g)^k}{\sin\varphi_g}\ \psi_0(g,i,\varepsilon) \\ &= \frac{2}{q}(-1)^i \frac{(\cot\varphi_g)^k}{\sin\varphi_g}\ \cos\left(\varphi_g(\varepsilon-2i-1)\right)\sin\left(\varepsilon\varphi_g\right). \end{aligned} \tag{5.10}$$

If k is odd then

$$\begin{aligned} \bar{S}_{q,i}^{(2,g)}(-1,\varepsilon g^k) &= \frac{(-1)^i}{q} \cdot \frac{(\cot\varphi_g)^k}{\sin\varphi_g}\ \psi_1(g,i,\varepsilon) \\ &= \frac{2}{q}(-1)^i \frac{(\cot\varphi_g)^k}{\sin\varphi_g}\ \sin\left(\varphi_g(\varepsilon+2i+1)\right)\sin\left(\varepsilon\varphi_g\right). \end{aligned} \tag{5.11}$$

We omit the proof of Lemma 5.4 since we simply use prosthaphaeresis formulas in order to obtain the product forms in (5.10) and (5.11). Observe that the sign of $\bar{S}_{q,i}^{(2,g)}(-1,\varepsilon g^k)$ is basically determined by the parity of i.

Corollary 5.5.

$$\left| \sum_{j=0}^{k-\nu} \bar{S}_{q,i_j}^{(2,g)}(-1,\varepsilon_j g^j) \right| \le \frac{2}{q} \cdot \frac{(\cot\varphi_g)^k}{\sin\varphi_g} \cdot (\cot\varphi_g)^{-\nu} \left(1 - \frac{1}{\cot\varphi_g}\right)^{-1}.$$

Proof. From Lemma 5.4 we get

$$\begin{aligned} \sum_{j=0}^{k-\nu} \left| \bar{S}_{q,i_j}^{(2,g)}(-1,\varepsilon_j g^j) \right| &\le \frac{2}{q} \cdot \frac{1}{\sin\varphi_g} \sum_{j=0}^{k-\nu} (\cot\varphi_g)^j \\ &= \frac{2}{q} \cdot \frac{1}{\sin\varphi_g} \frac{(\cot\varphi_g)^{k-\nu} - 1/\cot\varphi_g}{1 - 1/\cot\varphi_g}. \end{aligned}$$

□

5.5. **Proof of Theorem 1.2.** We can give a more accurate estimate from Lemma 5.4 in the case $g = 2$, namely

$$\left|\bar{S}^{(2,2)}_{q,i_j}(-1, \varepsilon_j 2^j)\right| \leq \frac{2}{q}\left(\cot\varphi_2\right)^j = \frac{2}{q}\cdot 3^{j/2} \quad \text{and}$$

$$\left|\sum_{j=0}^{k-\nu} \bar{S}^{(2,2)}_{q,i_j}(-1, \varepsilon_j 2^j)\right| \leq \frac{2}{q}\cdot 3^{(k-\nu)/2}\left(1 - \frac{1}{\sqrt{3}}\right)^{-1}. \tag{5.12}$$

The estimate (5.12) has been used in the proof of the first part of Theorem 1 in [3]. We include the formula while correcting a minor misprint (see Lemma 5 therein).

Proof of Theorem 1.2. The table below gives the values of $\bar{S}^{(2,2)}_{3\kappa,3\nu+j}(-1, 2^k)$ for $k \geq 2$ calculated from Lemma 5.4:

j	k even	k odd
0	$\frac{2}{q}\sqrt{3}^k$	$\frac{\sqrt{3}}{q}\sqrt{3}^k$
1	$-\frac{1}{q}\sqrt{3}^k$	$-\frac{\sqrt{3}}{q}\sqrt{3}^k$
2	$-\frac{1}{q}\sqrt{3}^k$	0

The first statement of Theorem 1.2 now follows exactly from the lines of the proof of Lemma 5 in [3]. For the second statement we distinguish several cases. First let k be even.

