ON NODES OF GIVEN DEGREE IN RANDOM TREES

MICHAEL DRMOTA
TECHNICAL UNIVERSITY OF VIENNA

ABSTRACT. Let T be a plane rooted tree with n nodes which is regarded as family tree of a Galton-Watson branching process conditioned on the total progeny. We discuss the process $L_n^{(d)}(t)$ which is then number of nodes of degree d in layer t. It is shown that the process $n^{-1/2}L_n^{(d)}(n^{1/2}t)$ converges weakly to Brownian excursion local time. This is done via characteristic functions which are obtained by means of generating functions arising from the combinatorial setup and complex contour integration.

1. INTRODUCTION

Consider a class $\mathcal{A}$ of plane rooted trees. Define for each $T \in \mathcal{A}$ the size $|T|$ by the number of nodes T consists of and a weight

$$\omega(T) = \prod_{k\ge 0} \varphi_k^{n_k(T)}, \tag{1.1}$$

where $(\varphi_k; k \ge 0)$ are non-negative numbers and $n_k(T)$ is the number of nodes $v \in T$ with out-degree k. Furthermore set

$$a_n = \sum_{T:|T|=n} \omega(T).$$

Then the corresponding generating function (GF) $a(z) = \sum_{n\ge 0} a_n z^n$ satisfies the functional equation

$$a(z) = z\varphi(a(z)), \tag{1.2}$$

where $\varphi(t) = \sum_{k\ge 0} \varphi_k t^k$. According to Meir and Moon [13] we will call such a family of trees *simply generated.* Now equip the sets $\mathcal{A}_n = \{T \in \mathcal{A} : |T| = n\}$ with the probability distribution induced by the weight function $\omega(T)$. Then we call each tree $T \in \mathcal{A}$ a random tree.

As Pavlov [14] and Aldous [1] pointed out there is a natural correspondence between simply generated random trees and Galton-Watson branching processes: Let X be a branching process with offspring distribution ξ determined by

$$\mathbf{P}\{\xi = k\} = \frac{\tau^k \varphi_k}{\varphi(\tau)},$$

where τ is an arbitrary nonnegative number within the circle of convergence of $\varphi(t)$. Then the distribution of X conditioned on the total progeny $|X|$ is determined by $\mathbf{P}\{X = T||X| = n\}$ and it is easily seen that this distribution coincides with that induced by (1.1). Furthermore it is obvious to see that there occurs no loss of generality if only critical branching processes are considered (compare with [12]). The condition for a branching process to be critical, $\mathbf{E}\xi = 1$, translated into the "language of trees" is $\tau\varphi'(\tau) = \varphi(\tau)$ and the variance of ξ is given by

$$\sigma^2 = \frac{\tau^2\varphi''(\tau)}{\varphi(\tau)}. \tag{1.3}$$

Date: December 11, 1996.
1991 *Mathematics Subject Classification.* Primary: 60J80, Secondary: 05C05.
Key words and phrases. random trees, branching processes, Brownian excursion, local time.
This research was supported by the Austrian Science Foundation, grant P10187-PHY.

Consider a simply generated tree $T \in \mathcal{A}_n$. We denote by $L_T^{(d)}(k)$ the number of nodes of degree d at distance k from the root (where the distance of two nodes v and w is defined as usual by the number of edges of the path connecting v and w). If T is a random tree then $L_T^{(d)}(k)$ becomes a random variable denoted by $L_n^{(d)}(k)$. For non-integer k we define $L_n^{(d)}(k)$ by linear interpolation:

$$L_n^{(d)}(t) = (\lfloor t\rfloor + 1 - t)L_n^{(d)}(\lfloor t\rfloor) + (t - \lfloor t\rfloor)L_n^{(d)}(\lfloor t\rfloor + 1), \quad t \ge 0.$$

We will show that the scaled process

$$l_n^{(d)}(t) = \frac{1}{\sqrt{n}} L_n^{(d)}(t\sqrt{n}), \quad t \ge 0,$$

weakly converges to Brownian excursion local time as n tends to infinity.

Theorem 1. *Let $\varphi(t)$ be the GF of a family of random trees. Besides, let $W(t)$ denote Brownian excursion of duration 1 and $l(t)$ its local time, i.e.*

$$l(t) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^1 I_{[t,t+\varepsilon]}(W(s))\, ds$$

Assume that $\varphi(t)$ has a positive or infinite radius of convergence R and $g = \gcd\{i|\varphi_i > 0\} = 1$. Furthermore suppose that the equation

$$t\varphi'(t) = \varphi(t)$$

has a minimal positive solution $\tau < R$ and that σ^2 defined by (1.3) is finite. Then the process $l_n^{(d)}(t)$ converges weakly to Brownian excursion local time, exactly that means

$$l_n^{(d)}(t) \xrightarrow{w} c_d \frac{\sigma}{2} l\left(\frac{\sigma}{2} t\right)$$

in $C[0,\infty)$, as $n \to \infty$, in which $c_d = \varphi_{d-1}\tau^{d-1}/\varphi(\tau)$.

Remark 1. It is well known that the expected value of the number of nodes of degree d in trees of size n is approximately $c_d n$. This property is reflected by the processes $l_n^{(d)}(t)$. In [6] it is shown that the process $l_n(t) = \frac{1}{\sqrt{n}} L_n(t\sqrt{n})$ converges to $\frac{\sigma}{2} l\left(\frac{\sigma}{2} t\right)$, where $L_n(k)$ denotes the total number of nodes at distance k to the root (in trees of size n).

Remark 2. The case $g > 1$ can be treated analogously. All limit theorems throughout this paper remain unchanged except that we have to require $n \equiv 1 \mod d$. Thus we may restrict ourselves to $g = 1$.

