SOME RECENT DEVELOPMENTS ON THE SARNAK CONJECTURE

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Abstract. In this article we survey some recent developments on the Sarnak conjecture in the context of product spaces, of prime number theorems, of polynomial subsequences, and of morphic sequences. We also present a new result on products of automatic sequences.

1. Introduction

1.1. **History - Möbius disjointness.** The Möbius function μ (defined by $\mu(1) = 1$, $\mu(p_1 p_2 \cdots p_k) = (-1)^k$ for different prime numbers p_j and $\mu(n) = 0$ else) is a very important multiplicative function which captures the multiplicative structure of the natural numbers very well. Therefore, we expect it to be independent from sequences with simple additive structure. One way to capture this independence is the so called Möbius orthogonality. We say that a bounded sequence of complex numbers $(a(n))_{n\in\mathbb{N}}$ is *orthogonal to the Möbius function* μ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} a(n)\mu(n) = 0. \tag{1}$$

One of the first formalizations of the *Möbius randomness law* was proposed by Iwaniec and Kowalski [32], which states that any "reasonable" sequence of complex numbers $(a(n))_{n\in\mathbb{N}}$ is orthogonal to μ. This Möbius randomness law proved to be true in almost all instances and also very useful as a guiding idea, but it does not provide an answer to the question which sequences are "reasonable".

This vague randomness law was posed in a much more concrete way by Peter Sarnak in 2010. Let T be a continuous map of a compact metric space X. Following Peter Sarnak [50], we say that T, or, more precisely, the topological dynamical system[1] (X, T) is Möbius disjoint

[1]For the reader's convenience we recall some definitions for dynamical systems in the Appendix.

(or Möbius orthogonal) if:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} f(T^n x)\mu(n) = 0 \tag{2}$$

holds for every $f \in C(X)$ and $x \in X$. This means that every sequence of the form $a(n) = f(T^n x)$ is orthogonal to μ.

Conjecture 1.1 (Sarnak conjecture (2010) [50])**.** *Each zero entropy continuous map T of a compact metric space X is Möbius disjoint.*

Thus, Sarnak's conjecture states that any sequence of the form $a(n) = f(T^n x)$, with the assumptions above, is "reasonable".

Zero entropy dynamical systems are also called deterministic and the corresponding sequences $a(n) = f(T^n x)$ are called deterministic sequences.

The Sarnak conjecture was already confirmed for several instances, for example for nilsequences and automatic sequences [29, 46]. The (possibly) simplest non-trivial examples of sequences of these kinds are Sturmian words (for nilsequences) or the Thue-Morse sequence (for automatic sequences).

As a general reference for this progress we mention the 2018 article [21] by Ferenczi, Kułaga-Przymus and Lemańczyk that gives an overview on Sarnak's conjecture, focusing on ergodic theory aspects.

The purpose of this article is to present some more recent progress on product spaces, on so-called prime number theorems, on polynomial subsequences and on morphic sequences. Topics that are thoroughly discussed in [21] will only be sketched.

1.2. **Related Problems.**

1.2.1. *Logarithmic Version.* We say that a topological dynamical system (X, T) is logarithmically Möbius disjoint (or logarithmically Möbius orthogonal) if

$$\lim_{N\to\infty} \frac{1}{\log(N)} \sum_{n\le N} \frac{1}{n} f(T^n x)\mu(n) = 0 \tag{3}$$

holds for every $f \in C(X)$ and $x \in X$. The weaker version of the Sarnak conjecture is called the logarithmic Sarnak conjecture.

Conjecture 1.2 (Logarithmic Sarnak conjecture)**.** *Each zero entropy continuous map T of a compact metric space X is logarithmically Möbius disjoint.*

1.2.2. *Chowla Conjecture.* The Chowla conjecture formalizes the conjecturally expected independence of the shifts of the Möbius function.

Conjecture 1.3. *For any fixed integer m and exponents $a_1, a_2, \ldots, a_m \geq 0$ with at least one of the a_i odd, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \mu(n+1)^{a_1} \ldots \mu(n+m)^{a_m} = 0.$$

A tempting interpretation of this conjecture is that every pattern of length m appears with the same frequency. However, this is not completely accurate as the appearances of 0's are highly correlated and do not contribute in the sum above. A better interpretation is that every pattern of length m has the same asymptotic frequency conditioned on the pattern of zeros.

