A BIVARIATE ASYMPTOTIC EXPANSION OF COEFFICIENTS OF POWERS OF GENERATING FUNCTIONS

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Abstract. The aim of this paper is to give a bivariate asymptotic expansion of the coefficient $y_{nk} = [x^n]y(x)^k$, where $y(x) = \sum y_n x^n$ has a power series expansion with non-negative coefficients $y_n \geq 0$. Such expansions are known for $\frac{k}{n} \in [a,b]$ with $a > 0$. In the first part we provide two versions of full asymptotic series expansions for y_{nk} and in the second part we show how to generalize these expansions to the case $\frac{k}{n} \in [0,b]$ if $y(x)$ has an algebraic singularity of the kind $y(x) = g(x) - h(x)\sqrt{1 - x/x_0}$. A concluding section provides extensions to multivariate asymptotic expansions and applications to multivariate generating functions. As a byproduct we get a remarkable identity for Catalan numbers.

1. Introduction

In many purposes it is interesting to know something about the coefficient

$$y_{nk} = [x^n]y(x)^k, \tag{1.1}$$

where $y(x) = \sum\limits_{n\geq 0} y_n x^n$ is the generating function of a non-negative sequence $y_n \geq 0$. For example, if $y(x)$ is the probability generating function of a discrete random variable X with $\mathbf{P}[X = n] = y_n$ $(n \geq 0)$ then $y(x)^k$ is the probability generating function of the sum $Z = X_1 + \cdots + X_k$, where X_i $(i = 1, \ldots, k)$ are independent identically distributed random variables with $\mathbf{P}[X_i = n] = y_n$. It is well known that asymptotic expansions for y_{nk} can be used to prove the central limit theorem for discrete random variables (see [GK, p 74 ff]). There it suffices to know the behaviour of y_{nk} for $n = \mathbf{E}X \cdot k + \mathcal{O}(k^{1/2+\varepsilon})$. Furthermore y_{nk} has an obvious combinatorial interpretation in the case where $y_n \in \mathbf{Z}$ denotes the number of certain combinatorial objects charaterized by $n \geq 0$ (see [MM][1]).

In section 2 we will recall and generalize these results, where $\frac{n}{k}$ is contained in a fixed interval $[a,b]$ with $a > 0$. The asymptotic main term for y_{nk} is given by

$$\frac{y(\rho)^k}{\rho^n \sqrt{2\pi\sigma^2 k}}, \tag{1.2}$$

where $\rho > 0$ satisfies

$$\frac{n}{k} = \frac{\rho y'(\rho)}{y(\rho)} \tag{1.3}$$

[1]Actually the author was motivated to write this paper by reading [MM] and [Go1].

and σ^2 is calculated by

$$\sigma^2 = \frac{n}{k} - \left(\frac{n}{k}\right)^2 + \frac{\rho^2 y''(\rho)}{y(\rho)}. \tag{1.4}$$

The main purpose is to provide two versions of full asymptotic series expansions for y_{nk}.

It is clear that such asymptotic expansions need not remain true for "small" k because we generally need no regularity assumption for y_n except $y_n \ge 0$. However there are examples, where the main asymptotic term (1.2) is significant for each k.

Let $y(x) = (1 - \sqrt{1-4x})/2$. Then the main asymptotic term is given by

$$\frac{f(\rho)^k}{\rho^n \sqrt{2\pi\sigma^2 k}} = \frac{k}{(2\pi n(2n-k)(n-k))^{\frac{1}{2}}} \frac{(2n-k)^{2n-k}}{(n-k)^{n-k} n^n} \tag{1.5}$$

(which is surely significant for $\frac{k}{n} \in \left[\frac{1}{4}, 1-\varepsilon\right]$). On the other hand we get by Lagrange's inversion formula

$$\begin{aligned} y_{nk} &= [x^n] y(x)^k \\ &= \frac{k}{n} [u^{n-k}] \left(\frac{u}{y^{-1}(u)}\right)^n \\ &= \frac{k}{n} [u^{n-k}] (1-u)^{-n}. \end{aligned} \tag{1.6}$$

Now $\varphi(u) = \frac{u}{y^{-1}(u)} = \frac{1}{1-u}$ has again non-negative coefficients and the asymptoic main term of $\frac{k}{n}[u^{n-k}]\varphi(u)^n$ (which is surely significant for $\frac{k}{n} \in \left[-\frac{1}{2}, \frac{1}{2}\right]$) has exactly the same shape as (1.5). Hence (1.5) is the asymptotic main term for $k \le (1-\varepsilon)n$. Furthermore in this example we can check our considerations directly by using (1.6) and Stirling's formula to get

$$\begin{aligned} y_{nk} &= \frac{k}{n} [u^{n-k}] (1-u)^{-n} \\ &= \frac{k}{n} \binom{2n-k-1}{n-k} \\ &= \frac{k}{(2\pi n(2n-k)(n-k))^{\frac{1}{2}}} \frac{(2n-k)^{2n-k}}{(n-k)^{n-k} n^n} \left(1 + \mathcal{O}\left(\frac{1}{n-k}\right)\right) \end{aligned} \tag{1.7}$$

for $k < n$. In [MM] the relation $y_{nk} = \frac{k}{n}[u^{n-k}]\varphi(u)^n$ has been used to obtain asymptotic expansions by a hidden saddle point method. Furthermore note that $y(x)$ is the solution of the functional equation $y = x\varphi(y)$. It is well known that solutions $y(x)$ of functional equations of the kind $F(x,y) = 0$, satisfying some regularity conditions, usually have an algebraic singularity $x_0 > 0$ on the real axis of the kind

$$y(x) = g(x) - h(x)\sqrt{1 - \frac{x}{x_0}}, \tag{1.8}$$

where $g(x)$ and $h(x)$ are analytic and non-zero around x_0.

In Section 3 we will discuss functions having such an algebraic singularity and will show that the asymptotic main term (1.2) is significant for "small" k in this case, too.

A concluding section provides extensions to multivariate asymptotic expansions and applications to multivariate generating functions.

It should be mentioned that the frame work of asymptotic series expansions for $[x^n]y(x)^k$ was established (probably) essentially by I. J. Good [Go1], [Go2]. But especially in [Go2] only a formal setting is used and no error term resp. a range for n, k is provided. An overview of results concerning $[x^n]y(x)^k$ can be found in [Ga]

2. Two Versions of Asymptotic Expansions

First we state and discuss a little bit modified theorem due to [Go1].