Remark 3. It is also possible to consider the step function process $\frac{1}{\sqrt{n}} L_n^{(d)}(\lfloor t\sqrt{n}\rfloor)$. The reason that we decided to work with a linear interpolated process instead of a step function process is that the proof of tightness (section 4) is essentially shorter for the first one since all "trajectories" of the process are continuous functions in $\mathrm{C}[0,\infty)$. More precisely, there is a similar tightness condition for the space $\mathrm{D}[0,\infty)$ in which step functions are allowed (see [4]) and in fact, by using direct (but messy) extensions of the method presented in section 4 we are also able to prove the same assertion for the step function process.

Since the distribution of $\sup_{t\ge 0} l(t)$ is the same as that of $2 \sup_{0\le t\le 1} W(t)$ (see [3] or [1]), which is indeed a beta distribution Theorem 1, immediately implies the following property for the maximal width of trees.

Corollary . *Under the assumption of Theorem 1 we have*

$$\sup_{t\ge 0} l_n^{(d)}(t) \xrightarrow{w} c_d \sigma \sup_{0\le t\le 1} W(t)$$

as $n \to \infty$.

2. Plan of the proof

In order to prove Theorem 1 we will use a proper variant of Prohorov's theorem [4, Theorem 12.3]. Thus we have to show that the finite dimensional distributions (fdd's) of $l_n^{(d)}(t)$ converges weakly to those of Brownian excursion local time and that these sequences are tight.

Consider a random tree $T \in \mathcal{A}_n$ and set

$$a_{kmn}^{(d)} = \sum_{T\in\mathcal{A}_n, L_T^{(d)}(k)=m} \omega(T).$$

Thus the distribution of $L_n^{(d)}(k)$ is given by

$$\mathbf{P}\left\{L_n^{(d)}(k) = m | T \in \mathcal{A}_n\right\} = \frac{a_{kmn}^{(d)}}{a_n}.$$

In order to obtain this distribution we use the immediate translation technique of combinatorial constructions into GFs which is widely used in combinatorial enumeration (for a description see e.g. [15]): If we denote vertices by $\circ$, then $\mathcal{A}$ may be described by the symbolic recursion

$$\mathcal{A} = \Psi(\mathcal{A}),$$

where the operator Ψ is defined by

$$\Psi(\mathcal{X}) = \{\circ\} \times (\varphi_0 \cup \varphi_1 \cdot \mathcal{X} \cup \varphi_2 \cdot \mathcal{X} \times \mathcal{X} \cup \cdots)$$

As $\cup$ and $\times$ correspond to sum and product we immediately get the functional equation (1.2).

Now let us mark all nodes of degree d in layer k and denote marked nodes by $\bullet$. Call the tree family obtained from $\mathcal{A}$ in that way $\mathcal{B}^{(d)}$ and its GF $a_k^{(d)}(z,u)$, i.e.

$$a_k^{(d)}(z,u) = \sum_{n,m\geq 0} a_{kmn}^{(d)} u^m z^n.$$

Then it is easy to see that

$$\begin{aligned}\mathcal{B}^{(d)} = \Psi^k\Big(&\varphi_0 \cdot \{\circ\} \cup \varphi_1 \cdot \{\circ\} \times \mathcal{A} \cup \cdots \cup \varphi_{d-2} \cdot \{\circ\} \times \mathcal{A}^{d-2} \\ &\cup \varphi_{d-1} \cdot \{\bullet\} \times \mathcal{A}^{d-1} \cup \varphi_d \cdot \{\circ\} \times \mathcal{A}^d \cup \cdots\Big)\end{aligned}$$

holds (compare also with [8]). Due to the correspondence

$$\begin{aligned}\circ &\leftrightarrow z \\ \bullet &\leftrightarrow uz\end{aligned}$$

we immediately get

$$\begin{aligned}a_k^{(d)}(z,u) &= y_k(z, z\varphi(a(z)) + (u-1)z\varphi_{d-1}a(z)^{d-1}) \\ &= y_k(z, (u-1)z\varphi_{d-1}a(z)^{d-1} + a(z))\end{aligned}$$

where

$$\begin{aligned}y_0(z,u) &= u \\ y_{i+1}(z,u) &= z\varphi(y_i(z,u)), \quad i \geq 0.\end{aligned} \tag{2.1}$$

Thus the distribution of $L_n^{(d)}(k)$ can be obtained by extracting the coefficient of $z^n u^m$ of $a_k^{(d)}(z,u)$ which we denote by $[z^n u^m]a_k^{(d)}(z,u)$. In order to prove Theorem 1 we need only weak limit theorems and thus it suffices to work with characteristic functions. The characteristic function of $\frac{1}{\sqrt{n}}L_n^{(d)}(k)$ is

$$\phi_{kn}^{(d)}(t) = \frac{1}{a_n}[z^n]a_k^{(d)}(z, e^{it/\sqrt{n}}) = \frac{1}{a_n}[z^n]y_k\left(z, z(e^{it/\sqrt{n}} - 1)\varphi_{d-1}a(z)^{d-1} + a(z)\right).$$