The Chowla conjecture is still mostly open. The case $m = 1$ is, of course, known. This case is equivalent to the prime number theorem. Moreover, the cases $m = 2$ and m odd are also known to be true for all scales N outside of a set of zero logarithmic density, thanks to an impressive result by Tao and Teräväinen [56].

It was already observed by Sarnak [50] that there is a strong relation between the Chowla and the Sarnak conjecture.

Theorem 1.4. *The Chowla conjecture implies Sarnak's conjecture.*

An ergodic proof can be found in [18].

1.2.3. *Logarithmic Sarnak and Chowla Conjecture.* Similarly to the Sarnak Conjecture it is possible to formulate the corresponding logarithmic version of the Chowla conjecture.

Conjecture 1.5. *For any fixed integer m and exponents $a_1, a_2, \ldots, a_m \geq 0$ with at least one of the a_i odd, we have*

$$\lim_{N\to\infty} \frac{1}{\log(N)} \sum_{n\leq N} \frac{1}{n} \mu(n+1)^{a_1} \ldots \mu(n+m)^{a_m} = 0.$$

On the one hand the logarithmic Chowla conjecture is clearly weaker than the original Chowla conjecture. On the other hand there are several strong results on the logarithmic version. For example, Tao proved that the logarithmic version of the Chowla conjecture holds for $m = 2$ [54], and, together with Teräväinen, he proved that the Chowla conjecture holds for odd m [57]. Furthermore, Tao proved [55] that the logarithmic versions of the Chowla conjecture and the Sarnak conjecture are equivalent. This has also some important implications observed by Frantzikinakis [23], for a thorough discussion see [21, Section 3.4]. Furthermore, it was observed by Gomilko, Kwietniak and Lemańczyk [27] that the logarithmic Chowla conjecture implies the Chowla conjecture along a subsequence.

2. Product dynamical systems

2.1. **Formulation of the problem.** It is well known that for two (deterministic) topological dynamical systems (X,T) and (Y,S), the product space $(X\times Y, T\times S)$ is again a (deterministic) topological dynamical system, the so called *product dynamical system*. Following the Sarnak conjecture, we expect that the product dynamical system of two deterministic topological dynamical systems (X,T) and (Y,S), is Möbius disjoint, i.e.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\le N} h(T^n x, S^n y)\mu(n) = 0,$$

holds for every $h\in C(X\times Y), x\in X$ and $y\in Y$. This clearly implies the Möbius disjointness for (X,T) and (Y,S), as we can choose $h(x,y)=f(x)$ or $h(x,y)=g(y)$ for any $f\in C(X), g\in C(Y)$.

On the other hand we can ask – independently of the Sarnak conjecture – whether the Möbius disjointness for (X,T) and (Y,S) implies the Möbius disjointness for the product dynamical system $(X\times Y, T\times S)$?

2.2. **Known Results.** Let $q\ge 2$ be a given integer. A complex valued sequence $f(n)$ of modulus 1 is called *strongly q-multiplicative* if

$$f(aq+b)=f(a)f(b)$$

holds for all non-negative integers a and for all integers $b\in\{0,1,\dots,q-1\}$. For example, if $s_q(n)$ denotes the q-ary sum-of-digits function then $f(n)=\mathrm{e}(\theta s_q(n))$[2] is strongly q-multiplicative for all real θ. Such sequences are deterministic and they are orthogonal to the Möbius function [42].[3] The most prominent sequence of this form is the (± 1 version of the) Thue-Morse sequence $T(n)=(-1)^{s_2(n)}$.

At least for such sequences there is a kind of product property in the sense of Möbius orthogonality.

[2] We use the abbreviation $\mathrm{e}(x)=\exp(2\pi i x)$ for $x\in\mathbb{R}$.

[3] Actually the authors of [42] prove a prime number theorem for strongly q-multiplicative sequences by Vaughan's method. However, the same method applies for Möbius orthogonality.

Theorem 2.1 ([12]). *Suppose that $q_1, q_2 \geq 2$ are co-prime integers. Then for all strongly q_1-multiplicative sequences $f(n)$ and for all strongly q_2-multiplicative sequences $g(n)$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} f(n)g(n)\mu(n) = 0.$$

On the one hand the methods of [12] are not sufficient to cover more than two factors. On the other hand [12] provides a prime number theorem for $f(n)g(n)$, too, if $f(n)$ and $g(n)$ are not of the form $\mathrm{e}(\ell_i n/(q_i-1))$ for some integers ℓ_i.

