# CORRELATIONS ON THE STRATA OF A RANDOM MAPPING

Michael Drmota

Technical University of Vienna

Abstract. L. Mutafchiev [Mu] observed that $\mu_n(k)$ the number of points in random mappings on $\{1,\dots,n\}$ with distance $k$ to cyclic points is asymptotically identically distributed with the number of cyclic points $\mu_n(0)$ if $k = o(\sqrt{n})$ by using the fact that $(\mu_n(k) - \mu_n(0))/\sqrt{n} \to 0$ in probability. The main purpose of this paper is to provide a local limit theorem for the joint distribution of $\mu_n(k_1), \mu_n(k_2)$. Furthermore we prove a local version of Mutafchiev's result. All results are obtained by applying singularity analysis of multivariate generating functions.

## 1. Introduction

Let $\mathcal{R}_n$ denote the set of all mappings $\varphi : \{1,\dots,n\} \to \{1,\dots,n\}$. Obviously there are $n^n$ different mappings and any $\varphi \in \mathcal{R}_n$ can be represented by its functional graph $G_\varphi$, where $(i, \varphi(i))$, $i = 1,\dots,n$, are the (directed) edges of $G_\varphi$. Each component of $G_\varphi$ consists of exactly one cycle of length $\geq 1$ and each cyclic point is the root of a (labeled) tree. Hence there is a unique path from a non-cyclic point to the correcponding root which is the next cyclic point. The length of the path is the distance to cyclic points. For any $k \geq 0$ let $\mu_n(k)(\varphi)$ denote the number of nodes in $G_\varphi$ with distance $k$ to cyclic points.

Throughout the paper we will assume that the mappings $\varphi \in \mathcal{R}_n$ are equally likely, i.e. $\mathbf{P}(\varphi) = n^{-n}$. Usually a mapping $\varphi \in \mathcal{R}_n$ is called random mapping in such a context. Mutafchiev [Mu] has proved the following theorem for the random variable $\mu_n(k)$.

**Theorem.** *If $n \to \infty$ and $k = o(\sqrt{n})$ then the distribution of $\mu_n(k)/\sqrt{n}$ tends weakly to the Rayleigh distribution. (The density of the Rayleigh distribution is given by $xe^{-\frac{1}{2}x^2}$, $0 < x < \infty$.)*

It should be noted that the case of cyclic points ($k = 0$) is due to B. Harris [Ha] whereas the disjoint case $k/\sqrt{n} \to \alpha > 0$ is due to

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Proskurin [Pr]. Mutafchiev [Mu] derived his result by observing that $(\mu_n(k) - \mu_n(0))/\sqrt{n} \to 0$ in probability as $n \to \infty$. Let us first give an interpretation of this property. Since the graph of a "typical" random mapping consists of $m = \Theta(\sqrt{n})$ trees ([Ha,Ko]) these trees constitute a representative sample of realizations of a Galton-Watson branching process [Se] defined by the probabilities $p_k = e^{-1}\frac{1}{k!}$ $(k \ge 0)$. (A particle produces $k$ children with probability $p_k$. See [Ko] for a precise description of the relations between random mappings and Glaton-Watson branching processes.) Since

$$\sum_{k\ge 0} kp_k = 1$$

(which means that this branching process is critical) the average number of children of $m$ initial particles is exactly $m$ and so on. From this point of view the (asymptotic) identical distribution of $\mu_n(k)$ is quite natural.

The aim of this paper is to provide a more precise insight into this phenomenon. We will provide a local limit theorem of the joint distribution of $\mu_n(k_1), \mu_n(k_2)$ as $n \to \infty$, where $k_1, k_2 \ge 0$ are arbitrary but fixed. From this and from a local version of Mutafchiev's theorem it follows that the conditional limiting distribution of $\mu_n(k_2)$ where it is assumed that $\mu_n(k_1) = m_1$ is asymptotically normal with mean value $\sim m_1$ and variance $\sim |k_1 - k_2| m_1$.

It should be further mentioned that we use a completely different method, namely multivariate generating functions and singularity analysis.