In a similar way we get the characteristic function for the higher dimensional distributions: The characteristic function of $\left(\frac{1}{\sqrt{n}}L_n^{(d)}(k_1), \frac{1}{\sqrt{n}}L_n^{(d)}(k_2)\right)$ $(k_1 < k_2)$ is given by

$$\begin{aligned}
&\phi^{(d)}_{k_1k_2n}(t_1,t_2)\\
&\quad= \frac{1}{a_n}[z^n]y_{k_1}\Big(z, z(e^{it_1/\sqrt{n}}-1)\varphi_{d-1}y_{k_2-k_1-1}\Big(z, z(e^{it_2/\sqrt{n}}-1)\varphi_{d-1}a(z)^{d-1}+a(z)\Big)^{d-1}\\
&\qquad\qquad + y_{k_2-k_1}\Big(z, z(e^{it_2/\sqrt{n}}-1)\varphi_{d-1}a(z)^{d-1}+a(z)\Big)\Big).
\end{aligned}$$

and that of $\left(\frac{1}{\sqrt{n}}L_n^{(d)}(k_1), \dots, \frac{1}{\sqrt{n}}L_n^{(d)}(k_p)\right)$ $(k_1 < k_2 < \cdots < k_p)$ by

$$\begin{aligned}
&\phi^{(d)}_{k_1\cdots k_pn}(t_1,\dots,t_p)\\
&= \frac{1}{a_n}[z^n]y_{k_1}\Bigg(z,z\left(e^{it_1/\sqrt{n}}-1\right)\varphi_{d-1}\\
&\qquad \times y_{k_2-k_1-1}\left(z,\cdots+y_{k_p-k_{p-1}}\left(z,z\left(e^{it_p/\sqrt{n}}-1\right)\varphi_{d-1}a(z)^{d-1}+a(z)\right)\dots\right)^{d-1}\\
&\qquad + y_{k_2-k_1}\left(z,\cdots+y_{k_p-k_{p-1}}\left(z,z\left(e^{it_p/\sqrt{n}}-1\right)\varphi_{d-1}a(z)^{d-1}+a(z)\right)\right)\dots\Bigg).
\end{aligned}$$

In order to extract the desired coefficient asymptotically we will use Cauchy's integral formula with a suitably chosen integration contour and approximate the integrand there. Therefore we need a detailed knowledge of the behaviour of the recursion (2.1), see [6, 7]:

Lemma 1. *Let z_0 be the point on the circle of convergence of $a(z)$ which lies on the positive real axis. Set $z = z_0\left(1+\frac{x}{n}\right)$ and $\alpha = z\varphi'(a(z))$. Furthermore assume that $|u-a(z)| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ and $\frac{x}{n}\to 0$ in such a way that $|\arg(-x)| < \pi$ and*

$$\left|1-\sqrt{\frac{-x}{n}}\right| \le 1+\frac{C}{\sqrt{n}}$$

are satisfied. Then $y_k(z,u)$ admits the local representation

$$y_k(z,u) = a(z) + \frac{(u-a(z))\alpha^k}{\frac{\sqrt{-x/n}+\sigma(\tau-u)/\tau\sqrt{2}}{2\sqrt{-x/n}} + \frac{\sqrt{-x/n}-\sigma(\tau-u)/\tau\sqrt{2}}{2\sqrt{-x/n}}\alpha^k + \mathcal{O}\left(\sqrt{\frac{|x|}{n}}\right)},$$

uniformly for $k = \mathcal{O}(\sqrt{n})$.

Remark 4. As $a(z)$ has only non-negative coefficients and $g = \gcd\{i|\varphi_i > 0\} = 1$ the function $a(z)$ has only one singularity on the circle of convergence which is located at z_0. $y_k(z,u)$ is closely related to $a(z)$ and hence it has also exactly one singularity near z_0 the nature of which is described by the above result.

By means of this lemma we will be able to derive the characteristic function of the limiting distribution (see section 3):

Theorem 2. *Let $k_i = \kappa_i\sqrt{n}, i = 1,\dots,k$ where $0 < \kappa_1 < \cdots < \kappa_p$. Then the characteristic function $\phi_{\kappa_1\dots\kappa_p}(t_1,\dots,t_p) = \lim_{n\to\infty}\phi_{k_1\cdots k_pn}(t_1,\dots,t_p)$ of the limiting distribution of $\left(\frac{1}{\sqrt{n}}L_n^{(d)}(k_1), \dots, \frac{1}{\sqrt{n}}L_n^{(d)}(k_p)\right)$ satisfies*

$$\phi_{\kappa_1\dots\kappa_p}(t_1,\dots,t_p) = 1 + \frac{\sigma}{i\sqrt{2\pi}}\int_\gamma f_{\kappa_1,\dots,\kappa_p}(x, c_dt_1,\dots,c_dt_p)e^{-x}\,dx, \tag{2.2}$$

where

$$f_{\kappa_1,\dots,\kappa_p}(x,t_1,\dots,t_p) = \Psi_{\kappa_1}(x,t_1,\Psi_{\kappa_2-\kappa_1}(\dots\Psi_{\kappa_{k-1}-\kappa_{k-2}}(x,t_{k-1},\Psi_{\kappa_p-\kappa_{k-1}}(x,t_p,0))\cdots)$$

with

$$\Psi_\kappa(x,t,y) = \frac{\sqrt{-x}e^{-\kappa\sigma\sqrt{-x/2}}(it+y)}{\sqrt{-x}e^{\kappa\sigma\sqrt{-x/2}} - (it+y)\frac{\sigma}{\sqrt{2}}\sinh\left(\kappa\sigma\sqrt{-\frac{x}{2}}\right)}.$$

and $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ *defined by*

$$\begin{aligned} \gamma_1 &= \left\{s \middle| |s| = 1 \text{ and } \Re s \le 0\right\}, \\ \gamma_2 &= \left\{s \middle| \Im s = 1 \text{ and } \Re s \ge 0\right\}, \\ \gamma_3 &= \overline{\gamma}_2. \end{aligned} \tag{2.3}$$

is a Hankel contour

Remark 5. The name "Hankel contour", "Hankel integral", etc. originate from Hankel's representation of the Gamma function,

$$\frac{1}{2\pi i}\int_\gamma (-s)^{-\alpha}e^{-s}\,ds = \frac{1}{\Gamma(\alpha)},$$

and have become usual due to the quite frequent occurrence of integration contours similar to γ in asymptotical problems in combinatorics.

