Predecessors in Random Mappings

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Abstract

Let $\mathcal{F}_n$ be the set of random mappings $\varphi : \{1,\ldots,n\} \to \{1,\ldots,n\}$ (such that every mapping is equally likely). For $x \in \{1,\ldots,n\}$ the elements $\bigcup_{k\ge 0} \varphi^{-k}(\{x\})$ are called the predecessors of x. Let N_r denote the random variable which counts the number of points $x \in \{1,\ldots,n\}$ with exactly r predecessors. In this paper we identify the limiting distribution of N_r as $n \to \infty$. If $r = r(n) = o(n^{\frac{2}{3}})$ then the limiting distribution is Gaussian, if $r \sim Cn^{\frac{2}{3}}$ then it is Poisson, and in the remaining case $rn^{-\frac{2}{3}} \to \infty$ it is degenerate. Furthermore it is shown that N_r is a Poisson approximation if $r \to \infty$.

1 Introduction

Let $\mathcal{F}_n$ be the set of mappings $\varphi : \{1,\ldots,n\} \to \{1,\ldots,n\}$. In what follows we will assume the uniform random model, i.e. each mapping $\varphi \in \mathcal{F}_n$ appears with probability n^{-n}. We will call the elements of $\mathcal{F} = \bigcup_{n\ge 0} \mathcal{F}_n$ random mappings. If $X : \mathcal{F} \to \mathbf{R}$ is an arbitrary function then $X_n = X|\mathcal{F}_n$ may considered as a sequence of random variables and it may asked whether X_n has a proper limiting distribution or not.

We are interested in the number of points N_r with exactly r predecessors. For $\varphi \in \mathcal{F}_n$ and $x \in \{1,\ldots,n\}$ the elements

$$V(x) = \bigcup_{k\ge 0} \varphi^{-k}(\{x\}) \tag{1}$$

are called predecessors. Thus, for $\varphi \in \mathcal{F}_n$,

$$N_{r,n}(\varphi) = |\{x \in \{1,\ldots,n\} : |V(x)| = r\}|. \tag{2}$$

In [11, 1] it was shown that

$$\mathbf{E}N_{r,n} \sim np_r,$$

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where

$$p_r = \frac{r^{r-1}}{r!}e^{-r} = \frac{r^{-\frac{3}{2}}}{\sqrt{2\pi}}(1+\mathcal{O}(r^{-1})), \tag{3}$$

but the limiting distribution was left open. For constant $r \geq 1$ Drmota and Soria [5] showed that

$$\frac{N_{r,n} - np_r}{\sqrt{n(p_r - 2rp_r^2)}} \xrightarrow{d} \mathcal{N}(0,1) \qquad (n \to \infty). \tag{4}$$

We extend this result to the case of non-constant $r = r(n)$.

Theorem 1 *Let N_r denote the number of predecessors in random mappings.*

- *If $np_r \to \infty$, i.e. $r = r(n) = o(n^{\frac{2}{3}})$ as $n \to \infty$, then (4) holds.*
- *If $np_r \to C$ as $n \to \infty$ for some real constant $C > 0$ then $N_{r,n}$ has a Poisson limiting distribution, i.e. for any $k \geq 0$*

$$\lim_{n\to\infty} \mathbf{Pr}[N_{r,n} = k] = e^{-C}\frac{C^k}{k!}.$$

- *If $np_r \to 0$ as $n \to \infty$ then $N_{r,n}$ degenerates, i.e. $\lim\limits_{n\to\infty} \mathbf{Pr}[N_{r,n} = 0] = 1$.*

In fact, we will present two completely different approaches. First we use a combinatorial counting argument (compare with [10]) to obtain explicit expressions for moments. By the use of these expressions we can only prove the case $np_r = \mathcal{O}(1)$ (Propositions 2 and 3). Second, we generalize the generating function methods of [5] to settle the general case (Proposition 5). This second method can be applied for more general situations too (Theorem 3).

If $r = r(n) \to \infty$ and $np_r \to \infty$ then $\mathbf{E}N_{r,n} \sim \mathbf{V}N_{r,n} \sim np_r$ as $n \to \infty$. Thus one might expect that $N_{r,n}$ is not only asymptotically normal but is a Poisson approximation too (see [2]). In fact, a more detailed analysis reveals that this is really true. We obtain local limit theorems with error terms (see Proposition 6) and estimates for the total variation

$$d_{TV}(N_{r,n}, \mathrm{Po}(np_r)) = \frac{1}{2}\sum_{k\geq 0}\left|\mathbf{Pr}[N_{r,n} = k] - e^{-np_r}\frac{(np_r)^k}{k!}\right|$$

even if $np_r = \mathcal{O}(1)$.

Theorem 2 *Suppose that $r = r(n) \to \infty$ as $n \to \infty$. Then $N_{r,n}$ is a Poisson approximation, i.e. $\lim_{n\to\infty} d_{TV}(N_{r,n}, \mathrm{Po}(np_r)) = 0$. More precisely, if $r \leq n^{\frac{3}{4}}$ then the total variation $d_{TV}(N_{r,n}, \mathrm{Po}(np_r))$ can be estimated by*

$$d_{TV}(N_{r,n}, \mathrm{Po}(np_r)) \leq C \min\left(\max(r^{-\frac{1}{3}}, n^{-\frac{1}{3}}r^{\frac{1}{2}}), \max(nr^{-\frac{11}{6}}, r^{-\frac{1}{3}})\right)$$

and if $n^{\frac{3}{4}} \le r \le n$ then we have

$$d_{TV}(N_{r,n}, \mathrm{Po}(np_r)) \le C\, n^{\frac{1}{2}} r^{-1},$$

where $C > 0$ is an absolute constant.

We remark that it is very useful to identify a mapping $\varphi \in \mathcal{F}_n$ with its functional graph G_φ with vertex set $V = \{1, \ldots, n\}$ and edges $E = \{(i, \varphi(i)) : 1 \le i \le n\}$. Obviously a component of this (directed labelled) graph consists of a cycle of trees. The roots of these trees are the cyclic points.

2 An Explicit Approach

The basic tool is the following explicit formula. In what follows $\binom{\alpha}{m}$ denotes the binomial coefficient $\frac{1}{m!}\alpha(\alpha-1)\cdots(\alpha-m+1)$.

Proposition 1 *Set*

$$A_M(r,j) = \begin{cases} r^{r-2}(n-(M-1)r) & \text{for } j = 1, \\ \sum\limits_{i=0}^{r-j} \binom{r-j}{i}(i+j)!\, r^{r-j-i-1} & \text{for } j > 1. \end{cases} \tag{5}$$

Then for any $m \ge 1$

$$\mathbf{E}\binom{N_{r,n}}{m} = \frac{n!}{n^n} \sum_{\sum jm_j = m} \frac{(n-Mr)^{n-Mr}}{(n-Mr)!} \prod_{j=1}^{r} \frac{1}{m_j!} \left(\frac{A_M(r,j)}{j!(r-j)!}\right)^{m_j}, \tag{6}$$

in which the sum is taken over all r-tuples $(m_1, \ldots, m_r)$ of non-negative integers m_j such that $\sum\limits_{j=1}^{r} jm_j = m$ and $Mr \le n$, where $M = \sum\limits_{j=1}^{r} m_j$.