## 2. The Combinatorial Background

As mentioned in the introduction the functional graph $G_\varphi$ of a mapping $\varphi \in \mathcal{R}_n$ can be interpreted as a set of cycles of rooted labeled trees. Furthermore these rooted labeled trees can be recursively described as a root followed by a set of trees. It is well known (cf. [VF,HP]) that such combinatorial relations can be transformed into terms of the corresponding (exponential) generating functions. Let

$$w(x) = \sum_{n\ge 1} \frac{n^{n-1}}{n!} x^n \tag{2.1}$$

be the exponential generating function of rooted labeled trees of size $n$ and

$$r(x) = \sum_{n\ge 0} \frac{n^n}{n!} x^n \tag{2.2}$$

the exponential generating function of (random) mappings on $\{1,\dots,n\}$. Then

$$r(x) = \exp\left(\log \frac{1}{1-w(x)}\right) = \frac{1}{1-w(x)} \tag{2.3}$$

and

$$w(x) = x\exp(w(x)). \tag{2.4}$$

In order to verify (2.3) and (2.4) you only have to check that if $c(x)$ is the exponential generating function of a combinatorial structure then

$$\log \frac{1}{1-c(x)} = \sum_{m\geq 1} \frac{1}{m} c(x)^m$$

corresponds to the cycle construction and

$$\exp(c(x)) = \sum_{m\geq 0} \frac{1}{m!} c(x)^m$$

corresponds to the set construction. (Conversely we can use the functional equations (2.3) and (2.4) to obtain (2.1) and (2.2) by Lagrange inversion.)

The essential thing for our purpose is that it is also possible to mark a parameter in such a combinatoral construction which leads to a bivariate generating function according to the size of this parameter in (random) mappings on $\{1,\dots,n\}$. For example, every tree corresponds to cyclic point. Hence

$$r(x,u) = \exp\left(\log \frac{1}{1-uw(x)}\right) = \frac{1}{1-uw(x)} = \sum_{n,m} y_{nm} \frac{x^n}{n!} u^m$$

is the bivariate generating function of a (random) mapping on $\{1,\dots,n\}$, where cyclic points are marked, i.e. $y_{nm}$ is the number of (random) mappings $\varphi$ on $\{1,\dots,n\}$ where the corresponding functional graph $G_\varphi$ has $m$ cyclic points. In the same fashion $x\exp(uw(x))$ corresponds to those trees where nodes with distance 1 to the root are marked. Thus

$$r_1(x,u) = \frac{1}{1-x\exp(uw(x))}$$

counts those (random) mappings on $\{1,\dots,n\}$ with $\mu_n(1) = m$. Obviously we can mark two (or more) different parameters during one procedure. This leads to the following observation, where we use the notation $L_d(x,v)$ defined by

$$L_0(x,v) = v \quad \text{and by}$$
$$L_{d+1}(x,v) = x\exp(L_d(x,v)).$$

**Proposition 1.** *Let $k \ge 0$, resp. $0 \le k_1 \le k_2$ be given. Then*

$$r_k(x,u) = \frac{1}{1 - L_k(x, uw(x))} = \sum_{n,m} y_{nm}^{(k)} \frac{x^n}{n!} u^m \tag{2.5}$$

*resp.*

$$\begin{aligned} r_{k_1,k_2}(x,u_1,u_2) &= \frac{1}{1 - L_{k_1}(x, u_1 L_{k_2-k_1}(x, u_2 w(x)))} \\ &= \sum_{n,m_1,m_2} y_{nm_1m_2}^{(k_1,k_2)} \frac{x^n}{n!} u_1^{m_1} u_2^{m_2} \end{aligned} \tag{2.6}$$

*is the exponential generating function of the numbers $y_{nm}^{(k)}$ resp. $y_{nm_1m_2}^{(k_1,k_2)}$ of those (random) mappings $\varphi$ on $\{1,\dots,n\}$ such that $\mu_n(k) = m$ resp. such that $\mu_n(k_1) = m_1$ and $\mu_n(k_2) = m_2$.*

*Proof.* By using the recursive description for labeled trees we obtain by induction that the exponential generating function $L_k(x, uw(x))$ corresponds to those trees where nodes with distance $k$ to the root are marked. Similarly $L_{k_1}(x, u_1 L_{k_2-k_1}(x, u_2(w(x)))$ corresponds to those trees where nodes with distance $k_1$ are marked by $u_1$ and nodes with distance $k_2 = k_1 + (k_2 - k_1)$ are marked by $u_2$. Hence (2.5) and (2.6) follow. □

In the next section we will use these representations and Cauchy's formula in order to obtain a multivariate asymptotic expansion for the numbers $y_{nm}^{(k)}$ resp. $y_{nm_1m_2}^{(k_1,k_2)}$.