For other results in this direction see e.g. [5] and the case of disjoint minimal self-joining systems or [48] as the Cartesian product of unipotent translations on homogeneous (of real connected Lie groups) remains in the same class of dynamical systems.

2.3. **New Results.** We want to show next that the problem for the Möbius orthogonality on product spaces has a partial solution in the context of automatic sequences. More precisely we show that Theorem 2.1 extends to products of primitive automatic sequences.

Automatic sequences can be defined in various ways (see [2]). In the context of dynamical systems, it is usual to define automatic sequences by using substitutions. A sequence $(a(n))_{n\geq 0}$ is called λ-automatic if there exist finite alphabets $\mathcal{A}, \mathcal{B}$, a substitution $\eta : \mathcal{B} \to \mathcal{B}^\lambda$ and a coding $\pi : \mathcal{B} \to \mathcal{A}$ such that $a(n) = \pi(x(n))$, where $(x(n))$ is a fixed point of the extension of η to $\mathcal{B}^{\mathbb{N}}$ by concatenation.

For example, consider the alphabet $\mathcal{B} = \{a, b\}$ and the substitution $\eta(a) = ab$, $\eta(b) = ba$. Then the fixed point of the extension of η to $\mathcal{B}^{\mathbb{N}}$ (that starts with the letter a) is given by

$$(x(n))_{n\geq 0} = (abbabaabbaababbabaababbaabbabaab\ldots)$$

The coding $\pi(a) = 1$ and $\pi(b) = -1$ defines then the Thue-Morse sequence $T(n)$. (It is an easy exercise that we also have $T(n) = (-1)^{s_2(n)}$, where $s_2(n)$ denotes the binary sum-of-digits function.)

Alternatively, an automatic sequence $a(n)$ can be defined as the output sequence of a finite automaton (where every state has λ outgoing edges indexed by $0, 1, \ldots, \lambda-1$) when (for every $n \geq 0$) the input is the digital expansion of n in base λ. For every automatic sequence $a(n)$ there exists a unique minimal automaton that generates $a(n)$.

An automatic sequence $a(n)$ is called primitive if the associated dynamical system is minimal. Equivalently, this holds if the associated minimal automaton is strongly connected and if the input 0 to the initial state is mapped to the initial state (see [49] for an in depth treatment of the dynamical systems associated with automatic sequences). For example, the Thue-Morse sequence is primitive.

It was shown in [46] that all automatic sequences are orthogonal to the Möbius function and in [39] that primitive automatic sequences are orthogonal to any bounded, aperiodic, multiplicative function. The next (and new) theorem extends this property to products of primitive automatic sequences.

Theorem 2.2. *Let $d \geq 2$ and $a_j(n)$, $1 \leq j \leq d$, be primitive λ_j-automatic sequences, where λ_j, $1 \leq j \leq d$ are pairwise coprime. Then $(a_1(n) \cdots a_d(n))_{n\geq 0}$ is orthogonal to any bounded, aperiodic, multiplicative function $m : \mathbb{N} \to \mathbb{C}$, i.e.*

$$\sum_{n\leq N} a_1(n) \cdots a_d(n) \cdot m(n) = o(N). \tag{4}$$

In particular, it is orthogonal to the Möbius function $\mu(n)$.

The proof is given in the Appendix, where we restrict ourselves to the case $d = 2$. However, in contrast to Theorem 2.1 the generalization to $d > 2$ is immediate.

3. Prime number theorems

3.1. **Classical Prime number theorems.** A Möbius-randomness principle is often closely related to a Prime Number Theorem (PNT), i.e., an asymptotic formula for the sum $\sum_{n\leq N} \Lambda(n)a(n)$, where $\Lambda(n)$ denotes the von Mangoldt function defined by $\Lambda(p^k) = \log p$ for $p \in \mathbb{P}$ and 0 else. This in turn gives us information about $\sum_{p\leq N} a(p)$ via summation by parts. More precisely, if we have (for a bounded sequence $a(n)$ and some constant c)

$$\sum_{n\leq N} \Lambda(n)a(n) = c\,N + o(N) \tag{5}$$

then, equivalently, we have

$$\sum_{p\leq N} a(p) = \frac{c\,N}{\log N} + o\left(\frac{N}{\log N}\right).$$

For example, if $a(n) = 1$ then the prime number theorem $\pi(N) \sim N/\log N$ is equivalent to $\sum_{n\leq N} \Lambda(n) \sim N$. Furthermore, the Dirichlet prime number theorem is equivalent to (5) for periodic sequences $a(n)$.