Theorem 1. *Let $y_n \ge 0$ be a sequence of non-negative real numbers such that*

$$d = gcd\{n \,|\, y_n > 0\} = 1 \tag{2.1}$$

and $y(x) = \sum\limits_{n\ge 0} y_n x^n$ be the generating function of y_n with positive radius of convergence $R > 0$. Let $a > \lim\limits_{r\to 0} \frac{ry'(r)}{y(r)}$ and $b < \lim\limits_{r\to R} \frac{ry'(r)}{y(r)}$. Then the coefficient $y_{nk} = [x^n]y(x)^k$ can be approximated by the following asymptotic series expansion ($M \ge 0$)

$$y_{nk} = \frac{y(\rho)^k}{\rho^n\sqrt{2\pi\sigma^2 k}}\left(1 + \sum_{1\le K\le M} \frac{a_K}{k^K} + \mathcal{O}\left(k^{-(M+1)}\right)\right) \quad (k\to\infty) \tag{2.2}$$

uniformly for $\frac{n}{k} \in [a, b]$, where $\rho = \rho_{\frac{n}{k}}$ is (uniquely) defined by

$$\frac{n}{k} = \frac{\rho y'(\rho)}{y(\rho)} \tag{2.3}$$

and $\sigma^2 = \sigma^2(\rho)$ is given by

$$\begin{aligned}\sigma^2 &= \frac{\rho y'(\rho)}{y(\rho)} - \left(\frac{\rho y'(\rho)}{y(\rho)}\right)^2 + \frac{\rho^2 y''(\rho)}{y(\rho)}\\ &= \frac{n}{k} - \left(\frac{n}{k}\right)^2 + \frac{\rho^2 y''(\rho)}{y(\rho)}.\end{aligned} \tag{2.4}$$

Set

$$\kappa_l = \kappa_l(\rho) = \left[\frac{\partial^l}{\partial u^l}\log y(\rho e^u)\right]_{u=0}, \quad (l\ge 1) \tag{2.5}$$

then $a_K = a_K(\rho)$ can be calculated by

$$a_K = \sum_{S\ge 0}(-1)^S \frac{(2S)!}{(2\kappa_2)^S S!} \sum_{\substack{p_3+p_4+\cdots=S-K\\ 3p_3+4p_4+\cdots=2S}} \frac{1}{p_3!}\left(\frac{\kappa_3}{3!}\right)^{p_3} \frac{1}{p_4!}\left(\frac{\kappa_4}{4!}\right)^{p_4}\cdots. \tag{2.6}$$

Remark 1. Note that it is no real restriction that $d = \gcd\{n \,|\, y_n > 0\} = 1$. If $d > 1$ then our theorem applies for $\overline{y}(x) = y(x^{1/d})$. Furthermore the lower bound

$A = \lim_{r\to 0} \frac{ry'(r)}{y(r)}$ for a is not only natural in a formal sense. Moreover A is the zero-order of $y(x)$ at $x = 0$. Hence $y_{nk} = 0$ for $\frac{n}{k} < A$. $\square$

Remark 2. It is clear that $\kappa_l = \kappa_l(\rho)$ are the coefficients of the Thiele expansion

$$y(\rho e^u) = y(\rho) \exp\left(\kappa_1 t + \frac{\kappa_2}{2!} t^2 + \cdots\right). \tag{2.7}$$

Note that $\kappa_1(\rho) = \frac{n}{k}$ and $\kappa_2(\rho) = \sigma^2$. In order to show that $\sigma^2 > 0$ observe that $y_n \geq 0$ implies

$$y(\rho) - |y(\rho e^{it})| \geq ct^2$$

(for some $c > 0$ and sufficiently small t). Hence $\kappa_2 = \sigma^2 > 0$. On the other hand we get

$$\frac{d}{dr}\left(\frac{ry'(r)}{y(r)}\right) = \frac{y'(r)}{y(r)} + \frac{ry''(r)}{y(r)} - \frac{ry'(r)^2}{y(r)^2} = \frac{\kappa_2}{r} > 0,$$

which implies that the mapping $r \to \frac{ry'(r)}{y(r)}$ is strictly increasing. Thus it is clear that $\rho = \rho_{\frac{n}{k}}$ is uniquely defined by (2.3) for $\frac{n}{k} \in [a, b]$. $\square$

Remark 3. Since $a_K = a_K(\rho)$ depends continuously on ρ it is apparent that the asymptotic series expansion (2.2) can be rewritten to

$$y_{nk} \sim \frac{y(\rho)^k}{\rho^n \sqrt{2\pi\sigma^2 k}} \left(1 + \sum_{N \geq 1} \frac{b_N}{n^N}\right) \quad (n \to \infty), \tag{2.8}$$

where $b_N = \kappa_1^N a_N$. For the moment this is only a trivial remark but it will turn out that both asymptotic series expansions, (2.2) and (2.8), can be proven independently. Furthermore the proof of (2.8) can be used to extend the result to "small" k if $y(x)$ has an algebraic singularity (see Section 3). $\square$

Remark 4. As mentioned in the introduction it is sometimes sufficient to know something about y_{nk}, where $n = \mu k + r$ and μ is fixed (or where $k = \overline{\mu} n + \overline{r}$ and $\overline{\mu}$ is fixed). Since (2.2) is a bivariate asymptotic expansion we immediately get by Taylor approximation that the asymptotic main term of y_{nk} must be given by

$$\frac{y(\rho_\mu)^k}{\rho_\mu^n \sqrt{2\pi\sigma^2 k}} \exp\left(-\frac{r^2}{2\sigma^2 k}\right)$$

resp. by

$$\frac{y(\rho_{\overline{\mu}^{-1}})^k}{(\rho_{\overline{\mu}^{-1}})^n \sqrt{2\pi\sigma^2 k}} \exp\left(-\frac{\overline{r}^2}{2\overline{\mu}^3 \sigma^2 n}\right).$$

However, it is possible to get full asymptotic series expansions for y_{nk} in these cases generalizing (2.2) and (2.8). The results are stated in the following Theorem. $\square$

Theorem 2. *Let y_n and $y(x) = \sum_{n\geq 0} y_n x^n$ be as in Theorem 1. Then the coefficient $y_{nk} = [x^n]y(x)^k$ can be expanded by the asymptotic series expansion ($M \geq 0$)*

$$y_{nk} = \frac{y(\rho)^k}{\rho^n\sqrt{2\pi\sigma^2 k}}\left(\exp\left(-\frac{r^2}{2\sigma^2 k}\right)\left(1+\sum_{1\leq L\leq M}\sum_{K-\frac{R}{2}=\frac{L}{2}} a_{K,R}\frac{r^R}{k^K}\right) + \mathcal{O}\left(k^{-\frac{M+1}{2}}\right)\right) \quad (k\to\infty) \tag{2.9}$$

uniformly for all $\mu \in [a,b]$ and all real r such that $n = \mu k + r$ is a positive integer, where $\rho = \rho_\mu$ is defined by

$$\mu = \frac{\rho y'(\rho)}{y(\rho)},$$

$\sigma^2 = \sigma^2(\rho)$ is given by

$$\sigma^2 = \mu - \mu^2 + \frac{\rho^2 y''(\rho)}{y(\rho)},$$

and $a_{K,R}$ can be calculated by

$$a_{K,R} = \sum_{S\geq 0}(-1)^S\frac{(2S+R)!}{2^S R! S! \kappa_2^{S+R}} \cdot \cdot \sum_{\substack{p_3+p_4+\cdots=S+R-K\\ 3p_3+4p_4+\cdots=2S+R}} \frac{1}{p_3!}\left(\frac{\kappa_3}{3!}\right)^{p_3}\frac{1}{p_4!}\left(\frac{\kappa_4}{4!}\right)^{p_4}\cdots. \tag{2.10}$$