Proof. Fix m different points $x_1, \ldots, x_m \in \{1, \ldots, n\}$. Our aim is to find an expression for the probability p_{mr} of those mappings with $|V(x_j)| = r$ $(1 \le j \le m)$. Since this probability does not depend on the choice of $x_1, \ldots, x_m$ we obtain

$$p_{mr} = \sum_{k \ge m} \frac{\binom{n-m}{k-m}}{\binom{n}{k}} \mathbf{Pr}[N_{r,n} = k] = \sum_{k \ge m} \frac{\binom{k}{m}}{\binom{n}{m}} \mathbf{Pr}[N_{r,n} = k] = \binom{n}{m}^{-1} \mathbf{E}\binom{N_{r,n}}{m}. \tag{7}$$

Actually, we show that

$$\begin{aligned} p_{mr} \;&=\; n^{-n} \sum_{\sum jm_j = m} \frac{(n-m)!}{(n-Mr)! \prod_j ((r-j)!)^{m_j}} \frac{m!}{\prod_j (m_j!(j!)^{m_j})} \cdot \\ &\quad \cdot \prod_j A_M(r,j)^{m_j}\, (n-Mr)^{n-Mr} \end{aligned} \tag{8}$$

which is equivalent to (6).

Observe that if $|V(x_k)| = |V(x_l)| = r$ then either $V(x_k)$ and $V(x_l)$ are disjoint or coincide. If they coincide then x_k and x_l are cyclic points in a component of size r. Hence the general situation in the case $|V(x_j)| = r$ $(1 \le j \le m)$ is as follows. There is a partition of $\{x_1, \ldots, x_m\}$ in m_1 singletons, m_2 (unordered) pairs, …, m_r (unordered) r-tuples such that $\sum_{j=1}^{r} jm_j = m$ and such that the predecessor sets of such an j-tuple $(1 \le j \le r)$ coincide but do not coincide with any other predecessor set. This means that there are m_1 points in $\{x_1, \ldots, x_m\}$ such that their predecessor sets are disjoint and do not coincide another predecessor set, there are m_2 pairs (x_{i_1}, x_{i_2}) of points in $\{x_1, \ldots, x_m\}$ such that $V(x_{i_1}) = V(x_{i_2})$ and so on.

First, assume that $m_1 = m$. This is the simplest case and we can argue as follows. We start with the points $x_1, \ldots, x_m$ and count those mappings $\varphi \in \mathcal{F}_n$ such that $V(x_j)$ $(1 \le j \le r)$ are disjoint and have size r. There are

$$\frac{(n-m)!}{(n-mr)!((r-1)!)^m} \tag{9}$$

ways of choosing $m(r-1)$ candiditates for predecessors of $x_1, \ldots, x_m$ and partitioning them into m disjoint groups corresponding to $x_1, \ldots, x_m$. If we interprete mappings as directed labelled graphs then x_j $(1 \le j \le r)$ and its predecessors constitute a labelled tree with root x_j and size r. There are r^{r-2} trees of this kind. Now the root of each tree chooses its image in $n-(m-1)r$ ways since the predecessor sets should be disjoint. Furthermore there are $(n-mr)^{n-mr}$ mappings of the remaining $n - mr$ elements. This proves the first term

$$\begin{aligned} n^{-n} &\frac{(n-m)!}{(n-mr)!((r-1)!)^m}(r^{r-2})^m(n-(m-1)r)^m(n-mr)^{n-mr} \\ &= n^{-n}\frac{(n-m)!}{(n-mr)!((r-1)!)^m}\frac{m!}{m!}A_m(r,1)^m(n-mr)^{n-mr} \end{aligned}$$

of (8) corresponding to the case $m_1 = m$.

Now consider the general case, where $\sum_{j=1}^{r} jm_j = m$. Instead of (9) there are

$$\frac{(n-m)!}{(n-Mr)!\prod_j((r-j)!)^{m_j}}\frac{m!}{\prod_j(m_j!(j!)^{m_j})} \tag{10}$$

possible ways of choosing candidates for predecessors and partitioning them into $M = \sum_{j=1}^{r} m_j$ groups. If we have an j-tuple $y_1, \ldots, y_j$ of $\{x_1, \ldots, x_m\}$ with $j \ge 2$ with coinciding predecessor sets of size r then $y_1, \ldots, y_j$ are cyclic points of a component of size r (of the related directed graph). Since there are lr^{r-l-1}

connected labelled graphs with r points and exacly one cycle of size l with given points $z_1, \ldots, z_l$ ([3]) it is clear that

$$\begin{aligned} A_M(r,j) &= \sum_{l=j}^{r} \binom{r-j}{l-j} l r^{r-l-1} (l-1)! \\ &= \sum_{i=0}^{r-j} \binom{r-j}{i} (i+j)! r^{r-j-i-1} \end{aligned}$$

is the number of ways to construct a component of size r containing exactly $y_1, \ldots, y_j$ as cyclic points. The m_1 points can be treated as above and the remaining $n - Mr$ points can be mapped arbitrarily. Hence the general term of (8) follows immediately from (10) and the above considerations. □

Before we can apply Proposition 1 we must study the behaviour of $A_M(r,j)$ in more detail.

Lemma 1 *Let*

$$A(r,j) = \sum_{i=0}^{r-j} \binom{r-j}{i} (i+j)! r^{r-j-i-1} \qquad (0 \le j \le r).$$

Then for any fixed $j \ge 0$

$$A(r,j) = r^{r-\frac{1}{2}(j+1)} 2^{\frac{1}{2}(j-1)} \Gamma\left(\frac{j+1}{2}\right) \left(1 + \mathcal{O}(r^{-\frac{1}{2}})\right). \tag{11}$$

For $j = 1$ *we have exactly* $A(r,j) = r^{r-1}$*. Furthermore*

$$\sum_{j=0}^{r} \frac{A(r,j)}{j!(r-j)!} x^j = \sum_{k=0}^{r} \frac{r^{r-k-1}}{(r-k)!} (x+1)^k. \tag{12}$$

Please note that $A(r,j)$ and $A_M(r,j)$ only differ for $j = 1$.

Proof. Using the equality

$$(i+j)! r^{-j-i-1} = \int_0^\infty e^{-rx} x^{i+j} \, dx$$

one can obtain

$$\begin{aligned} A(r,j) &= r^r \sum_{i=0}^{r-j} \binom{r-j}{i} \int_0^\infty e^{-rx} x^{i+j} \, dx \\ &= r^r \int_0^\infty e^{-rx} x^j (1+x)^{r-j} \, dx = r^r I(r,j), \end{aligned}$$

where

$$I(r,j) = \int_0^\infty e^{-rx} \left(\frac{x}{1+x}\right)^j (1+x)^r \, dx = \int_0^\infty g(x) e^{r h(x)} \, dx$$

in which $h(x) = \log(1+x) - x$ has a saddle point at $x = 0$. By standard saddle point approximation the asymptotic main term of $I(r,j)$ (for $r \to \infty$) evaluates to

$$\int_0^\infty e^{-\frac{1}{2}rx^2} x^j \, dx = r^{-\frac{1}{2}(j+1)} 2^{\frac{1}{2}(j-1)} \Gamma\left(\frac{j+1}{2}\right)$$

whereas the error term will be of order $\mathcal{O}(r^{-\frac{1}{2}(j+2)})$. This proves (11).

Equation (12) follows directly by comparing the coefficient of x^j on both sides. $\square$

We are now in the position to derive rather good estimates for mean value and variance.