3.2. **Prime number theorems for dynamical systems.** We say that a dynamical system (X,T) satisfies a PNT if there is an asymptotic formula for the sums $\sum_{n\leq N}\Lambda(n)f(T^nx)$ or equivalently for the sums $\sum_{p<N}f(T^px)$ when $N\to\infty$ for all $f\in C(X)$ and $x\in X$. [4] But specifying more to the dynamical context we say that a PNT holds if the limit

$$\lim_{N\to\infty}\frac{1}{\pi(N)}\sum_{p<N}f(T^px) \tag{6}$$

exists for all $f\in C(X)$ and $x\in X$. If so, by the Riesz theorem, for all $f\in C(X)$,

$$\lim_{N\to\infty}\frac{1}{\pi(N)}\sum_{p<N}f(T^px)=\int_X f\,d\nu_x$$

for some probability Borel measure ν_x on X. However, the fact that there is no reason for ν_x be T-invariant makes a use of traditional dynamical tools often useless, and proving a PNT in a concrete dynamical system, if possible, turns out to be an interesting interplay between dynamics and number theory, typically forcing dynamical estimates to be effective. Note that we do not insist on ν_x to be one and the same measure, and indeed the simplest situation of finite cyclic rotation for which a PNT holds via Dirichlet's theorem, shows that ν_x may depend on x. On the other hand, we will see also that in some natural cases the dynamics of prime "orbits" $\{T^px:\ p\in\mathbb{P}\}$ is as "equidistributed" as of the genuine orbits $\{T^nx:\ n\in\mathbb{N}\}$ which may make all the measures ν_x being just one T-invariant measure which appears while computing limits of the traditional Birkhoff sums. We recall that a topological system (X,T) is *uniquely ergodic* if there is exactly one T-invariant probability measure ν. In this situation, it is classical that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n<N}f(T^nx)=\int_X f\,d\nu$$

for each $f\in C(X)$ and $x\in X$ (and the convergence is uniform in x). In other words all orbits $\{T^nx:\ n\geq 1\}$ are *equidistributed* with respect to ν. One may ask if in such a context also prime orbits are

[4] We stress here that we insist on taking into account the prime orbits for **all** $x\in X$. Otherwise, if, for example, we only require to have the result for a "typical" point then the limit $\lim_{N\to\infty}\frac{1}{\pi(N)}\sum_{p<N}f(T^px)$ does exist for a.e. point $x\in X$ with respect to any T-invariant measure for any $f\in L^q$ with $q>1$ (see [4, 61]). Via the Calderon principle the a.e. PNT is reduced to harmonic analysis problems, namely, to study properties of some discrete singular operators defined on functions on $\mathbb{Z}$ with finite supports.

equidistributed. But the fact that the set $\mathbb{P}$ is of zero upper Banach density implies that we can easily construct a symbolic system with exactly one invariant measure (being Dirac at the unique fixed point) for which along one special orbit a PNT fails, see [21] for details. One way to remove this phenomenon of a single "exotic" orbit, is to assume that we consider only *minimal* systems. However, somewhat against expectations R. Pavlov [47] in 2008 provided constructions of symbolic minimal and uniquely ergodic systems (even totally ergodic) for which a PNT fails. Let us add that in Pavlov's counterexamples to PNT the entropy remained undetermined. In 2016 P. Sarnak gave a talk at CIRM [51] in which he postulated to find a sufficient *dynamical* condition for a PNT to hold. Note that in contrast to Sarnak's conjecture which postulates for a concrete class of dynamical systems (namely, deterministic systems) to have a concrete property (namely, to be orthogonal to the Möbius function), for a PNT it does not seem to be clear for which class of systems we postulate it to hold. So far examples for which a PNT holds which we will discuss shortly are (with one exception) either symbolic (of arithmetic origin) or algebraic. To increase the difficulty in precising the class of dynamical systems in which one expects a PNT to hold let us discuss some results from a recent paper [25]. In this article one deals with a PNT in the class of regular Toeplitz subshifts, that is the easiest non-trivial class beyond the class of periodic systems (in which a PNT holds).[5] Recall that a sequence $x \in A^{\mathbb{Z}}$ is Toeplitz if for every $n \in \mathbb{Z}$ there is $r_n \geq 1$ such that $x(n) = x(n + \ell r_n)$ for all $\ell \in \mathbb{Z}$. All Toeplitz subshifts are minimal and the regularity (which expresses a good approximation of the resulting x by periodic sequences) is added to obtain unique ergodicity, e.g. [11]. Somewhat surprisingly, it turns out that a PNT already fails in this class (while the Möbius orthogonality holds [17]). On the other hand, if one adds some speed in the approximation of x by periodic sequences, then a PNT holds. This is an illustration of a "principle" that if a PNT holds then the Möbius orthogonality does while if the Möbius orthogonality holds with some speed then also a PNT holds. On the base of the results, the authors of [25] formulated a (working) conjecture:

Conjecture 3.1. *Each ergodic and aperiodic measure-theoretic dynamical system has a minimal and uniquely ergodic topological model in which a PNT fails.*

[5] There are automatic sequences which are Toeplitz.

3.3. **Known and Expected Results.** In a search of a class for which we might expect a PNT to hold let us go through known cases which mostly rely on the method of bilinear forms (type I and type II sums that appear in the context of Vaughan's method [58, 10] for bounding sums of the form $\sum_{n\leq N}\Lambda(n)a(n)$).

3.3.1. *Subshifts of arithmetic origin.* Besides periodic sequences a PNT holds for rotations on the torus [59], for nilsystems [29], and for certain enumeration systems [5, 28]. Actually the question for a PNT for enumeration systems goes already back to Gelfond [26] who asked (among several questions) to estimate $\#\{p \leq N : s_q(p) \equiv l \bmod m\}$. Mauduit and Rivat [43] answered this question, showing that the sequence $a(n) = e(\alpha s_q(n))$ satisfies for all real α and all bases $q \geq 2$ such that $(q-1)\alpha \in \mathbb{R}\setminus\mathbb{Z}$ a PNT of the form:

$$\sum_{n\leq N}\Lambda(n)\, e(\alpha s_q(n)) = O(N^{1-\delta}),$$

where $\delta > 0$ and the implied constant depends on q and α. In particular,

$$\#\{p \leq N : T(p) = 1\} = \frac{\pi(N)}{2} + O(N^{1-\delta})$$

for some $\delta > 0$, where $T(n) = (-1)^{s_2(n)}$ denotes the Thue-Morse sequence. This result was then generalized to several directions: to q-multiplicative functions [42], to the Rudin-Shapiro sequence [44], and also to more general classes of automatic sequences (see [1, 46]).

Theorem 3.2. ([46]) *Every primitive automatic sequence $a(n)$ satisfies a PNT.*

3.3.2. *Symbolic models for finite rank systems.* The articles [6] and [22] deal with sufficient conditions for a PNT in natural symbolic models of finite rank transformations, that is, they put some conditions on the number of cuts into columns and the number of spacers put over the columns. In particular, a PNT holds when the number of columns is bounded, with a fixed sufficiently large number of spacers put over the columns.[6] In particular, both papers provide the first examples of minimal, uniquely ergodic systems which are weakly mixing (in fact, even mildly mixing) for which a PNT holds.

Problem 3.3. *Find a mixing system satisfying a PNT, in particular, is there a (symbolic) rank one system which is mixing and satisfies a PNT?*

[6]The criteria do not apply to the classical Chacon constructions.

A partial answer to this problem was given by A. Kanigowski [33] who established a PNT along a subsequence for a class of mixing Kochergin flows on T^2.

3.3.3. *Dynamical systems of algebraic systems (unipotent actions on homogeneous spaces).* As already mentioned, Vinogradov's theorem [59] says that all irrational rotations on the circle satisfy a PNT. This was largely generalized to nil-systems by Green and Tao [29]. In [52], the authors studied the orbits of the horocycle flow (the modular case) at prime times for points x whose orbits are equidistributed. They showed that the prime orbit for such an x visits every open set of volume $> 1/10$ and posed the following

Problem 3.4. *Does a PNT hold for horocycle flows?* [7]

3.3.4. *Smooth systems of zero entropy.* One more natural playground on which one might expect a PNT to hold is the class of smooth systems of zero entropy. The natural question (including the nil and the horocycle case) would be:

Problem 3.5. *Does a PNT hold for parabolic systems?*

3.3.5. *Analytic skew products.* An approach recently developed in [34] does not use the method of bilinear forms, but instead it relies on a strong approximation of the system by a sequence of periodic systems, relating the sums along primes to several Birkhoff sums. The authors of [34] consider so-called analytic skew products on the 2-dimensional torus

$$T_{\alpha,g}(x,y) = (x+\alpha, y+g(x)) \bmod 1,$$

where α is irrational and g is a 1-periodic real-analytic function. These kind of sequences have many interesting dynamical properties (see [37]). In particular, the orthogonality of the Möbius function was established by Liu and Sarnak [41] under some additional conditions on g (and proved in full generality in [60]).

Theorem 3.6 ([34])**.** *Let α be irrational and g a 1-periodic real-analytic function of zero mean. If $T_{\alpha,g}$ is uniquely ergodic then it satisfies a PNT.*

There is no simple criterion that ensures unique ergodicity of $T_{\alpha,g}$. However, if $g(x) = \sum_m a_m e(mx)$ denotes the Fourier series of g and if there exists a subsequence q_{n_k} of the best approximation denominators of α with $\|q_{n_k}\alpha\|/a_{q_{n_k}} \to 0$ (as $k \to \infty$) then it follows that $T_{\alpha,g}$ is uniquely ergodic (see [34] for more details).

[7] In fact, it is even unknown whether prime orbits of a generic point are dense.

It should be further noted that if we drop the assumption of analyticity and we consider only *continuous* $T_{\alpha,g}$ then, as shown in [34], a PNT need not hold.

3.4. **Logarithmic Prime number theorems.** Similarly to the Sarnak (and Chowla) conjecture we can also formulate a logarithmic version of the PNT, namely that

$$\sum_{n\le N} \frac{\Lambda(n)}{n} a(n) = c\log N + o(\log N) \tag{7}$$

(for some constant c) which is clearly weaker than the PNT since there are automatic sequences that do not satisfy a PNT, but we generally have:

Theorem 3.7 ([1]). *Every automatic sequence $a(n)$ satisfies the logarithmic PNT.*

Theorem 3.7 is stated in [1] in a slightly different but equivalent form, namely that every letter of an automatic sequence has a logarithmic density along primes. Actually [1] provides a criterion when an automatic sequence $a(n)$ satisfies not only (7) but also a PNT. It is convenient to formulate this criterion in terms of densities. To each automatic sequence $a(n)$ we can construct a unique minimal automaton that generates $a(n)$. The underlying di-graph of this automaton decomposes into one or several strongly connected components. Some of them are final in the sense that there is no edge to another strongly connected component. These final strongly connected components can be now considered to generate further automatic sequences $a^{(\ell)}(n)$ that satisfy a PNT according to Theorem 3.2, in particular, every letter has a density along the sequence of primes. The original sequence $a(n)$ has then densities along primes if and only if the corresponding densities of all sequences $a^{(\ell)}(n)$ (related to final strongly connected components) coincide.

Thus, the problem of PNT's for automatic sequences is completely solved.

In contrast to automatic sequences, where the logarithmic PNT holds, there seems to be a problem with deterministic systems given by one exotic orbit. We can proceed, for example as in [21, Remark 1.1] by choosing first $(c_{p^k}) \in \{-1,1\}^{\mathbb{N}}$ and then setting $b_{p^k} = c_{p^k}$ and $b_n = 1$ for the remaining n. Then we take the corresponding subshift X_b together with the continuous function $f(z) := 1 - z(1)$. The subshift has exactly one invariant measure, namely the Dirac measure at the fixed point $111\dots$ (so it is of zero entropy). Now, we consider

$$\frac{1}{\log N}\sum_{n\le N}\frac{\Lambda(n)}{n}f(S^n b)=\frac{1}{\log N}\sum_{n\le N}\frac{\Lambda(n)}{n}(1-b_n),$$

whose only non zero terms are of the form $\log p/p^k$ whenever $c_{p^k}=-1$. We can now choose (c_{p^k}) so that along different subsequences (N_j) we obtain different limits. Thus, the logarithmic PNT does not hold.

These properties lead us directly to the following problem.