Remark 6. In [6] it is shown that $\phi_{\kappa_1\ldots\kappa_p}(t_1,\ldots,t_p)$ (given in (2.2)) with $\sigma = 2$ and $c_d = 1$ is exactly the finite dimensional characteristic function of Brownian excursion local time.

Remark 7. Note that by using the method presented in this section the above theorem can only be established for the step function process $\frac{1}{\sqrt{n}}L_n\left(\lfloor t\sqrt{n}\rfloor\right)$. However, in section 4 we will show the inequality

$$\mathbf{P}\left\{\left|L_n\left(\rho\sqrt{n}\right) - L_n\left((\rho+\theta)\sqrt{n}\right)\right| \ge \varepsilon\sqrt{n}\right\} \le C\frac{\theta^2}{\varepsilon^4}$$

for some $C > 0$. This implies

$$\mathbf{P}\left\{\left|L_n^{(d)}(t) - L_n^{(d)}\left(\lfloor\rho\sqrt{n}\rfloor\right)\right| \ge \varepsilon\sqrt{n}\right\} \le \frac{C}{\varepsilon^\beta n} \to 0, \quad \text{as } n \to \infty.$$

Therefore it suffices to prove the theorem for the step function process.

We now turn to the problem of tightness By [4, Theorem 12.3] tightness of $l_n^{(d)}(t) = n^{-1/2}L_n^{(d)}(t\sqrt{n})$, $0 \le t \le A$, follows from tightness of $L_n^{(d)}(0)$ (which is obvioulsy satisfied) and from an estimate of the form

$$\mathbf{P}\left\{|L_n^{(d)}(\rho\sqrt{n}) - L_n^{(d)}((\rho+\theta)\sqrt{n})| \ge \varepsilon\sqrt{n}\right\} \le C\frac{\theta^\alpha}{\varepsilon^\beta} \tag{2.4}$$

for some $\alpha > 1$, $\beta \ge 0$, and $C > 0$ uniformly for $0 \le \rho \le \rho + \theta \le A$. We will derive (2.4) from the following property:

Theorem 3. *There exists a constant $C > 0$ such that*

$$\mathbf{E}\left(L_n^{(d)}(k) - L_n^{(d)}(k+h)\right)^4 \le C\,h^2 n \tag{2.5}$$

holds for all non-negative integers n, k, h.

Obviously Theorem 3 proves (2.4) for $\alpha = 2$ and $\beta = 4$ if $\rho\sqrt{n}$ and $\theta\sqrt{n}$ are non-negative integers. However, it is an easy exercise (see [11] or [10]) that (in the case of linear interpolation) an estimate of type (2.4) can be extended to arbitrary $\rho, \theta \ge 0$ (probably with a different constant C).

Therefore tightness follows from Theorem 3. Since

$$\begin{aligned} \mathbf{P}&\left\{L_n^{(d)}(k) = m, L_n^{(d)}(k+h) = l\right\} \\ &= \frac{1}{a_n}[z^n u^m v^l] y_k(z, z(u-1)\varphi_{d-1}y_{h-1}(z, z(v-1)\varphi_{d-1}a(z)^{d-1} + a(z)) \\ &\qquad + y_h(z, z(v-1)\varphi_{d-1}a(z)^{d-1} + a(z))) \end{aligned}$$

we obtain

$$\begin{aligned}\mathbf{P}\left\{L_n^{(d)}(k) - L_n^{(d)}(k+h) = m\right\} \\ &= \frac{1}{a_n}[z^n u^m] y_k(z,z(u-1)\varphi_{d-1} y_h(z, z(u^{-1}-1)\varphi_{d-1} a(z)^{d-1} + a(z)) \\ &\qquad + y_h(z, z(u^{-1}-1)\varphi_{d-1} a(z)^{d-1} + a(z)))\end{aligned}$$

and consequently

$$\mathbf{E}\left(L_n^{(d)}(k) - L_n^{(d)}(k+h)\right)^4 = \frac{1}{a_n}[z^n] H_{kh}(z), \tag{2.6}$$

in which

$$\begin{aligned}H_{kh}(z) = \Big(\frac{\partial}{\partial u} + 7\frac{\partial^2}{\partial u^2} + 6\frac{\partial^3}{\partial u^3} + \frac{\partial^4}{\partial u^4}\Big) & y_k(z,z(u-1)\varphi_{d-1} y_{h-1}(z, z(u^{-1}-1)\varphi_{d-1} a(z)^{d-1} + a(z)) \\ &+ y_h(z, z(u^{-1}-1)\varphi_{d-1} a(z)^{d-1} + a(z)))\Big|_{u=1}.\end{aligned}$$

Hence, $H_{kh}(z)$ is more or less explicit (see section 4) and a transfer lemma by Flajolet and Odlyzko [9] can be applied in order to prove Theorem 3.

3. Finite dimensional convergence

We will indicate how to prove Theorem 2 in the one and two dimensional case. It will be then clear how the general case can be proved.