Corollary 1 *Suppose that $r = r(n) \le n(1-\varepsilon)$ for some $\varepsilon > 0$ as $n \to \infty$. Then*

$$\mathbf{E} N_{r,n} = \frac{n!}{n^{n-1}} \frac{(n-r)^{n-r}}{(n-r)!} \frac{r^{r-1}}{r!} = n p_r \left(1 + \frac{r}{2n} + \mathcal{O}\left(\frac{r^2}{n^2}\right)\right) \tag{13}$$

and

$$\mathbf{V} N_{r,n} = n(p_r - 2rp_r^2) + \frac{1}{2} + \mathcal{O}(r^{-\frac{1}{2}})$$

as $n \to \infty$.

Proof. The asymptotic expansion (13) follows from (6) and an application of Stirling's formula:

$$\frac{n!}{n^n} \frac{(n-r)^{n-r}}{(n-r)!} = e^{-r} \left(1 + \frac{r}{2n} + \mathcal{O}\left(\frac{r^2}{n^2}\right)\right)$$

if $r \le n(1-\varepsilon)$. In the same manner we can prove (1) by using (11) for $j = 2$:

$$\frac{A(r,2)e^{-r}}{(r-2)!} = \frac{1}{2} + \mathcal{O}(r^{-\frac{1}{2}}). \qquad \square$$

We see that $\mathbf{E}N_{r,n}$ and $\mathbf{V}N_{r,n}$ are always of the same order of magnitude. We are now able to interpret the results of Theorem 1. If $\mathbf{E}N_{r,n} \to \infty$ then the limiting distribution is Gaussian; if $\mathbf{E}N_{r,n} \to$ constant > 0, then there is a discrete limiting distribution whereas the case $\mathbf{E}N_{r,n} \to 0$ the limit is degenerate. Now we can easily prove the latter case.

Proposition 2 *Suppose that $rn^{-\frac{2}{3}} \to \infty$ and $r \le n(1-\varepsilon)$ for some $\varepsilon > 0$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \mathbf{Pr}[N_{r,n} = 0] = 1. \tag{14}$$

Proof. If $rn^{-\frac{2}{3}} \to \infty$ and $r \le n(1-\varepsilon)$ then $\mathbf{E}N_{r,n} \to 0$. Hence (14) follows from

$$\mathbf{E}N_{r,n} = \sum_{k\ge 1} k\,\mathbf{Pr}[N_{r,n}=k] \ge 1 - \mathbf{Pr}[N_{r,n}=0]. \qquad \square$$

By considering formula (6) one would hope that the term corresponding to $m_1 = m$ and $m_j = 0$ for $j \ge 2$ is dominating. It is amazing that this is not always the case.

Lemma 2 *Suppose that $r \to \infty$ and $r = o(n^{\frac{2}{3}})$ as $n \to \infty$. Then*

$$\mathbf{E}N_{r,n}^m \sim (np_r)^m \qquad (n\to\infty) \tag{15}$$

for all $m \ge 1$ if and only if

$$r = \mathcal{O}\left(n^{\frac{1}{2}+\varepsilon}\right) \qquad (n\to\infty) \tag{16}$$

for all $\varepsilon > 0$.

Proof. Since $r = o(n^{\frac{2}{3}})$ ensures that $np_r \to \infty$ it is clear that (15) is equivalent to

$$\mathbf{E}\binom{N_{r,n}}{m} \sim \frac{(np_r)^m}{m!} \qquad (n\to\infty) \tag{17}$$

for all $m \ge 1$. From (11) we get

$$\frac{A(r,j)e^{-r}}{j!(r-j)!} = \Theta\left(r^{\frac{j}{2}-1}\right).$$

Hence by Stirling's formula and (6)

$$\begin{aligned}
\mathbf{E}\binom{N_{r,n}}{m} &= \sum_{\sum jm_j = m} \left(\frac{n}{r}\right)^{m_1} \prod_j \frac{1}{m_j!}\left(\frac{A(r,j)e^{-r}}{j!(r-j)!}\right)^{m_j} \cdot \left(1+\mathcal{O}\left(\frac{r}{n}\right)\right) \\
&= \frac{(np_r)^m}{m!} + \Theta\left(\sum_{\sum jm_j = m, m_1<m} \left(\frac{n}{r}\right)^{m_1} r^{\frac{n}{2}-M}\right).
\end{aligned}$$

Consequently (17) is equivalent to

$$\left(\frac{n}{r}\right)^{m_1} r^{\frac{n}{2}-M} = o\left(\left(\frac{n}{r^{\frac{3}{2}}}\right)^m\right) \tag{18}$$

for $m_1 < m$.

Now suppose that (16) holds and set $\varepsilon = \frac{1}{2m}$. If $m_1 < m$ then $M \ge m_1 + 1$ and hence

$$r^{2m-m_1-M} = o\left(n^{m-\frac{m_1}{2}-\frac{M}{2}+\frac{1}{2}}\right) = o\left(n^{m-m_1}\right)$$

which is the same as (18).

On the other hand, choose $M = 1$, i.e. $m_m = 1$. Thus (18) implies

$$r^{2m-m_1-M} = r^{2m-1} = o(n^m)$$

and hence $r = o(n^{\frac{1}{2}+\frac{1}{2}m})$ for all $m \geq 1$ which is the same as (16). This proves Lemma 2. □

Lemma 2 shows that the weak convergence property of Theorem 1 for $r = o(n^{\frac{2}{3}})$ and the asymptotic expansion for $\mathbf{E}N_{r,n}$ and $\mathbf{V}N_{r,n}$ need not imply asymptotic expansions for the moments $\mathbf{E}N_{r,n}^m$ and consequently not for the centralized moments $\mathbf{E}(N_{r,n} - \mathbf{E}N_{r,n})^m$. This indicates that the full proof of Theorem 1 requires a more refined analysis than using only (6) and direct asymptotic methods.

Lemma 3 *The probability generating function $f_{r,n}(u) = \mathbf{E}u^{N_{r,n}}$ is given by*

$$\begin{aligned} f_{r,n}(u) &= \sum_{m\geq 0} \mathbf{E}\binom{N_{r,n}}{m}(u-1)^m \\ &= \sum_{M\leq n/r} \frac{n!}{n^n}\frac{(n-Mr)^{n-Mr}}{(n-Mr)!}e^{Mr}\frac{1}{M!}\cdot \qquad (19) \\ &\quad \cdot\left((n-(M-1)r)p_r(u-1) + \sum_{j=2}^{r}\frac{A(r,j)e^{-r}}{j!(r-j)!}(u-1)^j\right)^M . \end{aligned}$$

Proof. (19) follows immediately from (6), by applying the multinomial formula, and by comparing the coefficients of $(u-1)^m$. □

We use this representation (19) to prove the following proposition.