## 3. Singulariy Analysis

Our main theorem is the following one.

**Theorem 1.** *Let $0 \le k_1 < k_2$ be arbitrary fixed integers. Then*

$$\begin{aligned} \mathbf{P}[\mu_n(k_1) = m_1, \mu_n(k_2) = m_2] &= \frac{y_{nm_1m_2}^{(k_1,k_2)}}{n^n} \\ = \frac{1}{n}\sqrt{\frac{m_1}{2\pi(k_2-k_1)}} \exp\left(-\frac{m_1^2}{2n} - \frac{(m_1-m_2)^2}{2m_1(k_2-k_1)}\right) &\left(1 + \mathcal{O}(n^{-\frac14})\right) \end{aligned} \tag{3.1}$$

*as $n \to \infty$ uniformly for $C_1\sqrt{n} \le m_i \le C_2\sqrt{n}$, $i = 1,2$, and $|m_1 - m_2| \le C_3 n^{\frac14}$, where $C_1 < C_2$, $C_3$ are arbitrary positive constants.*

In the same manner we get a local version of Mutafchiev's result [Mu].

**Theorem 2.** *Suppose that $k = k(n) = o(\sqrt{n})$ as $n \to \infty$. Then*

$$\mathbf{P}[\mu_n(k) = m] = \frac{y_{nm}^{(k)}}{n^n} = \frac{m}{n} \exp\left(-\frac{m^2}{2n}\right)(1 + o(1)) \tag{3.2}$$

*as $n \to \infty$ uniformly for $C_1\sqrt{n} \le m \le C_2\sqrt{n}$, where $C_1 < C_2$ are arbitrary positive constants. If $k \ge 0$ is fixed then the error term $o(1)$ can be replaced by $\mathcal{O}(n^{-1/2})$.*

Furthermore we get a normal limiting distribution for the conditional distribution.

**Corollary.** *Let $k_1, k_2 \ge 0$ be two different integers. Then*

$$\mathbf{P}[\mu_n(k_2) = m_2 | \mu_n(k_1) = m_1] \\ = \frac{1}{\sqrt{2\pi|k_1 - k_2|m_1}} \exp\left(-\frac{(m_2 - m_1)^2}{2m_1|k_1 - k_2|}\right)\left(1 + \mathcal{O}(n^{-\frac{1}{4}})\right)$$

*as $n \to \infty$ uniformly for $C_1\sqrt{n} \le m_i \le C_2\sqrt{n}$, $i = 1, 2$, and $|m_1 - m_2| \le C_3 n^{\frac{1}{4}}$, where $C_1 < C_2$, $C_3$ are arbitrary positive constants.*

*Proof of Theorem 1.* Basically we will use Cauchy's fomula

$$\frac{y_{nm_1m_2}^{(k_1,k_2)}}{n!} = \frac{1}{(2\pi i)^3} \int_\gamma \int_{\gamma_2} \int_{\gamma_1} r_{k_1k_2}(x, u_1, u_2) \frac{du_1}{u_1^{m_1+1}} \frac{du_2}{u_2^{m_2+1}} \frac{dx}{x^{n+1}},$$

where $\gamma, \gamma_2$, and $\gamma_1$ are properly chosen paths of integration around the origin. (They will be described explicitly in the sequel.) It will turn out that in the case $m_1, m_2 = \mathcal{O}(\sqrt{n})$, $m_1 - m_2 = \mathcal{O}(\sqrt{m_1})$ the dominating part of the above integral comes from the vicinity of $x = \frac{1}{e}$ and $u_1 = u_2 = 1$ which is also the region where the singularity branch determining the region of convergence of the power series $r_{k_1k_2}(x, u_1, u_2)$ is situated. First we will only concentrate on this part of the integration since the remaining parts can be estimated trivially and give exponentially small terms. We will sketch how to estimate the remaining parts. (See [FO] or [Gi] for a more detailed analysis of error terms in similar kinds of problems.)