($\kappa_l = \kappa_l(\rho)$ is defined in (2.5).) On the other hand we have ($M \geq 0$)

$$y_{nk} = \frac{k y(\rho)^k}{n\rho^n\sqrt{2\pi(-\overline{\kappa}_2)n}}\left(\exp\left(-\frac{\overline{r}^2}{2(-\overline{\kappa}_2)n}\right)\left(1+\sum_{1\leq L\leq M}\sum_{N-\frac{\overline{R}}{2}=\frac{L}{2}} b_{N,\overline{R}}\frac{r^R}{n^N}\right) + \mathcal{O}\left(n^{-\frac{M+1}{2}}\right)\right) \quad (n\to\infty) \tag{2.11}$$

uniformly for all $\overline{\mu} \in \left[\frac{1}{b},\frac{1}{a}\right]$ and all real $\overline{r}$ such that $k = \overline{\mu}n + \overline{r}$ is a positive integer, where $\rho = \rho_{\overline{\mu}^{-1}}$ is defined by

$$\overline{\mu} = \frac{y(\rho)}{\rho y'(\rho)},$$

$\overline{\kappa}_l(\rho)$ ($l \geq 1$) are defined by

$$\sum_{m\geq 1}\frac{\kappa_m(\rho)}{m!}\left(\sum_{l\geq 1}\frac{\overline{\kappa}_l(\rho)}{l!}t^l\right)^m = t, \tag{2.12}$$

and $b_{N,\overline{R}}$ are given by

$$b_{N,\overline{R}} = (-1)^{N+\overline{R}}\sum_{S\geq 0}(-1)^S\frac{(2S+\overline{R})!}{2^S\overline{R}!S!\overline{\kappa}_2^{S+\overline{R}}} \cdot \cdot \sum_{\substack{p_3+p_4+\cdots=S+\overline{R}-K\\ 3p_3+4p_4+\cdots=2S+\overline{R}}} \frac{1}{p_3!}\left(\frac{\overline{\kappa}_3}{3!}\right)^{p_3}\frac{1}{p_4!}\left(\frac{\overline{\kappa}_4}{4!}\right)^{p_4}\cdots. \tag{2.13}$$

Remark 5. Note that $a_{K,R}$ (resp. $b_{N,\overline{R}}$) are only present for $0 \le R \le \frac{3}{2}K$ (resp. for $0 \le \overline{R} \le \frac{3}{2}N$) and that both expansions, (2.9) and (2.11), generalize (2.2) and (2.8). Hence $b_K = b_{K,0} = \kappa_1^K a_{K,0} = \kappa_1^K a_K$ which is not apparent at a first glance. Furthermore if $y^{-1}(w)$ denotes the inverse function of $y(x)$ locally around $x = \rho$ then $\overline{\kappa}_l(\rho)$ can be calculated by

$$\overline{\kappa}_l(\rho) = \left[\frac{\partial^l}{\partial v^l} \log y^{-1}(y(\rho)e^v)\right]_{v=0} \tag{2.14}$$

which is of course equivalent to (2.12). Especially we have $\overline{\kappa}_1 = \kappa_1^{-1}$ and $\overline{\kappa}_2 = -\kappa_2\kappa_1^{-3}$. $\square$

Remark 6. Although the coefficients $a_{K,R}$ (resp. $b_{N,\overline{R}}$) are explicitely given it needs some effort to calculate them term by term. In [Go1]

$$a_1 = \frac{3\lambda_4 - 5\lambda_3^2}{24}$$

and

$$a_2 = \frac{168\lambda_3\lambda_5 + 385\lambda_3^4 - 630\lambda_3^2\lambda_4 - 24\lambda_6 + 105\lambda_4^2}{1152}$$

have been calculated, where λ_l abbreviates κ_l/σ^l. (Compare also with [Da], whereas (2.10) can be found in [GK].) Furthermore it should be noted that (see [Go2] for a formal setting)

$$\sum_{K,R} a_{K,R}\frac{r^R}{k^K} = \sum_{m\ge 0} (-1)^m k^{-\frac{m}{2}} \nu_m(k) H_m(k^{\frac{1}{2}} r),$$

where $H_m(x)$ denotes the m-th Hermite polynomial defined by

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m} e^{-x^2}$$

and $\nu_m(k)$ is given by

$$\sum_{m\ge 0} v_m(k)x^m = \exp\left(k\sum_{l\ge 3}\frac{\kappa_l}{l!}x^l\right)$$

or explicitely by

$$\nu_m(k) = \sum_{S\ge 0} A_{S,m}k^S,$$

where $A_{S,m}$ is defined in (2.16). $\square$

Proof. Since Theorem 1 is a special case of Theorem 2 we only have to prove the latter one. First we will prove (2.9). In this case we can proceed similarly to [GK] by using the usual saddle point method. Let $\mu \in [a,b]$ of equivalently $\rho = \rho_\mu \in [\rho_a, \rho_b]$, where $0 < \rho_a < \rho_b < R$. Then we have by Cauchy's formula

$$\begin{aligned} y_{nk} &= \frac{1}{2\pi i}\int_{|x|=\rho} \frac{y(x)^k}{x^{n+1}}\, dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} y(\rho e^{it})^k \rho^{-n} e^{-int}\, dt. \end{aligned} \tag{2.15}$$

Since $y_n \geq 0$ and $d = \gcd\{n \,|\, y_n > 0\} = 1$ we have $|y(x)| = y(|x|)$ if and only if x is real and non-negative. Furthermore $\kappa_l(\rho)$ is bounded for $\rho \in [\rho_a, \rho_b]$. Thus by continuity there exists some $\delta > 0$ and some α with $0 < \alpha < 1$ such that $|y(\rho e^{it})| \leq \alpha y(\rho)$ uniformly for $\rho \in [\rho_a, \rho_b]$ and $\delta \leq |t| \leq \pi$ and that

$$\left| \sum_{m \geq 3} \frac{\kappa_m(\rho)}{m!} (it)^m \right| < \frac{\sigma^2(\rho) t^2}{6}$$

uniformly for $\rho \in [\rho_a, \rho_b]$ and $|t| \leq \delta$. Hence we get for any positive $\varepsilon < \frac{1}{2}$

$$\begin{aligned}
y_{nk} &= \frac{1}{2\pi} \int_{-\delta}^{\delta} y(\rho e^{it})^k \rho^{-n} e^{-int}\, dt + \mathcal{O}\left(y(\rho)^k \rho^{-n} \alpha^k\right) \\
&= \frac{y(\rho)^k}{2\pi\rho^n} \left(\int_{-\delta}^{\delta} \exp\left(-irt - \frac{\sigma^2 k}{2} t^2 + \sum_{m \geq 3} \frac{\kappa_m(\rho)}{m!} k (it)^m \right) dt + \mathcal{O}\left(\alpha^k\right) \right) \\
&= \frac{y(\rho)^k}{2\pi\rho^n} \left(\frac{1}{\sqrt{\sigma^2 k}} \int_{-k^\varepsilon}^{k^\varepsilon} \exp\left(-i \frac{r}{2\sqrt{\sigma^2 k}} u - \frac{u^2}{2} + \sum_{m \geq 3} \frac{\kappa_m(\rho)}{\sigma^m m!} k^{1-\frac{m}{2}} (iu)^m \right) du \right. \\
&\qquad \left. + \mathcal{O}\left(\delta e^{-k^{2\varepsilon}/6}\right) + \mathcal{O}\left(\alpha^k\right) \right),
\end{aligned}$$

where we have estimated the integrals for $\delta \leq |t| \leq \pi$ and $k^{-1/2+\varepsilon}/\sigma \leq |t| \leq \delta$ and have substituted $u = \sqrt{\sigma^2 k} t$. Set

$$A_{P,Q} = \kappa_2^{-\frac{Q}{2}} \sum_{\substack{p_3 + p_4 + \cdots = P \\ 3p_3 + 4p_e + \cdots = Q}} \frac{1}{p_3!} \left(\frac{\kappa_3}{3!}\right)^{p_3} \frac{1}{p_4!} \left(\frac{\kappa_4}{4!}\right)^{p_4} \cdots . \tag{2.16}$$