Proposition 3 *Suppose that $np_r \to C$ as $n \to \infty$ for some real constant $C > 0$. Then $N_{r,n}$ has a (discrete) Poisson limiting distribution; more precisely for any $k \geq 0$*

$$\lim_{n\to\infty} \mathbf{Pr}[N_{r,n} = k] = e^{-C}\frac{C^k}{k!}. \qquad (20)$$

Proof. Let

$$\phi_n(t) = \mathbf{E}e^{itN_{r,n}} = f_{r,n}(e^{it}) \qquad (21)$$

denote the characteristic function of $N_{r,n}$. Using (12) we can obtain the following estimate

$$\begin{aligned} \left|\sum_{j=2}^{r}\frac{A(r,j)}{j!(r-j)!}(u-1)^j\right| &= \left|\sum_{k=0}^{r}\frac{r^{r-k-1}}{(r-k)!}u^k - \frac{A(r,0)}{r!} - \frac{A(r,1)}{(r-1)!}(u-1)\right| \\ &\leq \frac{1}{r}e^r + \mathcal{O}(e^r r^{-1}) + \mathcal{O}(e^r r^{-\frac{1}{2}}|u-1|). \qquad (22) \end{aligned}$$

For $|u| = 1$ it follows from (19) and (22) that for any $\varepsilon > 0$ there exists $M(\varepsilon)$ such that

$$\sum_{M > M(\varepsilon)} \frac{n!}{n^n} \frac{(n - Mr)^{n-Mr}}{(n - Mr)!} e^{Mr} \frac{1}{M!} \cdot$$
$$\cdot \left((n - (M-1)r) p_r (e^{it} - 1) + \sum_{j=2}^{r} \frac{A(r,j)}{j!(r-j)!} (e^{it} - 1)^j \right)^M < \varepsilon.$$

Furthermore

$$\lim_{n\to\infty} \sum_{M \le M(\varepsilon)} \frac{n!}{n^n} \frac{(n - Mr)^{n-Mr}}{(n - Mr)!} e^{Mr} \frac{1}{M!} \cdot$$
$$\cdot \left((n - (M-1)r) \frac{r^{r-2} e^{-r}}{(r-1)!} (e^{it} - 1) + \mathcal{O}(r^{-\frac{1}{2}}) \right)^M$$
$$= \sum_{M \le M(\varepsilon)} \frac{1}{M!} \left(C(e^{it} - 1) \right)^M.$$

Hence we obtain

$$\lim_{n\to\infty} \phi_n(t) = e^{C(e^{it}-1)}$$

which proves a weak convergence. Since $\mathbf{Pr}[N_{r,n} < 0] = 0$ and the limiting distribution is discrete that is equivalent to (20). This completes the proof of Proposition 3. □

The reason why we are not able to settle the general case by using this explicit formula (19) is that the characteristic function $\phi_n(t)$ or the moment generating function

$$M_n(t) = e^{-t\sqrt{np_r}} \sum_{m\ge 0} \mathbf{E}\binom{N_{r,n}}{m} \left(e^{t/\sqrt{np_r}} - 1\right)^m \tag{23}$$

of the normalized random variable

$$\frac{N_{r,n} - np_r}{\sqrt{np_r}} \tag{24}$$

usually contains alternating terms. (If we assume $r \to \infty$ then we can use $\sqrt{np_r}$ instead of $\sqrt{n(p_r - 2rp_r^2)}$ as a normalization factor.) Only in the case $t \ge 0$ and $r = o((n\log n)^{\frac{2}{5}})$ we can estimate the moment generating function. As in the proof of Proposition 3 we obtain for real u

$$\begin{aligned}\left| \sum_{j=2}^{r} \frac{A(r,j)}{j!(r-j)!} (u-1)^j \right| &= \left| \sum_{k=0}^{r} \frac{r^{r-k-1}}{(r-k)!} u^k - \frac{A(r,0)}{r!} - \frac{A(r,1)}{(r-1)!}(u-1) \right| \\ &\le \frac{1}{r} \left| u^{-r} e^{r/u} \right| + \mathcal{O}(e^r r^{-1}) + \mathcal{O}(e^r r^{-\frac{1}{2}} |u-1|).\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{r}u^{-r}e^{r/u-r} &= \frac{1}{r}\exp\left(r\left(\frac{t^2}{2np_r}\right)+\mathcal{O}\left(\frac{rt^3}{(np_r)^{\frac{3}{2}}}\right)\right)\\
&= r^{-1}e^{o(\log n)} = o(1)
\end{aligned}$$

for $u = e^{t/\sqrt{np_r}}$ and $r = o((n\log n)^{\frac{2}{5}})$ which leads to

$$\begin{aligned}
M_n(t) &= e^{-t\sqrt{np_r}}\sum_{M\le 4t\sqrt{np_r}}\left(1+\mathcal{O}(n^{-1})\right)\left(t\sqrt{np_r}+\frac{t^2}{2}+o(1)\right)^M + o(1)\\
&= e^{\frac{1}{2}t^2}+o(1) \qquad (25)
\end{aligned}$$

for $t > 0$. If $t < 0$ then there appears an alternating sum which should be very small although the terms inside the sum are very large (especially if $M \sim |t|\sqrt{np_r}$). The same difficulty appears if we try to use characteristic functions instead of moment generating functions.

3 An Implicit Approach

First we state an alternative representation of $f_{r,n}(u) = \mathbf{E}u^{N_{r,n}}$.

Lemma 4 ([5]) *Let $a_r(x,u)$ denote the solution of*

$$a_r(x,u) = xe^{a_r(x,u)} + (u-1)\frac{r^{r-1}}{r!}x^r, \qquad (26)$$

Then

$$\begin{aligned}
f_r(x,u) &= \sum_{n\ge 1}\frac{n^n}{n!}f_{r,n}(u)x^n \qquad (27)\\
&= \frac{1}{1-a_r(x,u)+(u-1)\frac{r^{r-1}}{r!}x^r}\exp\left(\frac{x^r}{r}\sum_{m=0}^{r}\frac{r^{r-m}}{(r-m)!}(u^m-1)\right).
\end{aligned}$$

Sketch of the proof. Let T denote a rooted tree such that all edges are directed to the root. If v is node of T then let T_v denote the (maximal) subtree of T with root v such that again all edges of T_v are directed to v. Now let $a_{r,nk}$ denote the number of labelled rooted trees T of size n such that for exactly k nodes v the subtree T_v consists of r nodes. Then

$$a_r(x,u) = \sum_{n,k} a_{r,nk}\frac{x^n}{n!}u^k$$

satisfies the functional equation (26). This follows from the fact that any labelled tree can be considered as a node followed by a sequence of trees and that we have to mark this node if the resulting tree has exactly size r. (Note that there are r^{r-1} different lablelled rooted trees of size r.)

Now a random mapping can be considered as a set of cycles of labelled rooted trees. Using $a_r(x,u)$ we can construct the generating function corresponding to a cycle of length m by

$$c_{r,m}(x,u) = \frac{1}{m}\left(xe^{a_r(x,u)}\right)^m + (u^m-1)x^r\frac{r^{r-m}}{r(r-m)!}.$$

Here we have to take into account that we need m additional marks if this cycle of trees has exactly r nodes. (Note that there are $mr!r^{r-m-1}/(r-m)!$ lablelled forests of size r with m components ([3]).) Hence

$$f_r(x,u) = \exp\left(\sum_{m\geq 1} c_{r,m}(x,u)\right)$$

which proves (27). □

We will use this representation (27) and Cauchy's formula to evaluate $f_{r,n}(u)$. For this purpose we will need some information about the dominating singularity of $f_r(x,u)$. First, let us discuss $a_r(x,u)$.