First, the characteristic function $\phi_{k,n}^{(d)}(t)$ of $\frac{1}{\sqrt{n}} L_n^{(d)}(k)$ is given by

$$\begin{aligned}\phi_{k,n}^{(d)}(t) &= \frac{1}{a_n}[z^n] y_k(z, z(e^{is/\sqrt{n}}-1)\varphi_{d-1} a(z)^{d-1} + a(z)) \\ &= \frac{1}{a_n}\int_\Gamma y_k\Big(z, z(e^{is/\sqrt{n}}-1)\varphi_{d-1} a(z)^{d-1} + a(z)\Big)\frac{dz}{z^{n+1}}\end{aligned}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ is given by

$$\begin{aligned}\Gamma_1 &= \left\{ z = z_0\left(1+\frac{x}{n}\right) \Big| \Re x \le 0 \text{ and } |x| = 1\right\} \\ \Gamma_2 &= \left\{ z = z_0\left(1+\frac{x}{n}\right) \Big| \Im x = 1 \text{ and } 0 \le \Re x \le \log^2 n\right\} \\ \Gamma_3 &= \overline{\Gamma}_2 \\ \Gamma_4 &= \left\{ z \,\Big|\, |z| = z_0\left|1+\frac{\log^2 n + i}{n}\right| \text{ and } \arg\left(1+\frac{\log^2 n + i}{n}\right) \le |\arg(z)| \le \pi\right\}.\end{aligned} \tag{3.1}$$

The integral over Γ_4 is asymptotically negligable (for details see [6]). Therefore we will concentrate on $\gamma' = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ which is a trucated Hankel contour.

It is well known that $a(z)$ has the local expansion

$$a(z) = \tau - \sqrt{2}\frac{\tau}{\sigma}\sqrt{1-\frac{z}{z_0}} + \cdots.$$

Hence

$$\alpha = z\varphi'(a(z)) = 1 - \sqrt{2}\sigma\sqrt{1-\frac{z}{z_0}} + \cdots$$

and we obtain (by substituting $z = z_0(1+x/n)$)

$$
\begin{aligned}
z(e^{is/\sqrt{n}}-1)\varphi_{d-1}a(z)^{d-1} &= n^{-1/2} z_0 is \varphi_{d-1}\tau^{d-1}\left(1+\mathcal{O}\left(\sqrt{\frac{|x|}{n}}\right)\right),\\
&= n^{-1/2}\tau i s c_d\left(1+\mathcal{O}\left(\sqrt{\frac{|x|}{n}}\right)\right),\\
\tau - a(z) &= \sqrt{2}\frac{\tau}{\sigma}\sqrt{\frac{-x}{n}}\left(1+\mathcal{O}\left(\sqrt{\frac{|x|}{n}}\right)\right),\\
\alpha^k &= \exp\left(-2\frac{k}{\sqrt{n}}\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)\left(1+\mathcal{O}\left(\frac{k|x|}{n}\right)\right),
\end{aligned}
$$

Thus, we can apply Lemma 1 for $z \in \gamma'$ and obtain

$$
y_k(z, z(e^{is/\sqrt{n}}-1)\varphi_{d-1}a(z)^{d-1} + a(z)) = a(z) + R_k,
$$

in which

$$
\begin{aligned}
R_k &= \frac{\dfrac{\tau i s c_d}{\sqrt{n}}\alpha^k + \mathcal{O}\left(\sqrt{\frac{|x|}{n}}\right)}{\dfrac{\sqrt{\frac{-x}{n}} + \frac{\sigma}{\sqrt{2}\tau}\left(\frac{\sqrt{2}\tau}{\sigma}\sqrt{\frac{-x}{n}} - \tau\frac{is}{\sqrt{n}}c_d\right)}{2\sqrt{\frac{-x}{n}}} + \dfrac{\sqrt{\frac{-x}{n}} - \frac{\sigma}{\sqrt{2}\tau}\left(\frac{\sqrt{2}\tau}{\sigma}\sqrt{\frac{-x}{n}} - \tau\frac{is}{\sqrt{n}}c_d\right)}{2\sqrt{\frac{-x}{n}}}\alpha^k + \mathcal{O}\left(\sqrt{\frac{|x|}{n}}\right)}\\
&\sim \frac{\tau}{\sqrt{n}}\frac{\sqrt{-x}\,isc_d\exp\left(-\kappa\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)}{\sqrt{-x}\exp\left(\kappa\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right) - \frac{\sigma}{\sqrt{2}}isc_d\sinh\left(\kappa\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)},
\end{aligned}
$$

where we have set $\kappa = k/\sqrt{n}$. This directly leads to

$$
[z^n]y_k \sim a_n + \frac{\tau z_0^{-n}}{n^{3/2}}\frac{1}{2\pi i}\int_\gamma \frac{\sqrt{-x}\,isc_d\exp\left(-\kappa\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)}{\sqrt{-x}\exp\left(\kappa\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right) - \frac{\sigma}{\sqrt{2}}isc_d\sinh\left(\kappa\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)}\,e^{-x}\,dx,
$$

where we have used the expansion

$$
z^{-n} = z_0^{-n}e^{-x}\left(1+\mathcal{O}\left(\frac{|x|^2}{n}\right)\right).
$$

Since

$$
a_n = [z^n]a(z) = \frac{\tau}{\sigma z_0^n\sqrt{2\pi n^3}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)
$$

Theorem 2 follows in the one dimensional case.

Next let us consider the two dimensional case: Denote by $\phi^{(d)}_{k,k+h,n}(s,t)$ the characteristic function of the joint distribution of $\frac{1}{\sqrt{n}}L_n^{(d)}(k)$ and $\frac{1}{\sqrt{n}}L_n^{(d)}(k+h)$ and set $l = k+h$. Then we have

$$
\begin{aligned}
\phi^{(d)}_{kln}(s,t) &= \frac{1}{a_n}[z^n]y_k\Big(z, z(e^{is/\sqrt{n}}-1)\varphi_{d-1}y_{h-1}\left(z, z(e^{it/\sqrt{n}}-1)\varphi_{d-1}a(z)^{d-1}+a(z)\right)^{d-1}\\
&\qquad + y_h\left(z, z(e^{it/\sqrt{n}}-1)\varphi_{d-1}a(z)^{d-1}+a(z)\right)\Big)\\
&= \frac{1}{a_n}\int_\Gamma y_k\Big(z, z(e^{is/\sqrt{n}}-1)\varphi_{d-1}y_{h-1}\left(z, z(e^{it/\sqrt{n}}-1)\varphi_{d-1}a(z)^{d-1}+a(z)\right)^{d-1}\\
&\qquad + y_h\left(z, z(e^{it/\sqrt{n}}-1)\varphi_{d-1}a(z)^{d-1}+a(z)\right)\Big)\frac{dz}{z^{n+1}}
\end{aligned}
$$