Then we have

$$\begin{aligned}
&\exp\left(\sum_{m \geq 3} \frac{\kappa_m(\rho)}{\sigma^m m!} k^{1-\frac{m}{2}} (iu)^m \right) \\
&\qquad = \sum_{\frac{Q}{2} - P \leq \frac{M}{2}} A_{P,Q}\, k^{P - \frac{Q}{2}} (iu)^Q + \mathcal{O}\left(u^M k^{\frac{-}{M+1}2}\right)
\end{aligned}$$

for sufficiently small u. Obviously $\sum_{\frac{Q}{2} - P \leq \frac{M}{2}}$ is a finite sum since $A_{P,Q} = 0$ for $Q > 3P$. Hence we get

$$\begin{aligned}
y_{nk} = \frac{y(\rho)^k}{2\pi\rho^n} &\left(\frac{1}{\sqrt{\sigma^2 k}} \int_{-\infty}^{\infty} \exp\left(-i \frac{r}{2\sqrt{\sigma^2 k}} u - \frac{u^2}{2} \right) \sum_{\frac{Q}{2} - P \leq \frac{M}{2}} A_{P,Q} k^{P - \frac{Q}{2}} (iu)^Q \, du \right. \\
&\qquad \left. + \mathcal{O}\left(k^{-\frac{M+2}{2}}\right) \right).
\end{aligned}$$

Since

$$\int_{-\infty}^{\infty} e^{-i\alpha u-\frac{u^2}{2}} u^L\,du = e^{-\frac{\alpha^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} (v-i\alpha)^L\,dv$$
$$= \sqrt{2\pi} e^{-\frac{\alpha^2}{2}} \sum_{0\le 2l\le L} \binom{L}{2l} \frac{(2l)!}{2^l l!} (-i\alpha)^{L-2l}$$

we immediately get (2.9).

In order to prove (2.11) we will use a different path of integration n Cauchy's formula (2.15). Since $y'(x) > 0$ for real $x > 0$ there exists an analytic function $z(w)$ which equals the inverse function $y^{-1}(w)$ of $y(x)$ locally around any $\rho \in [\rho_a, \rho_b]$. For $-\delta \le t \le \delta$ (where $\delta > 0$ will be chosen sufficiently small) we use the path

$$t \to w\left(y(\rho)e^{it}\right) = \rho \exp\left(\sum_{m\ge 1} \frac{\overline{\kappa}_m}{m!}(it)^m\right)$$

and then the cut circle $|x| = \overline{\rho}$, $\overline{\delta} \le |\arg(x)| \le \pi$, where $w(y(\rho)e^{i\delta}) = \overline{\rho}e^{i\overline{\delta}}$. Since $\overline{\kappa}_2 = -\sigma^2\overline{\mu}^3 < 0$ there exists some $\beta > 1$ such that $|w(\rho e^{it})| \ge \beta\rho$ uniformly for $\rho \in [\rho_a, \rho_b]$. Furthermore we can assume (by choosing $\delta > 0$ sufficiently small) that

$$\max_{\overline{\delta}\le t\le \pi} \left|y(\overline{\rho}e^{it})\right| = y(\overline{\rho}e^{i\overline{\delta}}).$$

Hence we get

$$y_{nk} = \frac{k}{2\pi n} \int_{-\delta}^{\delta} \frac{y(\rho)^k}{\rho^n} \exp\left(i\overline{r}t + \frac{\overline{\kappa}_2}{2}nt^2 - \sum_{m\ge 3} \frac{\overline{\kappa}_m}{m!} n(it)^m\right) dt$$
$$+ \mathcal{O}\left(\frac{y(\rho)^k}{\rho^n\beta^n}\right),$$

where we have used

$$\int \frac{y(x)^k}{x^{n+1}}\,dx = \frac{1}{n}\int \frac{y(x)^{k-1}}{x^n} y'(x)\,dx - \frac{y(x)^k}{nx^n}$$
$$= \frac{k}{n}\int \frac{w^{k-1}}{z(w)^n}\,dw - \frac{y(x)^k}{nx^n}.$$

So we can proceed as above to prove (2.11). □

3. Functions with algebraic singularities

Let $y(x)$ satisfy a functional equation $F(x,y) = 0$ such that

$$F(x_0, y_0) = 0$$
$$F_y(x_0, y_0) = 0$$
$$F_{yy}(x_0, y_0) \ne 0$$
$$F_x(x_0, y_0) \ne 0,$$

where $y_0 = y(x_0)$ and (x_0, y_0) is a regular point of the analytic function $F(x,y)$. Then by Weierstrass' preparation theorem it follows that $y(x)$ has a local representation of the form

$$y(x) = g(x) - h(x)\sqrt{1 - \frac{x}{x_0}}, \tag{3.1}$$

where $g(x)$ and $h(x)$ are analytic around x_0 and satisfy $g(x_0) = y_0$ and $h(x_0) = \left(\frac{x_0 F_x}{2F_{yy}}\right)^{1/2}$. Such functional equations quite often appear by transforming recurrent relations for a sequence y_n to relations for the generating function $y(x) = \sum_{n\geq 0} y_n x^n$. For example, let $F(x,y) = y - \frac{1}{1-y}$. Then $y_n = [x^n]y(x) = [x^n]\frac{1}{2}(1 - \sqrt{1-4x})$ denotes the number of planted plane rooted trees of size n and $y_{nk} = [x^n]y(x)^k$ the number of forests of size n with k components. As indicated in the introduction in this case it is possible to get a bivariate asymptotic expansion for y_{nk} which works even for "small" k, i.e. there is a uniform expansion valid for $\frac{k}{n} \leq 1-\varepsilon$ and not only for $\varepsilon \leq \frac{k}{n} \leq 1-\varepsilon$. In this section we will show how to obtain this result generally for functions $y(x)$ with an algebraic singularity of the kind (3.1).

Theorem 3. *Let $y_n \geq 0$ and $y(x) = \sum_{n\geq 0} y_n x^n$ be as in Theorem 1 such that there is only one singularity on the circle of convergence $|x| = x_0 > 0$ and that $y(x)$ can be locally represented as*

$$y(x) = g(x) - h(x)\sqrt{1 - \frac{x}{x_0}}, \tag{3.2}$$

where $g(x)$ and $h(x)$ are analytic functions around x_0 which are real and positive for real x. Furthermore suppose that $y(x)$ can be continued analytically to $|x| < x_0 + \delta$, $x \notin [x_0, x_0+\delta)$ (for some $\delta > 0$). Then the asymptotic series expansion (2.11) for $y_{nk} = [x^n]y(x)^k$ holds uniformly for $\overline{\mu} \in \left[0, \frac{1}{a}\right]$ and all real $\overline{r}$ such that $k = \overline{\mu}n + \overline{r}$ is a positive integer.