Lemma 5 *For sufficiently large $r \geq r_0$ the function $a_r(x,u)$ has an expansion*

$$\begin{aligned} a_r(x,u) &= a_r(u) - b_r(u)\sqrt{1-\frac{x}{\rho_r(u)}} \\ &\quad + \mathcal{O}\left(|1-x/\rho(u)|\right) \end{aligned} \tag{28}$$

uniformly for $|u| = 1$, $|x-\rho_r(u)| < \frac{1}{r}$, and $\arg(x/\rho_r(u)-1) \neq 0$, in which $a_r(u) = a$ and $\rho_r(u) = \rho$ are the solutions of the system of equations

$$a = \rho e^a + (u-1)\frac{r^{r-1}}{r!}\rho^r, \tag{29}$$

$$1 = \rho e^a, \tag{30}$$

and $b_r(u)$ is given by

$$b_r(u) = \sqrt{\frac{2\rho_r(u)e^{a_r(u)} + 2\rho_r(u)^r\frac{r^r}{r!}(u-1)}{\rho_r(u)e^{a_r(u)}}}.$$

Furthermore, the functions $a_r(u), \rho_r(u), b_r(u)$ ($|u| = 1$, $r \geq r_0$) can be uniformly expanded to

$$a_r(u) = 1 + p_r(u-1) - rp_r^2(u-1)^2 + \mathcal{O}(r^{-\frac{5}{2}}|u-1|^3) \tag{31}$$

$$\rho_r(u) = \frac{1}{e}\left(1 - p_r(u-1) + \frac{(2r+1)p_r^2}{2}(u-1)^2\right) + \mathcal{O}(r^{-\frac{5}{2}}|u-1|^3) \tag{32}$$

$$b_r(u) = \sqrt{2} + \mathcal{O}(r^{-\frac{1}{2}}|u-1|). \tag{33}$$

Proof. Set $c = a - 1$. If a, ρ are solutions of (29), (30) then c is a fixed point of

$$c = (u-1)p_r e^{-rc}.$$

By Banach's fixed point theorem it is easily checked that this equation has a solution satisfying $|c| \le \frac{1}{r}$ for $|u| \le 1+\varepsilon$ and $r \ge r_0$, where $\varepsilon > 0$ and r_0 is sufficiently large. Conversely $a = 1 + c$ and $\rho = e^{-1-c}$ is a solution of (29), (30).

Next let

$$F(a, x, u) = xe^a + (u-1)x^r \frac{r^{r-1}}{r!} - a. \tag{34}$$

Then the system of equations (29), (30) is the same as $F(a, \rho, u) = 0$ and $F_a(a, \rho, u) = 0$. Observe that

$$\begin{aligned} F_x &= e^a + r\rho^{r-1}(u-1)\frac{r^{r-1}}{r!} \ne 0 \quad \text{and} & (35)\\ F_{aa} &= \rho e^a = 1 \ne 0 & (36) \end{aligned}$$

for $|u| \le 1 + \varepsilon$. Hence

$$F_a F_{ax} - F_x F_{aa} = -F_x F_{aa} \ne 0$$

and it follows from the implicit function theorem ([9]) that the solutions $a = a_r(u)$, $\rho = \rho_r(u)$ are analytic for $|u| < 1 + \varepsilon$.

Next we will prove (31), (32), and (33). By implicit differentiation we obtain

$$\begin{aligned} \rho_r'(u) &= -\frac{F_u}{F_x}, \\ a_r'(u) &= -\frac{F_{ax}}{F_{aa}}\rho_r'(u), \\ \rho_r''(u) &= -\frac{F_{ax}a_r'(u) + F_{xx}\rho_r'(u) + 2F_{xu}}{F_x}\rho_r'(u), \\ r''(u) &= -\frac{F_{aax}}{F_{aa}}a_r'(u)\rho_r'(u) - \frac{F_{ax}}{F_{aa}}\rho_r''(u), \\ &\vdots \end{aligned}$$

which implies

$$\begin{aligned} \rho_r'(1) &= -\frac{p_r}{e}, & \rho_r'(u) &= \mathcal{O}(r^{-\frac{3}{2}}), \\ \rho_r''(1) &= \frac{2r+1}{e}p_r^2, & \rho_r''(u) &= \mathcal{O}(r^{-2}), \\ a_r'(1) &= p_r, & a_r'(u) &= \mathcal{O}(r^{-\frac{3}{2}}), \\ a_r''(1) &= -2rp_r^2, & a_r''(u) &= \mathcal{O}(r^{-2}) \end{aligned}$$

for $|u| < 1 + \varepsilon$ and in the same fashion

$$\rho_r'''(u) = \mathcal{O}(r^{-\frac{5}{2}}) \quad \text{and} \quad a_r'''(u) = \mathcal{O}(r^{-\frac{5}{2}}).$$

This proves (31) and (32). (33) is a direct consequence of (31) and (32).

Now, consider the Taylor series expansion of $F(a,x,u)$ around $a = a_r(u)$, $x = \rho_r(u)$, where $|u| < 1 + \varepsilon$:

$$\begin{aligned} F(a,x,u) &= F(a_r(u),\rho_r(u),u) + F_a(a_r(u),\rho_r(u),u)(a - a_r(u)) \\ &\quad + F_x(a_r(u),\rho_r(u),u)(x - \rho_r(u)) \\ &\quad + \frac{F_{aa}(a_r(u),\rho_r(u),u)}{2}(a - a_r(u))^2 + \cdots \\ &= F_x(a_r(u),\rho_r(u),u)(x - \rho_r(u)) \\ &\quad + \frac{F_{aa}(a_r(u),\rho_r(u),u)}{2}(a - a_r(u))^2 + \cdots \end{aligned} \tag{37}$$

If $a = a_r(x,u)$ is the solution of $F(a,x,u) = 0$ then by looking at (37) one would expect that the main term of $a_r(x,u) - a_r(u)$ is $-b_r(u)\sqrt{1 - x/\rho_r(u)}$. In fact, we will prove (28) by a bootstrapping procedure.

Observe that the solution $a_r(x,u)$ of (26) satisfies $a_r(x,u) = \mathcal{O}(1)$ if $|u| < 1 + \varepsilon$ and $|x| \le 1/e + 1/r$. Hence

$$\begin{aligned} F_x &= e^a + r x^{r-1}(u-1)\frac{r^{r-1}}{r!} = \Theta(1), \\ F_{aa} &= x e^a = \Theta(1), \\ F_{ax} &= e^a = \Theta(1), \\ F_{xx} &= r(r-1)x^{r-2}(u-1))\frac{r^{r-1}}{r!} = \mathcal{O}(r^{\frac{1}{2}}) \end{aligned}$$

if $|u| < 1 + \varepsilon$ and $|x - \rho_r(u)| < \frac{1}{r}$, where $\Theta(1)$ denotes a term which is bounded below and above by positive constants. Thus Taylor's theorem gives

$$\begin{aligned} 0 &= \Theta(1)(x - \rho_r(u)) + \Theta(1)(a_r(x,u) - a_r(u))^2 \\ &+ \Theta(1)(a_r(x,u) - a_r(u))(x - \rho_r(u)) + \mathcal{O}(r^{\frac{1}{2}}(x - \rho_r(u)^2) \end{aligned}$$

which implies

$$a_r(x,u) - a_r(u) = \mathcal{O}\left(\sqrt{|\rho_r(u) - x|}\right). \tag{38}$$

(Note that $(r^{\frac{1}{2}}(x - \rho_r(u)^2 = \mathcal{O}((x - \rho_r(u))^{\frac{3}{2}})$.) By a further application of Taylor's theorem and by using (38) we obtain

$$0 = F_x(x - \rho_r(u)) + \frac{1}{2}F_{aa}(a_r(x,u) - a_r(u))^2 + \mathcal{O}\left((x - \rho_r(u))^{\frac{3}{2}}\right)$$

and from this follows (28). □

Lemma 6 *Suppose that $|u| = 1$ and $r \geq r_0$. Then $x = \rho_r(u)$ is the only singularity of $a_r(x,u)$ on the boundary of the disc $|x| < |\rho_r(u)|$. Furthermore, there is an analytic continuation of $a_r(x,u)$ to the region $|x| < |\rho_r(u)|(1+\frac{1}{r})$, $|x-\rho(u)| \geq \frac{1}{r}$, where $a_r(x,u)$ satisfies*

$$|1 - xe^{a_r(x,u)}| \geq cr^{-\frac{1}{2}} \tag{39}$$

for some constant $c > 0$.