(1) If $n = (\mathbf{100}\ldots)_2$ then

$$\bar{S}^{(2,2)}_{3\kappa,3\nu+1}(n) \leq -\frac{1}{q}\sqrt{3}^k + \frac{2}{\sqrt{3}^3 q}\cdot\frac{\sqrt{3}^k}{1 - \sqrt{3}^{-1}} < 0.$$

(2) If $n = (\mathbf{101}\ldots)_2$ then

$$\bar{S}^{(2,2)}_{3\kappa,3\nu+1}(n) \leq -\frac{1}{q}\sqrt{3}^k - \frac{2}{q}\sqrt{3}^{k-2} + \frac{2}{\sqrt{3}^3 q}\cdot\frac{\sqrt{3}^k}{1 - \sqrt{3}^{-1}} < 0.$$

(3) If $n = (\mathbf{11}\ldots)_2$ then

$$\bar{S}^{(2,2)}_{3\kappa,3\nu+1}(n) \leq -\frac{1}{q}\sqrt{3}^k - \frac{1}{q}\sqrt{3}^k + \frac{2}{\sqrt{3}^2 q}\cdot\frac{\sqrt{3}^k}{1 - \sqrt{3}^{-1}} < 0.$$

If k is odd then we succeed with the same procedure by considering the cases $n = (\mathbf{10}\ldots)_2$, $n = (\mathbf{110}\ldots)_2$ and $n = (\mathbf{111}\ldots)_2$. □

5.6. **Proof of Theorem 1.5.** Let $g \geq 4$. We use the recursive relation (5.4) for the leading term $\bar{S}_{q,i_j}^{(2,g)}(-1,n)$ in order to derive a sufficient criterion for NLP's.

Lemma 5.6. *Let g and i be such that for all $1 \leq \varepsilon_1, \varepsilon_2 \leq g-1$ and $\varepsilon_1 \neq 0$ there hold*

a) $\psi_0(g,i,\varepsilon_1) + (\cot\varphi_g)^{-1}\,\psi_1(g,i-\varepsilon_1,\varepsilon_2) > R(g)$ *and*
b) $\psi_1(g,i,\varepsilon_1) + (\cot\varphi_g)^{-1}\,\psi_0(g,i+\varepsilon_1,\varepsilon_2) > R(g)$,

where

$$R(g) = 2\cdot(\cot\varphi_g)^{-2}\left(1-(\cot\varphi_g)^{-1}\right)^{-1}.$$

Then

(1) *If i is even then $(2,g,q)$ satisfies an $(i,0)$-NLP.*
(2) *If i is odd then $(2,g,q)$ satisfies an $(i,1)$-NLP.*

If ">" is replaced by "<" and "$R(g)$" by "$-R(g)$" in both a) and b) then

(1) *If i is even then $(2,g,q)$ satisfies an $(i,1)$-NLP.*
(2) *If i is odd then $(2,g,q)$ satisfies an $(i,0)$-NLP.*

Proof. Denote $\eta_j \in \{-1,0,1\}$. First, let k be even, then by using Lemma 5.4, Corollary 5.5 and the identity

$$\cos\left(\varphi_g(-2\varepsilon_1 g^k + C)\right) = (-1)^{\varepsilon_1}\cos\left(\varphi_g(-2\varepsilon_1 + C)\right)$$

we have

$$\begin{aligned}
\bar{S}_{q,i}^{(2,g)}(-1,n) &= \bar{S}_{q,i}^{(2,g)}(-1,\varepsilon_1 g^k) + (-1)^{\varepsilon_1}\bar{S}_{q,i-\varepsilon_1 g^k}^{(2,g)}(-1,\varepsilon_2 g^{k-1}) \\
&\quad + \sum_{j=0}^{k-2}\eta_j \bar{S}_{q,i_j}^{(2,g)}(-1,\varepsilon_j g^j) \\
&= \frac{(-1)^i}{q}\cdot\frac{(\cot\varphi_g)^k}{\sin\varphi_g}\Bigg(\sin\left(\varphi_g(2\varepsilon_1-2i-1)\right) + \sin\left(\varphi_g(2i+1)\right) \\
&\quad - \frac{\cos\left(\varphi_g(2\varepsilon_2-2\varepsilon_1+2i+1)\right)}{\cot\varphi_g} + \frac{\cos\left(\varphi_g(2\varepsilon_1-2i-1)\right)}{\cot\varphi_g} \\
&\quad + \frac{\delta}{(\cot\varphi_g)^2}\cdot\left(1-\frac{1}{\cot\varphi_g}\right)^{-1}\Bigg),
\end{aligned}$$

where $|\delta| \leq 2$. This gives the first inequality of Lemma 5.6. Now, let k be odd. Then since

$$\sin\left(\varphi_g(\pm 2\varepsilon_1 g^k + C)\right) = (-1)^{\varepsilon_1}\sin\left(\varphi_g(\mp 2\varepsilon_1 + C)\right)$$