Remark 7. Note that the case where $\overline{\mu} = 0$ and $k = \overline{r} \geq 1$ is fixed can be treated by Darboux's theorem [Db], too. Hence Theorem 3 generalizes Darboux's theorem in some sense. □

Proof. Since $y(x)$ satisfies the relation $(g(x) - y)^2 = h(x)\left(1 - \frac{x}{x_0}\right)$ it follows by the implicit function theorem that there exists an "inverse" function $z(w)$ of $y(x)$, which is analytic around $y(x_0)$ and satisfies $z'(y(x_0)) = 0$ and $z''(y(x_0)) \neq 0$. This implies that we can define $\overline{\kappa}_l(\rho)$ even for $\rho = x_0$ by

$$\overline{\kappa}_l(x_0) = \left[\frac{\partial^l}{\partial v^l}\log w(y(x_0 e^v))\right]_{v=0}.$$

Furthermore it shows that $\overline{\kappa}_l(\rho)$ is continuous at $\rho = x_0$. Hence we can consider the integral

$$\frac{k}{2\pi n}\int_{\substack{w=y(\rho)e^{it}\\ |t|\leq n^{-\frac{1}{2}+\varepsilon}}} \frac{w^{k-1}}{z(w)^n}\,dw$$

$$= \frac{k}{2\pi n}\int_{-n^{-\frac{1}{2}+\varepsilon}}^{n^{-\frac{1}{2}+\varepsilon}} \frac{y(\rho)^k}{\rho^n}\exp\left(i\overline{r}t + \frac{\overline{\kappa}_2}{2}nt^2 - \sum_{m\geq 3}\frac{\overline{\kappa}_m}{m!}n(it)^m\right)dt$$

and can derive an asymptotic series expansion of the kind (2.11) which is uniform for $\rho \in [x_0-\varepsilon, x_0]$. Furthermore we have

$$y_{nk} = \frac{k}{2\pi n} \int\limits_{\substack{w=y(\rho)e^{it}\\ |t|\le n^{-\frac12+\varepsilon}}} \frac{w^{k-1}}{z(w)^n}\,dw$$

$$- \frac{1}{2\pi n}\,\frac{y(x)^k}{x^n}\Bigg|_{x=\rho' e^{-i\delta'}}^{x=\rho' e^{i\delta'}} \tag{3.3}$$

$$+ \frac{1}{2\pi i} \int\limits_{\substack{|x|=\rho'\\ \delta'\le \arg(x)\le\pi}} \frac{y(x)^k}{x^{n+1}}\,dx, \tag{3.4}$$

where $\rho' = |z(y(\rho)e^{in^{1/2+\varepsilon}}|$ and $\delta' = \arg(z(y(\rho)e^{in^{1/2+\varepsilon}})$. First observe that $\rho' \ge \rho(1+cn^{2\varepsilon-1})$ (for some $c>0$) uniformly for $\rho\in[x_0-\varepsilon,x_0]$. Hence we can estimate (3.3) by

$$\mathcal{O}\left(\frac{y(\rho)^k}{n\rho^n} e^{-c'n^{2\varepsilon}}\right)$$

uniformly for $\rho \in [x_0-\varepsilon, x_0]$. In order to estimate (3.4) it would be useful verify a condition of the kind

$$\max_{\substack{|x|=\rho'\\ \delta'\le|\arg(x)|\le\pi}} |y(x)| = |y(\rho' e^{i\delta'})| \tag{3.5}$$

for all $\rho\in[x_0-\varepsilon,x_0]$ (where $\varepsilon>0$ is chosen appropriately). In fact this can be managed by the following observations.

First the representation (3.2) ensures that there exists some $\delta_1>0$ such that the mapping $t\to|y(re^{it})|$ $(0<t<\delta_1)$ strictly decreases for $r\in[x_0-\delta_1, x_0+\delta_1]$. Since $y_n\ge 0$ and $d=\gcd\{n\,|\,y_n>0\}=1$ there exists some $\delta_2<\delta_1$ such that

$$\max_{\substack{|x|=r\\ \delta_2\le|\arg(x)|\le\pi}} |y(x)| = \left|y(re^{i\delta_2})\right|$$

for all $r\le x_0$. Hence by continuity there are $\delta_3,\delta_4>0$ with $\delta_3<\delta_1$ such that

$$\max_{\substack{|x|\le x_0+\delta_3\\ \delta_2\le|\arg(x)|\le\pi}} |y(x)| \le y(x_0)(1-\delta_4).$$

Now choose $\delta_5<\delta_2$ such that

$$\max_{x_0\le r\le x_0+\varepsilon} \left|y(re^{i\delta_5})\right| > y(x_0)(1-\delta_4).$$

Thus we get (3.5) if $\rho'<x_0+\delta_3$ and $\delta'<\delta_5$ which is surely the case for sufficiently small but fixed $\varepsilon>0$ and $n\ge n_0(\varepsilon)$. This proves (2.11) for $\overline{\mu}\in[0,\varepsilon']$ for some $\varepsilon'>0$ whereas the case $\overline{\mu}\ge\varepsilon'$ is covered by Theorem 2. So we are done. $\square$

It should be noted that there exists a much simpler proof for the case $\overline{\mu}=0$ when you are not interested in the full asymptotic series expansion.

Theorem 4. *Let $y_n \ge 0$ and $y(x)$ as in Theorem 3. Then we have*

$$y_{nk} = \frac{kg(x_0)^{k-1}h(x_0)}{2n^{\frac{3}{2}}\sqrt{\pi}x_0^n}\left(\exp\left(-\frac{k^2}{4n}\left(\frac{h(x_0)}{g(x_0)}\right)^2\right) + \mathcal{O}\left(n^{-\frac{1}{2}}\right)\right) \tag{3.6}$$

uniformly for $k \le C\frac{n}{\log^2 n}$, where $C > 0$ is sufficiently small.