It is well known (cf. [HP]) that $w(x)$, the unique analytic solution of $w(x) = x\exp(w(x))$, has radius of convergence $\frac{1}{e}$ and that the local behaviour around $x = \frac{1}{e}$ is given by

$$w(x) = 1 - \sqrt{2}\sqrt{1 - ex} + w_2(1 - ex) + w_3(1 - ex)^{\frac{3}{2}} + \cdots . \tag{3.3}$$

Furthermore, $w(x)$ can be continued analytically to some region $|x| < \frac{1}{e}+\varepsilon$, $\arg\left(x-\frac{1}{e}\right) \ne 0$. With help of this property it is possible to choose the paths of integration $\gamma$, $\gamma_1$, and $\gamma_2$ in the following way.

Let $\gamma = \Gamma \cup \Gamma'$, where

$$\begin{aligned}\Gamma = &\left\{x = \frac{1}{e}\left(1+\frac{t}{n}\right) \,:\, |t| = 1, \Re t < 0\right\}\\ &\cup \left\{x = \frac{1}{e}\left(1+\frac{t}{n}\right) \,:\, \Im t = \pm 1, 0 \le \Re t \le \log^2 n\right\}\end{aligned}$$

is a part of a Hankel contour and

$$\Gamma' = \left\{x \in \mathbf{C} \,:\, |x| = \frac{1}{e}\left|1+\frac{i+\log^2 n}{n}\right|, |\arg x| \ge \left|\arg\left(1+\frac{i+\log^2 n}{n}\right)\right|\right\}.$$

Furthermore set $\gamma_2 = \{u_2 \in \mathbf{C} \,:\, |u_2| = 1\} = \Gamma_2 \cup \Gamma_2'$, where

$$\Gamma_2 = \left\{u_2 = e^{is/\sqrt{m_1}} \,:\, -\log^2 m_1 \le s_1 \le \log^2 m_1\right\}$$

If $x \in \Gamma$ and $u_2 \in \Gamma_2$ then let

$$\begin{aligned}\Gamma_1 = \Gamma_1(x,u_2) = &\left\{u_1 = U_1(x,u_2) - \frac{1}{m_1}e^{-i\vartheta} \,:\, -\pi < \vartheta < \pi\right\}\\ &\cup \left\{u_1 = U_1(x,u_2) - \frac{1}{m_1}e^{i\pi}t \,:\, 1 \le t \le \log^2 m_1\right\}\\ &\cup \left\{u_1 = U_1(x,u_2) - \frac{1}{m_1}e^{-i\pi}t \,:\, 1 \le t \le \log^2 m_1\right\}\end{aligned}$$

be the dominating part of $\gamma_1 = \Gamma_1 \cup \Gamma_1'$, where

$$\Gamma_1' = \{u_1 \in \mathbf{C} \,:\, |u_1| = |U_1(x,u_2) + (\log^2 m_1)/m_1|\}$$

and

$$\begin{aligned}U_1(x,u_2) = 1 - (u_2-1) - \frac{k_2-k_1-2}{2}(u_2-1)^2 + \sqrt{2}\sqrt{1-ex}\\ + \mathcal{O}\left(|1-ex| + |u_2-1|^3 + |1-ex|^{\frac{1}{2}}|u_2-1|\right)\end{aligned} \tag{3.4}$$

satisfies

$$L_{k_1}(x, U_1(x,u_2)L_{k_2-k_1}(x, u_2 w(x))) = 1. \tag{3.5}$$

The local expansion (3.4) for $U_1(x,u_2)$ can be determined easily by implicit differentiation and by using (3.3). If $x \notin \Gamma$ or $u_2 \notin \Gamma_2$ then

$$\gamma_1 = \{u_1 \in \mathbf{C} \,:\, |u_1| = 1 + (\log^2 m_1)/m_1\}.$$

At this point it should be noted that we could obtain (3.4) by considering the direct expansion

$$\begin{aligned} L_{k_1}&(x, u_1 L_{k_2-k_1}(x, u_2 w(x))) - 1 \\ &= (u_1-1) + (u_2-1) + \frac{k_1}{2}(u_1-1)^2 + \frac{k_2}{2}(u_2-1)^2 \\ &\quad + (k_1+1)(u_1-1)(u_2-1) - \sqrt{2}\sqrt{1-ex} \\ &\quad - \sqrt{2}(k_1+1)(u_1-1)\sqrt{1-ex} - \sqrt{2}(k_2+1)(u_2-1)\sqrt{1-ex} \\ &\quad + \mathcal{O}\left(|1-ex| + |u_1-1|^3 + |u_2-1|^3\right). \end{aligned} \tag{3.6}$$

and by solving the equation (3.5) asymptotically. Furthermore (3.6) is needed to estimate the error terms in the following integrals

As mentioned above we will first concentrate on the dominating part of integration $x \in \Gamma$, $u_2 \in \Gamma_2$, $u_1 \in \Gamma_1$.