where Γ is the integration path (3.1). For $z \in \gamma'$ we again use the substitution $z = z_0\left(1+\frac{x}{n}\right)$ and setting

$$y_h(z, z(e^{it/\sqrt{n}}-1)\varphi_{d-1}a(z)^{d-1}+a(z)) = a(z) + R_h$$

we obtain as above

$$R_h \sim \frac{\tau}{\sqrt{n}} \frac{itc_d\sqrt{-x}\exp\left(-\eta\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)}{\sqrt{-x}\exp\left(\eta\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right) - itc_d\frac{\sigma}{\sqrt{2}}\sinh\left(\eta\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)}, \tag{3.2}$$

in which $\eta = h/\sqrt{n}$. Similarly we have

$$y_k(z, z(e^{it/\sqrt{n}}-1)\varphi_{d-1}(a(z)+R_{h-1})^{d-1}+a(z)+R_h) = a(z) + R_k,$$

where

$$R_k \sim \frac{\tau}{\sqrt{n}} \frac{\exp\left(-\kappa\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)\left(isc_d\tau + \frac{\sqrt{n}}{\tau}R_h\right)}{\sqrt{-x}\exp\left(\kappa\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right) - \left(isc_d\tau + \frac{\sqrt{n}}{\tau}R_h\right)\frac{\sigma}{\sqrt{2}}\sinh\left(\kappa\frac{\sigma}{\sqrt{2}}\sqrt{-x}\right)} \tag{3.3}$$

and $\kappa = k/\sqrt{n}$. Thus Theorem 2 follows in the two dimensional case. The proof of the general case just requires an interative use of the preceding arguments.

4. Tightness

In this section we will show that the sequence of random variables $l_n^{(d)}(t) = n^{-1/2}L_n^{(d)}(t\sqrt{n})$, $(t \ge 0)$ is tight in $C[0,\infty)$.

Recall that

$$a_n = \frac{\tau}{\sigma z_0^n\sqrt{2\pi n^3}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)$$

Therefore, in view of Theorem 3, it suffices to prove

$$[z^n]H_{kh}(z) = \mathcal{O}\left(h^2 n^{-1/2} z_0^{-n}\right). \tag{4.1}$$

For this purpose we will use the following lemma due to Flajolet and Odlyzko from singularity analysis [9]:

Lemma 2. *Let $F(z)$ is analytic in a region*

$$\Delta = \{z : |z| < z_0 + \eta,\ |\arg(z - z_0)| > \vartheta,$$

in which z_0 and η are positive real numbers and $0 < \vartheta < \pi/2$. Furthermore suppose that there exists a real number $\alpha \notin \{-1, -2, \ldots\}$ such that

$$F(z) = \mathcal{O}\left((1 - z/z_0)^{-\alpha}\right) \qquad (z \in \Delta).$$

Then

$$[z^n]F(z) = \mathcal{O}\left(z_0^{-n}n^{\alpha-1}\right).$$

For example, it follows that if a function $f(z)$ satisfies

$$f(z) = \mathcal{O}\left(\frac{1}{1-\alpha}\right)$$

in Δ, where $\alpha = z\varphi'(a(z))$, then

$$[z^n]f(z) = \mathcal{O}\left(z_0^{-n}n^{-1/2}\right). \tag{4.2}$$

You only have to observe that α has the local expansion

$$\alpha = 1 - \sqrt{2}\sigma\sqrt{1 - \frac{z}{z_0}} + \mathcal{O}\left(1 - \frac{z}{z_0}\right).$$

We will also make use of the fact that

$$\sup_{z\in\Delta}|\alpha| = 1 \tag{4.3}$$

and that

$$\sup_{z\in\Delta}\left|\frac{1-\alpha^h}{1-\alpha}\right| = \sup_{z\in\Delta}\left|\sum_{j=0}^{h-1}\alpha^j\right| = h. \tag{4.4}$$

Now we derive a more or less explicit representation for $H_{kh}(z)$. From this and (4.2)–(4.4) the estimate (4.1) will follow immediately.

Lemma 3. *Set $\alpha = z\varphi'(a(z))$, $\beta = z\varphi''(a(z))$, $\gamma = z\varphi'''(a(z))$, and $\delta = z\varphi''''(a(z))$. Then we have*

$$\begin{aligned}
\frac{\partial y_k}{\partial u}(z,1) &= (z\varphi'(a(z)))^k = \alpha^k,\\
\frac{\partial^2 y_k}{\partial u^2}(z,1) &= \frac{\beta}{\alpha}\alpha^k\frac{1-\alpha^k}{1-\alpha},\\
\frac{\partial^3 y_k}{\partial u^3}(z,1) &= \frac{\gamma}{\alpha}\alpha^k\frac{1-\alpha^{2k}}{1-\alpha^2} + 3\frac{\beta^2}{\alpha}\alpha^k\frac{(1-\alpha^k)(1-\alpha^{k-1})}{(1-\alpha)(1-\alpha^2)},\\
\frac{\partial^4 y_k}{\partial u^4}(z,1) &= \frac{\delta}{\alpha}\alpha^k\frac{1-\alpha^{3k}}{1-\alpha^3}\\
&\quad + \left(2\beta\gamma(2+5\alpha+5\alpha^k+3\alpha^{k+1}) + 3\beta^3/\alpha\right)\alpha^k\frac{(1-\alpha^k)(1-\alpha^{k-1})}{(1-\alpha^2)(1-\alpha^3)}\\
&\quad + 3\beta^3(1+5\alpha)\alpha^k\frac{(1-\alpha^k)(1-\alpha^{k-1})(1-\alpha^{k-2})}{(1-\alpha)(1-\alpha^2)(1-\alpha^3)}.
\end{aligned}$$