Remark 8. Note that this theorem does not provide an asymptotic formula for all k. (In Theorem 3 the range for $\overline{r} = k$ is not restricted.) But this is no real restriction since the asymptotic main term is relevant just for $k = \mathcal{O}(\sqrt{n})$ □

Proof. For simplicity we will assume that the radius of convergence $x_0 = 1$. We will use Cauchy's formula

$$y_{nk} = \int_\gamma \frac{y(x)^k}{x^{n+1}}\,dx,$$

where the path of integration $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ consists of four parts:

$$\begin{aligned}
\gamma_1 &= \left\{ x = 1 + \frac{t}{n} \,\middle|\, |t| = 1, \Re t \le 0 \right\}\\
\gamma_2 &= \left\{ x = 1 + \frac{t}{n} \,\middle|\, 0 < \Re t \le \log^2 n, \Im t = 1 \right\}\\
\gamma_3 &= \overline{\gamma_2}\\
\gamma_4 &= \left\{ x \,\middle|\, |x| = \left|1 + \frac{\log^2 n + i}{n}\right|, \arg(1 + \frac{\log^2 n + i}{n}) \le |\arg(x)| \le \pi \right\}.
\end{aligned}$$

Let $k \le C\frac{n}{\log^2 n}$. Then we have for $x \in \gamma_1 \cup \gamma_2 \cup \gamma_3$

$$\begin{aligned}
y(x)^k x^{-(n+1)} &= g(x)^k\left(1 - \frac{h(x)}{g(x)}\sqrt{1-x}\right)^k x^{-(n+1)}\\
&= g(1)^k \exp\left(-\frac{k}{\sqrt{n}}\frac{h(1)}{g(1)}(-t)^{\frac{1}{2}} - t\right)\cdot\\
&\quad\cdot\left(1 + \mathcal{O}\left(\frac{|t|^2}{n}\right) + \mathcal{O}\left(\frac{k|t|}{n}\right) + \mathcal{O}\left(k\frac{|t|^{\frac{3}{2}}}{n^{\frac{3}{2}}}\right)\right).
\end{aligned}$$

Let $\lambda = \frac{k}{\sqrt{n}}\frac{h(1)}{g(1)}$. Then it is an easy exercise to obtain

$$\int_{\gamma'} e^{-\lambda\sqrt{-t}-t}\,dt = \frac{\lambda}{2\sqrt{\pi}}e^{-\frac{\lambda^2}{2}} + \mathcal{O}\left(e^{-\log^2 n}\right),$$

where $\gamma' = \{t\,|\,|t| = 1, \Re t \le 0\} \cup \{t\,|\,0 < \Re t \le \log^2 n, \Im t = \pm 1\}$, by using the substitution $u^2 = t$. Furthermore we have

$$\int_{\gamma'} \left|e^{-\lambda\sqrt{-t}-t}t^L\right|\,dt = \mathcal{O}_L(1)$$

uniformly for $\lambda \ge 0$ since $\Re\sqrt{-t} > 0$ for $t \in \gamma'$, and

$$\int_{\gamma_4} \frac{y(x)^k}{x^{n+1}}\,dx = \mathcal{O}\left(g(1)^k e^{-\left(1-\frac{k}{n}\frac{h(1)}{g(1)}\right)\log^2 n}\right)$$

since we can assume that

$$\sup_{|x|=r, x\neq r} |y(x)| = \lim_{t\to 0} \left|y(re^{it})\right|.$$

for $1 \le r \le 1+\delta$ (where $\delta > 0$ is chosen sufficiently small). Hence we get (3.6) uniformly for $k \le C\frac{n}{\log^2 n}$. □

As a byproduct of our asymptotic series expansions (2.9) and (2.11) applied to functions of the form (3.2) we get a remarkable identity for Catalan numbers

$$C_n = \frac{1}{n}\binom{2n-2}{n-1}. \tag{3.7}$$

Corollary. *For $K \ge 1$ we have*

$$\sum_{S\ge 1} (-1)^S \frac{(2S)!}{2^{2S} S!} \sum_{\substack{p_3+p_4+\cdots = S-K \\ 3p_3+4p_4+\cdots = 2S}} \frac{1}{p_3!} C_3^{p_3} \frac{1}{p_4!} C_4^{p_4} \cdots = 0. \tag{3.8}$$

Proof. Let $y(x) = 1-\sqrt{1-x}$. From $a_K(\rho) = \kappa_1(\rho)^{-K} b_K(\rho)$ and $\lim_{\rho\to 1} b_K(\rho) = b_K(1)$ it follows that

$$\lim_{\rho\to 1} a_K(\rho) = 0.$$

Now

$$\kappa_l(\rho) \sim 2^{1-2l} \frac{(2l-2)!}{(l-1)!} (1-\rho)^{\frac{1}{2}-l} \qquad (\rho \to 1)$$

implies

$$\kappa_2^{-\frac{S}{2}} \frac{1}{p_3!}\left(\frac{\kappa_3}{3!}\right)^{p_3} \frac{1}{p_4!}\left(\frac{\kappa_4}{4!}\right)^{p_4} \cdots \sim \frac{1}{2^{S+K}} (1-\rho)^{-\frac{K}{2}} \frac{1}{p_3!} C_3^{p_3} \frac{1}{p_4!} C_4^{p_4} \cdots \qquad (\rho \to 1)$$

for $p_3+p_4+\cdots = S-K$ and $3p_3+4p_4+\cdots = 2S$. So (3.8) follows from

$$\begin{aligned} 0 &= \lim_{\rho\to 1} a_K(\rho) \\ &\sim \lim_{\rho\to 1} (1-\rho)^{-\frac{K}{2}} \sum_{S\ge 0} (-1)^S \frac{(2S)!}{2^{2S+K} S!} \cdot \\ &\qquad \cdot \sum_{\substack{p_3+p_4+\cdots = S-K \\ 3p_3+4p_4+\cdots = 2S}} \frac{1}{p_3!} C_3^{p_3} \frac{1}{p_4!} C_4^{p_4} \cdots . \end{aligned}$$

□

4. Extensions and Generalizations

There are several possibilities to extend or generalize the asymptotic series expansions obtained above. For example in [Go1] an asymptotic expansion for $[x_1^{n_1}\cdots x_m^{n_m}]y(x_1,\ldots,x_m)^k$ that generalizes (2.2) is proved. It is clear that we can extend this result to the case $n_l = \mu_l k + r_l$ $(1 \le l \le m)$ in order to generalize (2.9) (compare also with [BR] and [Go2], where a formal setting without error terms has been established). We will use the notations $\mathbf{n} = (n_1,\ldots,n_m)$ for an m-dimensional multiindex of non-negative integers, $|\mathbf{n}|$ for the sum $n_1+\cdots+n_m$, $\mathbf{n}!$ for $n_1!n_2!\cdots n_m!$, and $\mathbf{x} = (x_1,\ldots,x_m)$ for an m-dimensional complex variable and the abbreviation $\mathbf{x}^{\mathbf{n}} = x_1^{n_1}\cdots x_m^{n_m}$.

Theorem 5. *Let $m \ge 1$, $\mathbf{n}$ an m-dimensional multiindex, and $y_{\mathbf{n}} \ge 0$ an (m-fold) sequence of non-negative real numbers such that almost all multiindices $\mathbf{n}$ can be represented as a finite linear combination of the set $\{\mathbf{n}\,|\,y_{\mathbf{n}}>0\}$ with positive integers as coefficients. Suppose that the generating function $y(x)=\sum_{\mathbf{n}} y_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}$ represents an analytic function around $\mathbf{x}=\mathbf{0}$ and let A be a compact subset of the region of convergence of $y(\mathbf{x})$ containing only points with real positive components. Then the coefficient $y_{\mathbf{n}k} = [\mathbf{x}^{\mathbf{n}}]y(\mathbf{x})^k$ can be approximated by the following asymptotic series expansion ($M \ge 0$)*