For $u_2 \in \Gamma_2$ and $x \in \Gamma$ we get

$$\begin{aligned} \frac{1}{2\pi i} \int\limits_{\Gamma_1} & r_{k_1 k_2}(x, u_1, u_2) \, \frac{du_1}{u_1^{m_1+1}} \\ &= U_1(x,u_2)^{-m_1-1}\left(1 + \mathcal{O}(|u_2-1| + |1-ex|^{1/2})\right). \end{aligned}$$

Hence, by using the substitutions $u_2 = e^{is/\sqrt{m_1}}$ and $x = \frac{1}{e}\left(1 + \frac{t}{n}\right)$ we immediately obtain

$$\begin{aligned} \frac{1}{(2\pi i)^2} & \int\limits_{\Gamma} \int\limits_{\Gamma_2} U_1(x,u_2)^{-m_1-1}\left(1 + \mathcal{O}(|u_2-1| + |1-ex|^{1/2})\right) \frac{du_2}{u_2^{m_2+1}} \, \frac{dx}{x^{n+1}} \\ &= \frac{1}{4\pi^2 i} \frac{e^n}{\sqrt{m_1} n} \times \\ &\quad \times \int\limits_{\overline{\Gamma}} \int\limits_{\overline{\Gamma}_2} \exp\left(-t - \sqrt{2}\frac{m_1}{\sqrt{n}}\sqrt{-t} - \frac{k_2-k_1}{2}s^2 - i\frac{m_2-m_1}{\sqrt{m_1}}s\right) ds\, dt \times \\ &\quad \times \left(1 + \mathcal{O}\left(\frac{m_1}{n} + m_1^{-\frac{1}{2}} + (\frac{m_1}{n})^{\frac{1}{2}}\right)\right) \\ &= \frac{e^n}{2\pi}\sqrt{\frac{m_1}{n^3(k_2-k_1)}} \exp\left(-\frac{m_1^2}{2n} - \frac{(m_1-m_2)^2}{2m_1(k_2-k_1)}\right)\left(1 + \mathcal{O}(n^{-\frac{1}{4}})\right), \end{aligned}$$

where we have used (see [Dr])

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{\overline{\Gamma}} &\exp\left(-t-\sqrt{2}\frac{m_1}{\sqrt{n}}\sqrt{-t}\right)\,dt \\
&= \frac{1}{2\pi i}\int_H \exp\left(-t-\sqrt{2}\frac{m_1}{\sqrt{n}}\sqrt{-t}\right)\,dt + \mathcal{O}(e^{-\log^2 n}) \\
&= \frac{1}{\pi i}\int_{-\infty}^{\infty} \exp\left(-u^2+i\sqrt{2}\frac{m_1}{\sqrt{n}}u\right)u\,du + \mathcal{O}(e^{-\log^2 n}) \\
&= \frac{m_1}{\sqrt{2\pi n}}\exp\left(-\frac{{m_1}^2}{2n}\right) + \mathcal{O}(e^{-\log^2 n}),
\end{aligned}
$$

(where $H = \{t \in \mathbf{C} : \Re t < 0, |t| = 1\} \cup \{t \in \mathbf{C} : \Re \geq 0, \Im t = \pm 1\}$ is a Hankel contour), and

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\log^2 m_1}^{\log^2 m_1} &\exp\left(-\frac{k_2-k_1}{2}s^2 - i\frac{m_2-m_1}{\sqrt{m_1}}s\right)\,ds \\
&= \frac{1}{\sqrt{2\pi(k_2-k_1)}}\exp\left(-\frac{(m_1-m_2)^2}{2m_1(k_2-k_1)}\right) + \mathcal{O}(e^{-\log^2 m_1}).
\end{aligned}
$$