we have

$$\begin{aligned}\bar{S}_{q,i}^{(2,g)}(-1,n) =& \frac{(-1)^i}{q} \cdot \frac{(\cot\varphi_g)^k}{\sin\varphi_g}\Bigg(-\cos\left(\varphi_g(2\varepsilon_1+2i+1)\right) + \cos\left(\varphi_g(2i+1)\right) \\ &+ \frac{\sin\left(\varphi_g(2\varepsilon_2-2\varepsilon_1-2i-1)\right)}{\cot\varphi_g} + \frac{\sin\left(\varphi_g(2\varepsilon_1+2i+1)\right)}{\cot\varphi_g} \\ &+ \frac{\delta}{(\cot\varphi_g)^2}\cdot\left(1-\frac{1}{\cot\varphi_g}\right)^{-1}\Bigg),\end{aligned}$$

where again $|\delta| \le 2$. This yields the second inequality. □

We are now ready to give the proof of Theorem 1.5.

Proof of Theorem 1.5. For convenience put

$$\alpha = \cos\left((2i+1)\varphi_g\right), \qquad \beta = \sin\left((2i+1)\varphi_g\right)$$

and consider the left hand side of inequality $a)$ in Lemma 5.6. Then by using trigonometric addition formulas we have

$$\begin{aligned}\psi_0(g,i,\varepsilon_1)+&(\cot\varphi_g)^{-1}\psi_1(g,i-\varepsilon_1,\varepsilon_2) = \\ &\alpha\left(\sin(2\varepsilon_1\varphi_g) + \frac{\cos(2\varepsilon_1\varphi_g)}{\cot\varphi_g} - \frac{\cos(2(\varepsilon_2-\varepsilon_1)\varphi_g)}{\cot\varphi_g}\right) \\ &+\beta\left(-\cos(2\varepsilon_1\varphi_g)+1+\frac{\sin(2\varepsilon_1\varphi_g)}{\cot\varphi_g}+\frac{\sin(2(\varepsilon_2-\varepsilon_1)\varphi_g)}{\cot\varphi_g}\right) \\ =:&\ \alpha\gamma_1+\beta\gamma_2.\end{aligned}$$

The same calculation for inequality $b)$ in Lemma 5.6 yields

$$\psi_1(g,i,\varepsilon_1) + (\cot\varphi_g)^{-1}\psi_0(g,i+\varepsilon_1,\varepsilon_2) = \alpha\gamma_2+\beta\gamma_1.$$

We distinguish two cases on the leading coefficient ε_1. First let $\varepsilon_1 \le \frac{g}{2}$. Then

$$\begin{aligned}\gamma_1 &\ge \sin(2\varphi_g) + \frac{\cos(2\varphi_g)}{\cot\varphi_g} - \frac{1}{\cot\varphi_g} = 2\sin(2\varphi_g) - 2\tan\varphi_g, \\ \gamma_2 &\ge -\cos(2\varepsilon_1\varphi_g)+1+\frac{\sin(2\varepsilon_1\varphi_g)}{\cot\varphi_g}+\frac{\sin(-2\varepsilon_1\varphi_g)}{\cot\varphi_g} \\ &\ge 1-\cos(2\varphi_g) = 2(\sin\varphi_g)^2.\end{aligned}$$

On the other hand, if $\varepsilon_1 > \frac{g}{2}$ then

$$\begin{aligned}\gamma_1 &\ge \sin((g-1)\varphi_g) + \frac{\cos((g-1)\varphi_g)}{\cot\varphi_g} - \frac{1}{\cot\varphi_g} = 1-\tan\varphi_g, \\ \gamma_2 &\ge 1-\cos((g+2)\varphi_g) + \frac{\sin((g+2)\varphi_g)}{\cot\varphi_g} - \frac{\sin(g\varphi_g)}{\cot\varphi_g} = 1+\sin\varphi_g.\end{aligned}$$