Proof From $y_{k+1}(z,u) = z\varphi(y_k(z,u))$ we directly obtain the recurring relations

$$\begin{aligned}
\frac{\partial y_{k+1}}{\partial u} &= z\varphi'(y_k)\frac{\partial y_r}{\partial u},\\
\frac{\partial^2 y_{k+1}}{\partial u^2} &= z\varphi''(y_k)\left(\frac{\partial y_k}{\partial u}\right)^2 + z\varphi'(y_k)\frac{\partial^2 y_k}{\partial u^2},\\
\frac{\partial^3 y_{k+1}}{\partial u^3} &= z\varphi'''(y_k)\left(\frac{\partial y_k}{\partial u}\right)^3 + 3z\varphi''(y_k)\frac{\partial y_k}{\partial u}\frac{\partial^2 y_k}{\partial u^2} + z\varphi'(y_k)\frac{\partial^3 y_k}{\partial u^3},\\
\frac{\partial^4 y_{k+1}}{\partial u^4} &= z\varphi''''(y_k)\left(\frac{\partial y_k}{\partial u}\right)^4 + 6z\varphi'''(y_k)\left(\frac{\partial y_k}{\partial u}\right)^2\frac{\partial^2 y_k}{\partial u^2}\\
&\quad + 3z\varphi''(y_k)\left(\frac{\partial^2 y_k}{\partial u^2}\right)^2 + 4z\varphi''(y_k)\frac{\partial y_k}{\partial u}\frac{\partial^3 y_k}{\partial u^3} + z\varphi'(y_k)\frac{\partial^4 y_k}{\partial u^4}.
\end{aligned}$$

Since $y_k(z,1) = a(z)$ for all $k \ge 0$ this system of recurring relations has the explicit solution for $u = 1$ which are stated in Lemma 3. □

Let

$$\begin{aligned}
h_{kh}(z,u) = y_k(z,&z(u-1)\varphi_{d-1}y_{h-1}(z,z(u^{-1}-1)\varphi_{d-1}a(z)^{d-1}+a(z))\\
&+ y_h(z,z(u^{-1}-1)\varphi_{d-1}a(z)^{d-1}+a(z)))
\end{aligned}$$

and set

$$\begin{aligned}
B = B(z,u) =& z(u-1)\varphi_{d-1}y_{h-1}(z,z(u^{-1}-1)\varphi_{d-1}a(z)^{d-1}+a(z)))\\
&+ y_h(z,z(u^{-1}-1)\varphi_{d-1}a(z)^{d-1}+a(z))).
\end{aligned}$$

Since $H_{kh}(z)$ is a linear combination of

$$\frac{\partial h_{kh}(z,1)}{\partial u},\ \frac{\partial^2 h_{kh}(z,1)}{\partial u^2},\ \frac{\partial^3 h_{kh}(z,1)}{\partial u^3},\ \frac{\partial^4 h_{kh}(z,1)}{\partial u^4}$$

we will concentrate on these functions. First observe that

$$\begin{aligned}
\frac{\partial h_{kh}(z,u)}{\partial u} &= \frac{\partial y_k}{\partial u}(z,u)\frac{\partial B}{\partial u}, \\
\frac{\partial^2 h_{kh}(z,u)}{\partial u^2} &= \frac{\partial^2 y_k}{\partial u^2}(z,u)\left(\frac{\partial B}{\partial u}\right)^2 + \frac{\partial y_k}{\partial u}(z,u)\frac{\partial^2 B}{\partial u^2}, \\
\frac{\partial^3 h_{kh}(z,u)}{\partial u^3} &= \frac{\partial^3 y_k}{\partial u^3}(z,u)\left(\frac{\partial B}{\partial u}\right)^3 + 3\frac{\partial^2 y_k}{\partial u^2}(z,u)\frac{\partial B}{\partial u}\frac{\partial^2 B}{\partial u^2} + \frac{\partial y_k}{\partial u}(z,u)\frac{\partial^3 B}{\partial u^3}, \\
\frac{\partial^4 h_{kh}(z,u)}{\partial u^4} &= \frac{\partial^4 y_k}{\partial u^4}(z,u)\left(\frac{\partial B}{\partial u}\right)^4 + 6\frac{\partial^3 y_k}{\partial u^3}(z,u)\left(\frac{\partial B}{\partial u}\right)^2\frac{\partial^2 B}{\partial u^2} \\
&\quad + 3\frac{\partial^2 y_k}{\partial u^2}(z,u)\left(\frac{\partial^2 B}{\partial u^2}\right)^2 + 4\frac{\partial^2 y_k}{\partial u^2}(z,u)\frac{\partial B}{\partial u}\frac{\partial^3 B}{\partial u^3} + \frac{\partial y_k}{\partial u}(z,u)\frac{\partial^4 B}{\partial u^4}.
\end{aligned}$$