As an example how to estimate the remaining integrals let us consider the case where $x \in \Gamma$, $u_2 \in \Gamma_2' = \gamma_2 \setminus \Gamma_2$, and $u_1 \in \gamma_1 = \{u_1 \in \mathbf{C} : |u_1| = 1 + (\log^2 m_1)/m_1\}$. First observe that

$$
\max_{x\in\Gamma, u_2\in\Gamma_2'} |L_{k_2-k_1}(x, u_2 w(x))| \leq 1 - c_1\frac{\log^4 m_2}{m_2}
$$

for some positive constant $c_1$. Hence

$$
\max_{x\in\Gamma, u_2\in\Gamma_2', u_1\in\gamma_1} |L_{k_1}(x, u_1 L_{k_2-k_1}(x, u_2 w(x)))| \leq 1 - c_2\frac{\log^4 m_2}{m_2}
$$

and we can estimate the following integral by

$$
\begin{aligned}
&\left|\frac{1}{(2\pi i)^3}\int_{\Gamma}\int_{\Gamma_2'}\int_{\gamma_1} r_{k_1k_2}(x,u_1,u_2)\,\frac{du_1}{u_1^{m_1+1}}\,\frac{du_2}{u_2^{m_2+1}}\,\frac{dx}{x^{n+1}}\right| \\
&\qquad \leq c_2 e^n m_2\left(1+\frac{\log^2 m_1}{m_1}\right)^{-m_1} \\
&\qquad \leq \frac{e^n}{n^3}.
\end{aligned}
$$

The other integrals can be tackled in a similar fashion. You always have to observe that the integrand is not too large and that the modulus of one variable (in the previous example it was $u_1$) is large enough.

This proves (3.1). $\square$

*Proof of Theorem 2.* As in the proof of Theorem 1 we can use Cauchy's formula

$$\frac{y_{nm}^{(k)}}{n!} = \frac{1}{(2\pi i)^2} \int_\gamma \int_{\gamma_1} r_k(x,u)\, \frac{du}{u^{m+1}}\, \frac{dx}{x^{n+1}}, \tag{3.7}$$

where $\gamma = \Gamma \cup \Gamma'$ as above and $\gamma_1 = \gamma_1(x) = \Gamma_1 \cup \Gamma_1'$ which can be described in the following way. If $x \in \Gamma$ then let $U = U(x)$ be the solution of $L_k(x, U(w(x))) = 1$ and set

$$\begin{aligned}\Gamma_1 = &\left\{u_1 = U_(x) - \frac{1}{n}e^{-i\vartheta} :\ -\pi < \vartheta < \pi\right\}\\ &\cup \left\{u = U(x) - \frac{1}{n}e^{i\pi}t :\ 1 \le t \le \sqrt{n}\log^2 n\right\}\\ &\cup \left\{u = U(x) - \frac{1}{n}e^{-i\pi}t :\ 1 \le t \le \sqrt{n}\log^2 n\right\}\end{aligned}$$

and

$$\Gamma_1' = \{u_1 \in \mathbf{C} \,:\, |u_1| = 1 + (\log^2 n)/\sqrt{n}\}.$$

Finally if $x \in \Gamma'$ then $\gamma_1 = \{u \in \mathbf{C} \,:, |u| = 1\}$. Observing that

$$U(x) = \frac{1}{w(x)}(1 + \mathcal{O}(k(1-ex))),$$

we can evaluate the dominating part of (3.7) to

$$\begin{aligned}&\frac{1}{(2\pi i)^2} \int_\Gamma \int_{\Gamma_1} r_k(x,u)\, \frac{du}{u^{m+1}}\, \frac{dx}{x^{n+1}}\\ &\qquad = \frac{1}{2\pi i} \int_\Gamma w(x)^{-m-1}(1 + \mathcal{O}(k(1-ex)))^m \frac{dx}{x^{n+1}}\\ &\qquad \sim \frac{me^n}{\sqrt{2\pi n^3}} \exp\left(-\frac{m^2}{2n}\right).\end{aligned}$$

The remaining integrals can be easily estimated. This proves (3.2). Furthermore, if $k \ge 0$ is fixed then we immediately obtain the proposed error term $1 + \mathcal{O}(n^{-1/2})$. $\square$

*Remark.* Obviously the above technique can be generalized to obtain a local limit theorem for the joint distribution of $\mu_n(k_1), \ldots \mu_n(k_l)$ $(l \ge 3)$. You only have to use a multivariate saddle point method instead of the single saddle point technique on $\Gamma_2$.

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Department of Discrete Mathematics
Technical University of Vienna
Wiedner Hauptstrasse 8-10/118
A-1040 Vienna, AUSTRIA
E-mail: mdrmota@ecx.tuwien.ac.at