Now, consider the case where $\alpha > 0$ and $\beta > 0$. Since for $x \in [0,1]$ it holds

$$2x(\sin(2\varphi_g)-\tan\varphi_g) + 2\sqrt{1-x^2}(\sin\varphi_g)^2 \ge 2(\sin(2\varphi_g)-\tan\varphi_g) > R(g)$$

and

$$x(1-\tan\varphi_g) + \sqrt{1-x^2}(1+\sin\varphi_g) \ge 1-\tan\varphi_g > R(g)$$

we have that $\alpha\gamma_1+\beta\gamma_2 > R(g)$ and $\alpha\gamma_2+\beta\gamma_1 > R(g)$ is satisfied whenever

$$i \in \bigcup_{\nu} \left[2\nu(g+1), \frac{g}{2}+2\nu(g+1)\right].$$

Now, let $\alpha<0$ and $\beta<0$. We use the same inequalities as before (multiplied by -1) and have $\alpha\gamma_1+\beta\gamma_2 < -R(g)$ and $\alpha\gamma_2+\beta\gamma_1 < -R(g)$. Thus,

$$i \in \bigcup_{\nu} \left[(2\nu+1)(g+1), \frac{g}{2}+(2\nu+1)(g+1)\right].$$

The application of Lemma 5.6 finishes the proof of Theorem 1.5.

□

6. Proof of Theorem 1.7

The idea of the proof is to show that

$$S_{p,0}^{(2,g)}(-1,g^k) = \frac{1}{p}\sum_{l=1}^{p-1}\prod_{j=0}^{k-1}\frac{1-\zeta_p^{lg^{j+1}}}{1+\zeta_p^{lg^j}}$$

is positive for infinitely many k and also negative for infinitely many k. The multiplicative subgroup $U=\{1,g,g^2,\ldots,g^{s-1}\}$ induces a partition of cosets $L_1,L_2,\ldots,L_t$ of the set $\{1,2,\ldots,p-1\}$. As above we define the *eigenvalues*

$$\lambda_l = \prod_{j=0}^{s-1}\frac{1-\zeta_p^{lg^{j+1}}}{1+\zeta_p^{lg^j}}.$$

Since $\lambda_{l_1}=\lambda_{l_2}$ if l_1 and l_2 belong to the same coset L we also use the short hand notation λ_L for λ_l if $l\in L$.

With help of this notations we get proper representations for $S_{p,0}^{(2,g)}(-1,g^{ks})$ and $S_{p,0}^{(2,g)}(-1,g^{ks-2})$ that will be used in the proof of Theorem 1.7:

$$\begin{aligned} S_{p,0}^{(2,g)}(-1,g^{ks}) &= \frac{s}{p}\sum_{r=1}^{t}\lambda_{L_r}^k, \\ S_{p,0}^{(2,g)}(-1,g^{ks-2}) &= \frac{1}{p}\sum_{r=1}^{t}\lambda_{L_r}^k\sum_{l\in L_r}\frac{(1+\zeta_p^l)(1+\zeta_p^{gl})}{(1-\zeta_p^{gl})(1-\zeta_p^{g^2l})}. \end{aligned}$$

In particular we use the following estimates:

Lemma 6.1. *For every r we have $\lambda_{L_r}^4>0$. Hence*

$$S_{p,0}^{(2,g)}(-1,g^{4ks}) > 0. \tag{6.1}$$

Furthermore

$$S_{p,0}^{(2,g)}(-1,g^{4ks-2}) \le \left(c_1 - c_2\frac{\sqrt{p}}{t\log p}\right)\frac{1}{t}\sum_{r=1}^{t}\lambda_{L_r}^{4k} \tag{6.2}$$

for some constants $c_1,c_2>0$ that only depend on g.

Proof. By definition it follows that λ_l is either real or imaginary. Hence $\lambda_l^4 > 0$. Thus, (6.1) follows immediately.

The proof of (6.2) requires several steps. First, we will prove that there are constants c_1, c_2 such that

$$\sum_{l\in L_r} \frac{(1+\zeta_p^l)(1+\zeta_p^{gl})}{(1-\zeta_p^{gl})(1-\zeta_p^{g^2 l})} \le c_1 s - c_2 \sum_{l\in L_r} \frac{p^2}{l^2}. \tag{6.3}$$

For the sake of shortness set

$$T_l = \frac{(1+\zeta_p^l)(1+\zeta_p^{gl})}{(1-\zeta_p^{gl})(1-\zeta_p^{g^2 l})}$$

By elementary calculations we have

$$\arg(T_l) = \frac{l\pi}{p}(1-g^2) + \pi.$$

If $|l \bmod p| \le \eta p$, where $\eta = 1/(4(g^2-1))$, then $|T_l| \gg p^2/l^2$ and consequently

$$\Re(T_l) \le -c_2 \frac{p^2}{l^2}$$

for some constant $c_2 > 0$. On the other hand, if $|l \bmod p| > \eta p$ then $\Re(T_l) \le |T_l| \le c_1$ for another constant $c_1 > 1$. Of course, this directly proves (6.3) (by assuming without loss of generality that $c_2 \le \eta c_1$).