Furthermore, simple calculations provide

$$\begin{aligned}
\frac{\partial B(z,1)}{\partial u} &= z\varphi_{d-1}a(z)^{d-1}(1-\alpha^h) = \mathcal{O}\left(1\right), \\
\frac{\partial^2 B(z,1)}{\partial u^2} &= -2z^2\varphi_{d-1}^2 a(z)^{2d-3}\alpha^{h-1} + 2z\varphi_{d-1}a(z)^{d-1}\alpha^h + z^2\varphi_{d-1}^2 a(z)^{2d-2}\beta\alpha^{h-1}\frac{1-\alpha^h}{1-\alpha} \\
&= \mathcal{O}\left(1\right) + \mathcal{O}\left(\frac{1-\alpha^h}{1-\alpha}\right) = \mathcal{O}\left(h\right), \\
\frac{\partial^3 B(z,1)}{\partial u^3} &= \mathcal{O}\left(1\right) + \mathcal{O}\left(\frac{1-\alpha^h}{1-\alpha}\right) + \mathcal{O}\left(\left(\frac{1-\alpha^h}{1-\alpha}\right)^2\right) = \mathcal{O}\left(h^2\right), \\
\frac{\partial^4 B(z,1)}{\partial u^4} &= \mathcal{O}\left(1\right) + \mathcal{O}\left(\frac{1-\alpha^h}{1-\alpha}\right) + \mathcal{O}\left(\left(\frac{1-\alpha^h}{1-\alpha}\right)^2\right) + \mathcal{O}\left(\left(\frac{1-\alpha^h}{1-\alpha}\right)^3\right) = \mathcal{O}\left(h^3\right).
\end{aligned}$$

Hence, we have uniformly for $z \in \Delta$

$$\begin{aligned}
\frac{\partial h_{kh}(z,1)}{\partial u} &= \mathcal{O}\left(1\right), \\
\frac{\partial^2 h_{kh}(z,1)}{\partial u^2} &= \mathcal{O}\left(\frac{(1-\alpha^k)(1-\alpha^h)}{1-\alpha}\right) + \mathcal{O}\left(h\right) = \mathcal{O}\left(h\right), \\
\frac{\partial^3 h_{kh}(z,1)}{\partial u^3} &= \mathcal{O}\left(\frac{(1-\alpha^h)^3}{(1-\alpha)^2}\right) + \mathcal{O}\left(\frac{1-\alpha^h}{1-\alpha}\,h\right) + \mathcal{O}\left(h^2\right) \\
&= \mathcal{O}\left(h^2\right), \\
\frac{\partial^4 h_{kh}(z,1)}{\partial u^4} &= \mathcal{O}\left(\frac{(1-\alpha^h)^4}{(1-\alpha)^3}\right) + \mathcal{O}\left(\frac{(1-\alpha^h)^2}{(1-\alpha)^2}\,h\right) + \mathcal{O}\left(\frac{h^2}{1-\alpha}\right) + \mathcal{O}\left(\frac{(1-\alpha^h)^4}{(1-\alpha)^3}\right) \\
&= \mathcal{O}\left(\frac{h^2}{1-\alpha}\right),
\end{aligned}$$

which implies

$$\begin{aligned}
[z^n]\frac{\partial h_{kh}(z,1)}{\partial u} &= \mathcal{O}\left(n^{-1}\right) = \mathcal{O}\left(h^2 n^{-1/2}\right), \\
[z^n]\frac{\partial^2 h_{kh}(z,1)}{\partial u^2} &= \mathcal{O}\left(hn^{-1}\right) = \mathcal{O}\left(h^2 n^{-1/2}\right), \\
[z^n]\frac{\partial^3 h_{kh}(z,1)}{\partial u^3} &= \mathcal{O}\left(h^2 n^{-1}\right) = \mathcal{O}\left(h^2 n^{-1/2}\right), \\
[z^n]\frac{\partial^3 h_{kh}(z,1)}{\partial u^3} &= \mathcal{O}\left(h^2 n^{-1/2}\right).
\end{aligned}$$

Thus, (4.1) follows and we have proved tightness.

References

[1] D. J. Aldous, The continuum random tree II: an overview, *Stochastic Analysis,* M. T. Barlow and N. H. Bingham, Eds., Cambridge University Press 1991, 23-70.
[2] D. J. Aldous, The continuum random tree III, *Ann. Prob. 21* (1993), 248-289.
[3] P. Biane and M. Yor, *Valeurs principales associees aux temps locaux Browniens,* Bull. Sci. Math. 111, (1987), 23–101.
[4] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, New York, 1968.
[5] J. W. Cohen and G. Hooghiemstra, Brownian excursion, the $M/M/1$ queue and their occupation times, *Mathematics of Operations Research 6,* 4 (1981), 608-629.
[6] M. Drmota and B. Gittenberger, On the profile of random trees, *Random Structures and Algorithms*, to appear.
[7] M. Drmota and P. Kirschenhofer, On generalized independent subsets of trees, *Random Structures and Algorithms 2* (1991), 187-208.
[8] M. Drmota and M. Soria, Marking in combinatorial constructions: generating functions and limiting distributions, *Theoretical Comp. Science 144* (1995), 67-99.
[9] P. Flajolet and A. M. Odlyzko, Singularity analysis of generating functions, *SIAM J. on Discrete Math. 3,* 2 (1990), 216-240.
[10] B. Gittenberger, On the contour of random trees, *SIAM J. Discrete Math*, submitted.
[11] W. Gutjahr and G. Ch. Pflug, The asymptotic contour process of a binary tree is a Brownian excursion, *Stochastic Processes and their Applications 41* (1992), 69-89.
[12] D. P. Kennedy, The Galton-Watson process conditioned on the total progeny, *J. Appl. Prob. 12* (1975), 800-806.
[13] A. Meir and J. W. Moon, On the Altitude of Nodes in Random Trees, *Canadian Journal of Mathematics 30* (1978), 997-1015.
[14] Pavlov
[15] J. S. Vitter and P. Flajolet, Average-Case Analysis of Algorithms and Data Structures, in *Handbook of Theoretical Computer Science,* J. van Leeuwen, Ed., vol. A: Algorithms and Complexity. North Holland, 1990, ch. 9, pp. 431-524.