Proof. First consider the tree function $a(x)$ which is the solution of the functional equation

$$a(x) = xe^{a(x)}.$$

It is well known (see [7]) that $a(x)$ is singular at $x = 1/e$. More precisely there is local representation of the kind

$$a(x) = 1 - \sqrt{2}\sqrt{1-ex} + \mathcal{O}(1-ex), \tag{40}$$

say for $|x - 1/e| \leq \varepsilon$. Since the coefficient

$$[x^n]a(x) = \frac{n^{n-1}}{n!} > 0$$

for all $n \geq 1$ it follows that

$$\left|xe^{a(x)}\right| < e^{-1+a(1/e)} = 1$$

for $|x| \leq 1/e$ but $x \neq 1/e$. Thus by the implicit function theorem ([9]) $a(x)$ is regular for $|x| \leq 1/e$ with the only exception $x = 1/e$. Furthermore, by compactness there exists $\delta > 0$ such that $a(x)$ can be analytically continued to the region $|x| \leq 1/e + \delta$, $|x - 1/e| > \varepsilon$. Hence by continuity and by (40) we obtain

$$|1 - xe^{a(x)}| \geq c_1 r^{-\frac{1}{2}}$$

for $|x| < |\rho_r(u)|(1+\frac{1}{r})$, $|x-\rho(u)| \geq \frac{1}{r}$, where $c_1 > 0$ is some constant.

If $|u| = 1$ and $|x| < |\rho_r(u)|(1+\frac{1}{r})$, $|x-\rho(u)| \geq \frac{1}{r}$ then

$$x^r(u-1)\frac{r^{r-1}}{r!} = \mathcal{O}(r^{-\frac{3}{2}}).$$

Hence

$$a_r(x,u) - a(x) = \mathcal{O}(r^{-1})$$

in this region. This proves (39). Furthermore, by the implicit function theorem $a_r(x,u)$ is analytic in this region. □

With the help of these analytic properties we are now in a position to obtain a proper asymptotic representation for $f_{r,n}(u)$, $|u| = 1$.

Proposition 4 *The probability generating function $f_{r,n}(u)$ has a uniform asymptotic representation*

$$f_{r,n}(u) = e^{-n}\rho_r(u)^{-n}\left(1+\mathcal{O}(r^{-\frac{1}{2}})\right) \tag{41}$$

for $|u|=1$ and $r \geq r_0$.

Proof. By using (27) and Cauchy's formula we have

$$f_{r,n}(u) = \frac{n!}{n^n}\frac{1}{2\pi i}\int_\Gamma f_r(x,e^u)\,\frac{dx}{x^{n+1}},$$

in which we can use the following path of integration $\Gamma = \Gamma_1 \cup \Gamma_2$:

$$\begin{aligned}
\Gamma_1 &= \left\{ z=\rho(u)\left(1+\frac{s}{n}\right) \,:\, s\in\gamma' \right\}, \\
\Gamma_2 &= \left\{ z = Re^{i(\vartheta - \arg(u))} \,:\, R = |\rho(u)|\left|1+\frac{\log^2 n + i}{n}\right|, \right. \\
&\qquad\qquad \left. \arg\left(1+\frac{\log^2 n+i}{n}\right) \leq |\vartheta| \leq \pi \right\}.
\end{aligned} \tag{42}$$

where $\gamma' = \{s : |s|=1, \Re s \leq 0\}\cup\{s \,:\, 0<\Re s<\log^2 n, \Im s = \pm 1\}$ is the major part of a Hankel contour $\gamma = \{s : |s|=1, \Re s\leq 0\}\cup\{s \,:\, 0<\Re s<\infty, \Im s=\pm1\}$.

For $z\in\Gamma$ and $|u|=1$ we trivially have

$$\exp\left(\frac{z^r}{r}\sum_{m=0}^{r}\frac{r^{r-m}}{(r-m)!}(u^m-1)\right) = 1+\mathcal{O}(r^{-1}).$$

Therefore this factor will have no importance. Now let us approximate

$$1-a_r(z,u)+(u-1)z^r\frac{r^{r-1}}{r!} = 1-ze^{a_r(z,u)}$$

for $z\in\Gamma_1$ and $|u|=1$. Using (28) and (30) we immediately obtain

$$\begin{aligned}
1-ze^{a_r(z,u)} &= 1-\rho_r(u)\left(1+\frac{s}{n}\right)e^{a_r(u)-b_r(u)\sqrt{-s/n}+\mathcal{O}(s/n)} \\
&= \sqrt{2}\sqrt{-\frac{s}{n}}\left(1+\mathcal{O}\left(\sqrt{\frac{|s|}{r}}\right)\right)
\end{aligned}$$

where we have used the substitution $z=\rho(u)\left(1+\frac{s}{n}\right)$ and the estimates $|s|\geq 1$, $u-1=\mathcal{O}(1)$, and $r\leq n$. Hence

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\Gamma_1} f_r(z,u)\frac{dz}{z^{n+1}} &= \frac{\rho_r(u)^{-n}}{\sqrt{2n}}\frac{1}{2\pi i}\int_{\gamma'}\frac{e^{-s}}{\sqrt{-s}}(1+\mathcal{O}(|s|^{\frac{1}{2}}r^{-\frac{1}{2}}))\,ds \\
&= \frac{\rho_r(u)^{-n}}{\sqrt{2\pi n}}(1+\mathcal{O}(r^{-\frac{1}{2}})),
\end{aligned}$$

where we have used

$$\frac{1}{2\pi i}\int_{\gamma'} \frac{e^{-s}}{\sqrt{-s}}(1+\mathcal{O}(|s|^{\frac{1}{2}}r^{-\frac{1}{2}})\,ds = \frac{1}{\sqrt{\pi}} + \mathcal{O}(r^{-\frac{1}{2}}).$$

By (28) and (39) we can estimate

$$\max_{z\in\Gamma_2,\,|u|=1} f_r(z,u) = \mathcal{O}(\sqrt{n})$$

which gives

$$\frac{1}{2\pi i}\int_{\Gamma_2} f_r(z,u)\,\frac{dz}{z^{n+1}} = \mathcal{O}\left(e^n\sqrt{n}e^{-\log^2 n}\right).$$

Thus

$$\frac{n!}{n^n}\frac{1}{2\pi i}\int_{\Gamma} f_r(x,e^u)\,\frac{dx}{x^{n+1}}, = e^{-n}\rho_r(u)^{-n}(1+\mathcal{O}(r^{-\frac{1}{2}})).$$

This completes the proof of Proposition 4. □

Now the proof of Theorem 1 can be finished.