The next step is to use Pólya-Vinogradov inequality (compare with [3] and [12, p. 86, Aufgabe 12 b]) to obtain for all cosets L_r

$$\#\{l \in L_r : |l \bmod p| \le 2tp^{1/2}\log p\} > p^{1/2}\log p.$$

Hence

$$\sum_{l\in L_r} \frac{p^2}{l^2} \ge \frac{p^{3/2}}{4t^2 \log p}$$

and consequently

$$\sum_{l\in L_r} T_l = \sum_{l\in L_r} \Re(T_l) \le c_1 \frac{p}{t} - c_2 \frac{p^{3/2}}{t^2 \log p}$$

which directly gives (6.2). □

We can now prove the first part of Theorem 1.7. If $p \in \mathbb{P}_t$ and $p > Ct^2(\log p)^2$ then we surely have

$$c_1 - c_2 \frac{\sqrt{p}}{t \log p} < 0$$

which shows that $S_{p,0}^{(2,g)}(-1, g^{4ks-2}) < 0$ for all k. Hence, $(2, g, p)$ does not satisfy a $(0,0)$-NLP.

We can also state this observation in the following way.

Lemma 6.2. *Suppose $g \geq 2$ is an even integer and p an odd prime. If $(2, g, p)$ satisfies a $(0,0)$-NLP then*

$$s = \mathrm{ord}_p(g) \leq C p^{1/2} \log p,$$

where $C > 0$ just depends on g.

Now a proper variation of a result of Erdős [5] (compare also with [3]) says:

Lemma 6.3. *For every even integer $g \geq 2$ and every sequence $\varepsilon_p \to 0$ (as $p \to \infty$) we have*

$$\#\{p \leq x : s = \mathrm{ord}_p(g) \leq p^{1/2+\varepsilon_p}\} = o\left(\frac{x}{\log x}\right).$$

Of course, a combination of these two lemmas directly proves the second part of Theorem 1.7. □

References

[1] J. Coquet, *A summation formula related to the binary digits*, Invent. Math. 73(1) (1983), 107–115.

[2] M. Drmota and M. Skałba, *Sign-changes of the Thue-Morse fractal function and Dirichlet L-series*, Manuscripta Math. 86(4) (1995), 519–541.

[3] M. Drmota and M. Skałba, *Rarified sums of the Thue-Morse sequence*, Trans. Amer. Math. Soc. 352(2) (2000), 609–642.

[4] J.-M. Dumont, *Discrépance des progressions arithmétiques dans la suite de Morse*, C. R. Acad. Sci. Paris Sér. I Math. 297(3) (1983), 145–148.

[5] P. Erdős, *Bemerkungen zu einer Aufgabe von Elementen*, Arch. Math. (Basel) 27(2) (1976), 159–163.

[6] S. Goldstein, K. A. Kelly and E. R. Speer, *The fractal structure of rarefied sums of the Thue-Morse sequence*, J. Number Theory 42(1) (1992), 1–19.

[7] P. J. Grabner, *A note on the parity of the sum-of-digits function*, in: Séminaire Lotharingien de Combinatoire (Gerolfingen, 1993), volume 1993/34 of Prépubl. Inst. Rech. Math. Av., Univ. Louis Pasteur, Strasbourg, 1993, 35–42.

[8] P. J. Grabner, T. Herendi and R. F. Tichy, *Fractal digital sums and codes*, Appl. Algebra Engrg. Comm. Comput. 8(1) (1997), 33–39.

[9] H. Leinfellner, *New results on rarefied sums of the Thue-Morse sequence*, in: Beiträge zur zahlentheoretischen Analysis, volume 338 of Grazer Math. Ber., Karl-Franzens-Univ. Graz, Graz, 1999, 9–30.

[10] D. J. Newman, *On the number of binary digits in a multiple of three*, Proc. Amer. Math. Soc. 21 (1969), 719–721.

[11] D. J. Newman and M. Slater, *Binary digit distribution over naturally defined sequences*, Trans. Amer. Math. Soc. 213 (1975), 71–78.

[12] I. M. Vinogradov, *Elemente der Zahlentheorie*, Oldenburg, München, 1956.

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