$$y_{\mathbf{n}k} = \frac{y(\rho)^k}{\rho^{\mathbf{n}}\sqrt{(2\pi)^m|\Sigma(\rho)|k}}\left(\exp\left(-\frac{1}{2k}\mathbf{r}\Sigma(\rho)^{-1}\mathbf{r}'\right)\cdot\right.$$
$$\cdot\left(1+\sum_{1\le L\le M}\ \sum_{K-\frac{S}{2}=\frac{L}{2}}\ \sum_{|\mathbf{R}|=S} a_{K,\mathbf{R}}\frac{\mathbf{r}^{\mathbf{R}}}{k^K}\right) \tag{4.1}$$
$$\left.+\mathcal{O}\left(k^{-\frac{M+1}{2}}\right)\right)\qquad (k\to\infty)$$

uniformly for all μ with $\rho_\mu \in A$ and all $\mathbf{r}$ with real components such that $\mathbf{n}=k\mu+\mathbf{r}$ is a multiindex, where $\rho=\rho_\mu=(\rho_1,\ldots,\rho_m)_{(\mu_1,\ldots,\mu_m)}$ is defined by

$$\mu_j = \frac{\rho_j y_{x_j}(\rho)}{y(\rho)}\qquad (j=1,\ldots,m) \tag{4.2}$$

and the $m\times m$-matrix $\Sigma=\Sigma(\rho)=(s_{ij}(\rho)_{1\le i,j\le m})$ is given by

$$s_{ij}(\rho) = \left[\frac{\partial^2}{\partial u_i\partial u_j}y(\rho_1e^{u_1},\ldots,\rho_me^{u_m})\right]_{\mathbf{u}=\mathbf{1}}.$$

Generally let $\kappa_{\mathbf{l}}(\rho)$ be defined by

$$\kappa_{\mathbf{l}}(\rho) = \left[\frac{\partial^{|\mathbf{l}|}}{\partial\mathbf{u}^{\mathbf{l}}}y(\rho_1e^{u_1},\ldots,\rho_me^{u_m})\right]_{\mathbf{u}=\mathbf{1}} \tag{4.3}$$

then $a_{K,\mathbf{R}}$ can be expressed formally by

$$\sum_{K,\mathbf{R}} a_{K,\mathbf{R}}\frac{\mathbf{r}^{\mathbf{R}}}{k^K} = \sum_{\mathbf{m}}(-1)^{\mathbf{m}}k^{-\frac{|\mathbf{m}|}{2}}\nu_{\mathbf{m}}(k)H_{\mathbf{m}}(k^{\frac{1}{2}}\mathbf{r},\Sigma^{-1}), \tag{4.4}$$

where $H_{\mathbf{m}}(\mathbf{x},\mathbf{C})$ denotes a generalized Hermite polynomial defined by

$$H_{\mathbf{m}}(\mathbf{x},\mathbf{C}) = (-1)^{\mathbf{m}} \exp\left(\frac{1}{2}\mathbf{x}\mathbf{C}\mathbf{x}'\right) \frac{\partial^{\mathbf{m}}}{\partial \mathbf{x}^{\mathbf{m}}} \exp\left(-\frac{1}{2}\mathbf{x}\mathbf{C}\mathbf{x}'\right)$$

and $\nu_{\mathbf{m}}(k)$ is given by

$$\sum_{\mathbf{m}} v_{\mathbf{m}}(k)\mathbf{x}^{\mathbf{m}} = \exp\left(k \sum_{|\mathbf{l}|\ge 3} \frac{\kappa_{\mathbf{l}}}{\mathbf{l}!}\mathbf{x}^{\mathbf{l}}\right)$$

or explicitely by

$$\nu_{\mathbf{m}}(k) = \sum_{S\ge 0} A_{S,\mathbf{m}} k^S,$$

where $A_{S,\mathbf{m}}$ is defined by

$$A_{S,\mathbf{m}} = \sum_{\substack{|\mathbf{l}_3|=3,|\mathbf{l}_4=4|,\dots \\ |\mathbf{p}_{\mathbf{l}_3}|+|\mathbf{p}_{\mathbf{l}_4}|+\cdots=S \\ 3\mathbf{p}_{\mathbf{l}_3}+4\mathbf{p}_{\mathbf{l}_4}+\cdots=\mathbf{m}}} \frac{1}{\mathbf{p}_{\mathbf{l}_3}!}\left(\frac{\kappa_{\mathbf{l}_3}}{3!}\right)^{\mathbf{p}_{\mathbf{l}_3}} \frac{1}{\mathbf{p}_{\mathbf{l}_4}!}\left(\frac{\kappa_{\mathbf{l}_4}}{4!}\right)^{\mathbf{p}_{\mathbf{l}_4}} \cdots .$$

Remark 9. Although the representation for $a_{K,\mathbf{R}}$ (due to [Go2]) is explicit, more detailed formulas are rather involved. For example, I. J. Good [Go1] has evaluated $a_{1,00}$ to

$$\begin{aligned} a_{1,00} = \frac{1}{24}(&3\lambda_{40} - 12\lambda_{31}\alpha + 6\lambda_{22}(1+2\alpha^2) - 12\lambda_{13}\alpha \\ &+ 3\lambda_{04} - 5\lambda_{30}^2 - 9\lambda_{21}^2(1+4\alpha^2) - 9\lambda_{12}^2(1+4\alpha^2) - 5\lambda_{03}^2 \\ &+ 30\lambda_{30}\lambda_{21}\alpha - 6\lambda_{30}\lambda_{12}(1+4\alpha^2) + 2\lambda_{30}\lambda_{03}\alpha(3+2\alpha^2) \\ &+ 18\lambda_{21}\lambda_{12}\alpha(3+2\alpha^2) - 6\lambda_{21}\lambda_{03}(1+4\alpha^2) + 30\lambda_{12}\lambda_{03}\alpha), \end{aligned}$$

where

$$\lambda_{rs} = \frac{\kappa_{rs}\kappa_{02}^{\frac{r}{2}}\kappa_{20}^{\frac{s}{2}}}{|\Sigma|^{\frac{r+s}{2}}},$$

$$\alpha = \frac{\lambda_{11}}{\sqrt{\lambda_{20}\lambda_{02}}},$$

and $|\Sigma| = \kappa_{20}\kappa_{02} - \kappa_{11}$. $\square$

Remark 10. It should be further mentioned that it also possible to evaluate the coefficient

$$[\mathbf{x}^{\mathbf{n}}]f(\mathbf{x})y(\mathbf{x})^k$$

by exactly the same methods when $f(\mathbf{x})$ is assumed to posses a convergent powers series expansion. This property leads to multivariate asymptotic expansions for the coefficients of multivariate generating function of special types. Let

$$a(\mathbf{x}) = \sum_{\mathbf{n}} a_{\mathbf{n}}\mathbf{x}^{\mathbf{n}} = \sum_{n\ge 0} \varphi_n(x_2,\dots,x_m)x_1^n$$

be a multivariate generating function and suppose for a moment that $x_2 > 0, \dots, x_m > 0$ are fixed. Then the radius of convergence of $a(\mathbf{x})$ (considered as a power series in x_1) is given by

$$R(x_2, \dots, x_m) = \left(\limsup_{n\to\infty} \varphi(x_2, \dots, x_m)^{\frac{1}{n}} \right)^{-1}.$$

Hence there will many cases where $\varphi(x_2, \dots, x_m)$ has an representation of the kind

$$\varphi(x_2, \dots, x_m) = c_n f(x_2, \dots, x_m) R(x_2, \dots, x_m)^{-n}(1 + o(1)).$$