Proposition 5 *Let $r\to\infty$ and $r = o(n^{\frac{2}{3}})$ as $n\to\infty$. Then (4) holds.*

Proof. Let $\phi_n(t)$ be the characteristic function of the normalized random variables (24), i.e.

$$\phi_n(t) = e^{-it\sqrt{np_r}} f_{r,n}(e^{it/\sqrt{np_r}}).$$

Since

$$\rho_r(u)^{-n} = e^n e^{i\sqrt{np_r}t-\frac{1}{2}t^2(1+\mathcal{O}(r^{-\frac{1}{2}}))+\mathcal{O}(t^3(np_r)^{-\frac{1}{2}})} \tag{43}$$

for $u = e^{it/\sqrt{np_r}}$ we directly obtain from (41)

$$\phi_n(t) = e^{-\frac{1}{2}t^2} + o(1).$$

Hence the limiting distribution is normal. □

Remark. It should be mentioned that it is possible to prove all parts of Theorem 1 just by using (27) and complex integration. The case of constant r was discussed in [5] and the preceding proof is essentially a variation of [5]. Furthermore, we can also adapt the proof of Proposition 5 to reprove Proposition 2 and 3. Instead of (43) we have to use

$$\begin{aligned}\rho_r(u)^{-n} &= e^n\left(1-p_r(u-1)+\mathcal{O}(r^{-2})\right)^{-n}\\ &= e^n e^{np_r(u-1)}\left(1+\mathcal{O}(nr^{-2})\right).\end{aligned}$$

Consequently

$$\mathbf{E}e^{itN_{r,n}} = f_{r,n}(e^{it}) = e^{C(e^{it}-1)} + o(1)$$

if $np_r \to C$, $0 \le C < \infty$. In this way we have also filled the gap $n(1-\varepsilon) < r \le n$ of Proposition 2. Therefore all parts of Theorem 1 are proved completely. $\Box$

In [5] it is shown that for any fixed $r \ge 0$

$$\mathbf{Pr}[N_{r,n} = k] = \frac{1}{\sqrt{2\pi n(p_r - 2rp_r^2)}} \exp\left(-\frac{(k-np_r)^2}{2n(p_r-2rp_r^2)}\right) + \mathcal{O}_r(n^{-\frac{3}{4}})$$

uniformly for $k \ge 0$, i.e. there is a local limit theorem, too. However, $\mathbf{E}N_{r,n} \not\sim \mathbf{V}N_{r,n}$ as $n \to \infty$. Therefore $N_{r,n}$ is surely no Poisson approximation.

But, if $r = r(n) \to \infty$ as $n \to \infty$ then $\mathbf{E}N_{r,n} \sim \mathbf{V}N_{r,n}$ and we might expect that $N_{r,n}$ a Poisson approximation. With the help of Proposition 4 we provide a local limit theorem for general r which really leads to the assertion that $N_{r,n}$ is a Poisson approximation if $r \to \infty$. Moreover, we estimate the total variation $d_{TV}(N_{r,n}, \mathrm{Po}(np_r))$.

Proposition 6 *Suppose that $r \le cn^{\frac{2}{3}}$ for some constant $c > 0$ then*

$$\mathbf{Pr}[N_{r,n} = k] = \frac{1}{\sqrt{2\pi np_r}} \exp\left(-\frac{(k-np_r)}{2np_r}\right) + \mathcal{O}(r^{\frac{1}{4}}n^{-\frac{1}{2}}) + \mathcal{O}(r^{\frac{3}{2}}n^{-1}) \qquad (44)$$

uniformly for $k \ge 0$ and $0 \le r \le cn^{\frac{2}{3}}$ as $n \to \infty$.

Furthermore, if $r \ge c\sqrt{n}$ for some constant $c > 0$ then

$$\mathbf{Pr}[N_{r,n} = k] = e^{-np_r}\frac{(np_r)^k}{k!} + \mathcal{O}(nr^{-2}) + \mathcal{O}(r^{-\frac{1}{2}}) \qquad (45)$$

uniformly for $k \ge 0$ and $r \ge c\sqrt{n}$ as $n \to \infty$.

Proof. Suppose that $r \le cn^{\frac{2}{3}}$ for some constant $c > 0$. By Cauchy's formula we directly obtain

$$\begin{aligned}
\mathbf{Pr}[N_{r,n} = k] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{r,n}(e^{it})e^{-ikt}\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(np_r-k)t - \frac{t^2}{2}np_r}\Big(1 + \mathcal{O}(nr^{-2}t^2) \\
&\qquad\qquad + \mathcal{O}(nr^{-\frac{3}{2}}t^3) + \mathcal{O}(r^{-\frac{1}{2}})\Big)\,dt \\
&= \frac{1}{\sqrt{2\pi np_r}} \exp\left(-\frac{(k-np_r)^2}{2np_r}\right) + \mathcal{O}(e^{-c_1 nr^{-\frac{3}{2}}}) \\
&\quad + \mathcal{O}(r^{\frac{1}{4}}n^{-\frac{1}{2}}) + \mathcal{O}(r^{\frac{3}{2}}n^{-1})
\end{aligned}$$

which proves (44).

If $r \geq c\sqrt{n}$ for some constant $c > 0$ then we obtain even more easily

$$\begin{aligned}\mathbf{Pr}[N_{r,n} = k] &= \frac{1}{2\pi i}\int_{|u|=1} f_{r,n}(u)u^{-k-1}\,du \\ &= \frac{1}{2\pi i}\int_{|u|=1} e^{np_r(u-1)}u^{-k-1}(1+\mathcal{O}(nr^{-2})+\mathcal{O}(r^{-\frac{1}{2}}))\,du \\ &= e^{-npr}\frac{(np_r)^k}{k!} + \mathcal{O}(nr^{-2}) + \mathcal{O}(r^{-\frac{1}{2}}).\end{aligned}$$

This completes the proof of Proposition 6. $\square$

Proposition 7 *The total variation of $d_{TV}(N_{r,n}, \mathrm{Po}(np_r))$ can be estimated by*

$$d_{TV}(N_{r,n}, \mathrm{Po}(np_r)) \leq \begin{cases} C\,r^{-\frac{1}{3}} & \text{if } 1 \leq r \leq n^{\frac{2}{5}}, \\ C\,n^{-\frac{1}{3}}r^{\frac{1}{2}} & \text{if } n^{\frac{2}{5}} \leq r \leq n^{\frac{4}{7}}, \\ C\,nr^{-\frac{11}{6}} & \text{if } n^{\frac{4}{7}} \leq r \leq n^{\frac{2}{3}}, \\ C\,r^{-\frac{1}{3}} & \text{if } n^{\frac{2}{3}} \leq r \leq n^{\frac{3}{4}}, \\ C\,n^{\frac{1}{2}}r^{-1} & \text{if } n^{\frac{3}{4}} \leq r \leq n, \end{cases} \tag{46}$$

where $C > 0$ is an absolute constant.

Proof. First, suppose that $r \leq n^{\frac{2}{5}}$. Set $\eta = r^{\frac{1}{6}}$. Since

$$\frac{(np_r)^k}{k!} = \frac{1}{\sqrt{2\pi np_r}}\exp\left(-\frac{(k-np_r)^2}{2np_r}\right)\left(1+\mathcal{O}\left(\frac{np_r|k-np_r|+|k-np_r|^3}{(np_r)^2}\right)\right)$$

we directly obtain from (44)

$$\sum_{|k-np_r|\leq\eta\sqrt{np_r}}\left|\mathbf{Pr}[N_{r,n}=k] - \frac{(np_r)^k}{k!}\right| = \mathcal{O}(\eta\sqrt{np_r}n^{-\frac{1}{2}}r^{\frac{1}{4}} + n^{-\frac{1}{2}}r^{\frac{3}{4}}) = \mathcal{O}(\eta^{-2}).$$

Furthermore, by a variant of Tschebyscheff's inequality

$$\mathbf{Pr}[|N_{r,n}-np_r| \geq \eta\sqrt{np_r}] \leq \frac{\mathbf{V}N_{r,n} + (\mathbf{E}N_{r,n}-np_r)^2}{\eta^2 np_r} = \mathcal{O}(\eta^{-2}).$$

A similar estimate holds for $\mathrm{Po}(np_r)$. Hence (46) holds for $r \leq n^{\frac{2}{5}}$.

In the same way we obtain (46) for $n^{\frac{2}{5}} \leq r \leq n^{\frac{4}{7}}$ resp. for $n^{\frac{4}{7}} \leq r \leq n^{\frac{2}{3}}$. You only have to choose η appropriately: $\eta = n^{\frac{1}{6}}r^{-\frac{1}{4}}$ resp. $\eta = n^{-\frac{1}{2}}r^{\frac{11}{12}}$.

If $n^{\frac{2}{3}} \leq r \leq n^{\frac{3}{4}}$ then we only have $\mathbf{V}N_{r,n} = \mathcal{O}(1)$. Using this and (45) yields

$$d_{TV}(N_{r,n}, \mathrm{Po}(np_r)) = \mathcal{O}(\eta r^{-\frac{1}{2}}) + \mathcal{O}(\eta^{-2}) = \mathcal{O}(r^{-\frac{1}{3}}),$$

where $\eta = r^{\frac{1}{6}}$.

In the case $n^{\frac{3}{4}} \le r \le n/2$ we use the inequality

$$\mathbf{Pr}[N_r \ge \eta] \le \frac{\mathbf{E}N_{r,n}}{\eta} = \mathcal{O}\left(\eta^{-1} n r^{-\frac{3}{2}}\right)$$

instead of Tschebyscheff's inequality and obtain

$$d_{TV}(N_{r,n}, \mathrm{Po}(np_r)) = \mathcal{O}(\eta r^{-\frac{1}{2}}) + \mathcal{O}\left(\eta^{-1} n r^{-\frac{3}{2}}\right) = \mathcal{O}(n^{\frac{1}{2}} r^{-1} (1 - r/n)^{-\frac{1}{4}}),$$

where $\eta = n^{\frac{1}{2}} r^{-\frac{1}{2}}$. Finally, if $n/2 \le r \le n$ then we trivially have

$$d_{TV}(N_{r,n}, \mathrm{Po}(np_r)) = \mathcal{O}(np_r) = \mathcal{O}(n^{\frac{1}{2}} r^{-1}).$$

□

It should be mentioned, too, that Arney and Bender [1] discussed a slightly more general case. Let D be a set of non-negative integers containing 0 and let $\overline{\mathcal{F}}_n$ denote those mappings $\varphi \in \mathcal{F}_n$ such that number of immediate predecessors of a point $|\varphi^{-1}(\{x\})|$ is not arbitrary but must be contained in D. Hence the exponential generating function for those mappings is given by

$$\overline{f}(x) = \sum_{n \ge 0} |\overline{\mathcal{F}}_n| \frac{x^n}{n!} = \frac{1}{1 - x\phi'(\overline{a}(x))},$$

where $\overline{a}(x)$ is the corresponding tree function and satisfies the functional equation

$$\overline{a}(x) = x \sum_{n \in D} \frac{\overline{a}(x)^n}{n!} = x\psi(\overline{a}(x)).$$

Let $\overline{\mathcal{F}} = \bigcup\limits_{n \ge 0} \overline{\mathcal{F}}_n$ and suppose that each $\varphi \in \overline{\mathcal{F}}_n$ is equally likely. Then the number of predecessors $\overline{N}_{r,n} = N_{r,n}|\overline{\mathcal{F}}$ can be interpreted as a sequence of random variables. Note that $\overline{\mathcal{F}}_n = \emptyset$ if $n \not\equiv 0 \bmod d$ and that $\overline{N}_{r,n}$ is surely very small if $r \not\equiv 1 \bmod d$, where

$$d = \gcd\{n \ge 1 \,:\, n \in D\}.$$

Let $\overline{f}_{r,nk}$ denote the number of mapping $\varphi \in \overline{\mathcal{F}}_n$ such that $\overline{N}_{r,n}(\varphi) = k$. Then

$$\begin{aligned} \overline{f}_r(x,u) &= \sum_{n,k} \overline{f}_{r,nk} \frac{x^n}{n!} u^k && (47) \\ &= \frac{1}{1 - x\psi'(\overline{a}_r(x,u))} \exp\left(x^r \sum_{m=1}^{r} \frac{b_{mr}}{m} (u^m - 1) \right), \end{aligned}$$

where $b_{mr} = [x^r]\left(x\psi'(\overline{a}(x))\right)^r$ and $\overline{a}_r(x,u)$ is the solution of

$$\overline{a}_r(x,u) = x\psi(\overline{a}_r(x,u)) + x^r(u-1)[z^r]\overline{a}(z)$$

(compare with [8]).

Suppose that D contains a number ≥ 2 and let $\beta > 0$ be defined by $\beta\psi'(\beta) = \psi(\beta)$, i.e. β is the solution of

$$\sum_{n\geq 1, n\in D} \frac{\beta^{n-1}}{n!} = 1.$$

Furthermore set

$$\lambda = \frac{\beta^2\psi''(\beta)}{\psi(\beta)}$$

and

$$\begin{aligned}\overline{p}_r &= \frac{\beta^{r-1}}{d\psi(\beta)^r}[x^r]\overline{a}(x) \\ &= \frac{r^{-\frac{3}{2}}}{\sqrt{2\pi\lambda}}(1+\mathcal{O}(r^{-1})) \quad (\text{for } r \equiv 1 \bmod d.)\end{aligned}$$

Then we can can prove an analogue to Theorems 1 and 2 by using the same methods as above. (Note that asymptotic expansions for $\mathbf{E}\overline{N}_{r,n}$ and $\mathbf{V}\overline{N}_{r,n}$, which are needed for large deviation estimates, can be obtained by using (47) and singularity analysis [6].)

Theorem 3 *Let $\overline{N}_r$ denote the number of predecessors in random mappings $\overline{\mathcal{F}}$ and suppose that D contains a number ≥ 2.*

- *If $n\overline{p}_r \to \infty$ $(r \equiv 1 \bmod d)$ as $n \to \infty$ then*

$$\frac{\overline{N}_{r,n} - n\overline{p}_r}{\sqrt{n(\overline{p}_r + \overline{p}_r^2(1-2r-\lambda^{-1}))}} \xrightarrow{d} \mathcal{N}(0,1) \quad (n \equiv 0 \bmod d, n \to \infty).$$

- *If $n\overline{p}_r \to C$ $(r \equiv 1 \bmod d)$ as $n \to \infty$ for some real constant $C > 0$ then $\overline{N}_{r,n}$ has a Poisson limiting distribution, i.e. for any $k \geq 0$*

$$\lim_{d|n,\, n\to\infty} \mathbf{Pr}[\overline{N}_{r,n} = k] = e^{-C}\frac{C^k}{k!}.$$

- *If $n\overline{p}_r \to 0$ $(r \equiv 1 \bmod d)$ as $n \to \infty$ then $\overline{N}_{r,n}$ degenerates, i.e.*

$$\lim_{d|n,\, n\to\infty} \mathbf{Pr}[\overline{N}_{r,n} = 0] = 1.$$

Furthermore, if $r \to \infty$ then $\overline{N}_{r,n}$ is a Poisson approximation. The total variation $d_{TV}(\overline{N}_{r,n}, \mathrm{Po}(n\overline{p}_r))$ $(r \equiv 1 \bmod d,\ d|n)$ can be estimated as the corresponding total variation in Theorem 2.

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