Hence we can use (4.1) to obtain a multivariate asymptotic expansion for $a_{\mathbf{n}}$. For example, let $a(\mathbf{x})$ be the solution of a functional equation $F(\mathbf{x}, a) = 0$ (where F is an analytic function satisfying some additional conditions, see [Dr]). Suppose that there are positive real $\mathbf{x}_0, a_0 = a(\mathbf{x}_0)$ such that

$$\begin{aligned} F(\mathbf{x}_0, a_0) &= 0, \\ F_a(\mathbf{x}_0, a_0) &= 0. \\ F_{aa}(\mathbf{x}_0, a_0) &\neq 0, \quad \text{and} \\ F_{x_1}(\mathbf{x}_0, a_0) &\neq 0. \end{aligned}$$

Then we get by Weierstrass' preparation theorem that $a(\mathbf{x})$ admits the representation

$$a(\mathbf{x}) = g(\mathbf{x}) - h(\mathbf{x}) \sqrt{1 - \frac{x_1}{\gamma(x_2, \dots, x_m)}} \tag{4.5}$$

locally "around" the hypersurface $x_1 - \gamma(x_2, \dots, x_m) = 0$. $\gamma(x_2, \dots, x_m)$ (and $\alpha(x_2, \dots, x_m)$) are the solutions of the equations

$$\begin{aligned} F(\gamma, x_2, \dots, x_m, \alpha) &= 0 \\ F_a(\gamma, x_2, \dots, x_m, \alpha) &= 0 \end{aligned}$$

which exist by the implict function theorem. (Surely $\gamma(x_{0,2}, \dots, x_{0,m}) = x_{0,1}$, compare also with (3.1).) Hence by [FO] (for details see [Dr])

$$\begin{aligned} \varphi_n(x_2, \dots, x_m) = \frac{h(\gamma(x_2, \dots, x_m), x_2, \dots, x_m)}{2\sqrt{\pi} n^{\frac{3}{2}}} \gamma(x_2, \dots, x_m)^{-n} \cdot \\ \cdot (1 + o(1)) \qquad (n \to \infty). \end{aligned}$$

□

The results of section 3 can be applied for functions admitting a reprensentation of the kind (4.5) to evaluate the coefficient $[\mathbf{x}^{\mathbf{n}}] y(\mathbf{x})^k$ even for small k. Theorem 4 can only be used for $k \geq \varepsilon n_j$ $(j = 1, \dots, m)$ for some $\varepsilon > 0$. On the other hand the method presented in Remark 10 only works for fixed $k \geq 1$. The following theorem fills the gap between.

Theorem 6. *Let $m \ge 1$ and let $y(\mathbf{x})$ be the generating function of the sequence $y_{\mathbf{n}}$ satisfying the assumptions of Theorem 5. Furthermore assume that $y(\mathbf{x})$ admits a represention of the kind (4.5), where $\gamma(x_2,\dots,x_m)$ is an analytic function which is positive and real for positiv and real $x_2,\dots,x_m$ and satisfies $\gamma_{x_j}(x_2,\dots,x_m) \ne 0$ for $j=2,\dots,m$. Additionaly assume that $(\gamma(x_2,\dots,x_m),x_2,\dots,x_m)$ is the only singulatiy of $y(\mathbf{x})$ for $|x_1| \le \gamma(x_2,\dots,x_m)$ if $x_2,\dots,x_m$ are positive and real and that $y(\mathbf{x})$ has an analytic continuation to $|x_1| \le \gamma(x_2,\dots,x_m)(1+\varepsilon)$, $x_1 \notin [\gamma(x_2,\dots,x_m),\gamma(x_2,\dots,x_m)(1+\varepsilon))$ for some $\varepsilon > 0$. Then the coefficient $y_{\mathbf{n}k} = [\mathbf{x}^{\mathbf{n}}]y(\mathbf{x})^k$ can be approximated by the following series expansion ($M \ge 0$)*

$$
\begin{aligned}
y_{\mathbf{n}k} = \frac{y(\rho)^k}{\rho^{\mathbf{n}}\sqrt{(2\pi)^m(-|\overline{\Sigma}(\rho)|)n_1}} \Bigg(& \exp\left(-\frac{1}{2k}\overline{\mathbf{r}}\overline{\Sigma}(\rho)^{-1}\overline{\mathbf{r}}'\right) \cdot \\
& \cdot \left(1 + \sum_{1\le L\le M}\ \sum_{N_1-\frac{S}{2}=\frac{L}{2}}\ \sum_{|\overline{\mathbf{R}}|=S} b_{N_1,\overline{\mathbf{R}}}\frac{\overline{\mathbf{r}}^{\overline{\mathbf{R}}}}{n_1^{N_1}}\right) \\
& + \mathcal{O}\left(n_1^{-\frac{M+1}{2}}\right)\Bigg) \qquad (n_1 \to \infty)
\end{aligned}
\tag{4.6}
$$

uniformly for $0 \le \overline{\mu}_1 \le b_1$, $a_j \le \overline{\mu}_j \le b_j$, $0 < a_j < b_j$ $(j=2,\dots,m)$ and for all $\overline{\mathbf{r}} \in \mathbf{R}^m$ such that $k = n_1\overline{\mu}_1 + \overline{r}_1$, $n_j = n_1\overline{\mu}_j + \overline{r}_j$ $(j=2,\dots,m)$ are positive integers, where b_1 and the intervals $[a_j,b_j]$ must be chosen in a way that there exists a (unique) solutions $\rho = (\rho_1,\dots,\rho_m) = \rho_{\overline{\mu}}$ of

$$
\begin{aligned}
\overline{\mu}_1 &= \frac{y(\rho)}{\rho_1 y_{x_1}(\rho)} \\
\overline{\mu}_j &= \frac{\rho_j y_{x_j}(\rho)}{\rho_1 y_{x_1}(\rho)} \qquad (j=2,\dots,m).
\end{aligned}
$$

Let $\overline{\kappa}_{\mathbf{m}} = \overline{\kappa}_{\mathbf{m}}(\rho)$ be defined by

$$
\sum_{|\mathbf{l}|\ge 1}\frac{\kappa_{\mathbf{l}}}{\mathbf{l}!}\left(\sum_{|\mathbf{m}|\ge 1}\frac{\overline{\kappa}_{\mathbf{m}}}{\mathbf{m}!}\mathbf{t}^{\mathbf{m}}\right)^{l_1} t_2^{l_2}\cdots t_l^{l_m} = t_1,
$$

where $\kappa_{\mathbf{l}} = \kappa_{\mathbf{l}}(\rho)$ are defined in (4.3), then $m\times m$-matrix $\overline{\Sigma}(\rho)$ is given by

$$
\overline{\Sigma}(\rho) = (\overline{\kappa}_{\mathbf{l}}(\rho))_{|\mathbf{l}|=2}
$$

and $b_{N_1,\overline{\mathbf{R}}}$ can be determined similarly to (4.4).

Note that (4.6) can be applied for fixed $k \ge 1$ (i.e. $\overline{\mu}_1 = 0, \overline{r}_1 = k$) and therefore generalizes the methods presented in [Dr] for the solutions of functional equations.

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