Problem 3.8. *Which deterministic sequences satisfy a logarithmic PNT?*

4. Polynomial Subsequences

Eisner [16] introduced (among other variants) the following polynomial version of Sarnak's conjecture.

Conjecture 4.1 (Polynomial Sarnak's conjecture for minimal systems I)**.** *Let (X,T) be a minimal topological dynamical system with entropy zero. Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\le N}f(T^{P(n)}x)\mu(n)=0 \tag{8}$$

holds for every $f\in C(X)$, every polynomial $P:\mathbb{N}\to\mathbb{N}_0$ and every $x\in X$.

It was already noticed in [16] that this conjecture is false without the minimality assumption.

Actually this conjecture is generally false. This was observed simultaneously and independently in [34] and [40]. The counterexamples in [34] are minimal and uniquely ergodic and they are based on the constructions of *continuous* (not analytic) counterexamples to a PNT in the class of skew products, while counterexamples in [40] are symbolic and minimal[8] and are based on the ideas of [31].

On the other hand there are several (good) reasons why this polynomial version might hold for many systems. First, it holds generically. Let (X,μ,T) be a standard invertible measure-theoretic dynamical system, $f\in L^q(X,\mu)$ for some $q>1$, and let P be an integer polynomial. Then (8) holds for almost all x (see [16, 18]).

Furthermore, if $a(n)=f(q^n x)$ is a nilsequence then a polynomial subsequence $a(P(n))=f(q^{P(n)}x))$ is again a nilsequence (see [9, 16, 30,

[8]These are Toeplitz subshifts and it is rather unclear if in the construction in [40] they can be made uniquely ergodic.

36]). Hence, the polynomial Sarnak conjecture is satisfied for transitive (equivalently, minimal) nilsystems.

Finally, we mention a recent result for strongly q-multiplicative functions [13] and the polynomial $P(n) = n^2$.

Theorem 4.2. *Let $q \geq 2$ be a given integer. Then, for all complex valued strongly q-multiplicative functions $g(n)$ of modulus 1, we have*

$$\lim_{N\to\infty} \sum_{n\leq N} g(n^2)\mu(n) = 0.$$

If $g(n)$ attains only finitely many values then it is a primitive q-automatic sequence. So the associated dynamical system is minimal as it is requested by the above conjecture.

The proof method for Theorem 4.2 uses in a first step a general theorem by Mauduit and Rivat [45]. This theorem says that if a function $f(n)$ (of modulus 1) satisfies a so-called *carry property* and a *Fourier property* then

$$\sum_{n\leq N} f(n^2) = o(N).$$

This result is applied for $f(n) = g(p^2 n)\overline{g(q^2 n)}$, where p and q are different prime numbers. Actually, the verification of the sufficient conditions (the *carry* and *Fourier property*) is the main difficulty in the proof. Thus one gets

$$\sum_{n\leq N} g(p^2 n^2)\overline{g(q^2 n^2)} = o(N).$$

At this stage we can apply a general principle by Kátai [35] (see also Bourgain, Sarnak and Ziegler [7] for a quantitative version). It says that if $(x_n)_{n\in\mathbb{N}}$ is a bounded complex valued sequence such that for every pair (p, q) of different prime numbers (that are sufficiently large) we have

$$\sum_{n<N} x_{pn}\overline{x_{qn}} = o(N), \tag{9}$$

then for all bounded multiplicative functions $m(n)$ it follows that

$$\sum_{n<N} x_n m(n) = o(N) \qquad (N \to \infty). \tag{10}$$

Of course this completes the proof of Theorem 4.2 by setting $x_n = g(n^2)$.

We expect that Theorem 4.2 generalizes to primitive automatic sequences (by combining the above mentioned methods of [13] with those of [46]).

5. Morphic Sequences

We have already considered automatic sequences which can be defined as codings of fixed points of substitutions (on sequences over a finite alphabet) of constant length (we recall that there are several equivalent ways to define automatic sequences [2]). One classical way of generalizing automatic sequences is to consider substitutions of non-constant length instead, which leads us to the notion of *morphic sequences*: they are obtained as fixed points of arbitrary substitutions (over a finite alphabet), followed by a coding. Thus, it is a natural question whether results for automatic sequences can be generalized to *morphic sequences*.

One of the simplest morphic sequences is the *Fibonacci word*

$$(x_n)_{n\geq 1} = (2 + \lfloor n\varphi \rfloor - \lfloor (n+1)\,\varphi \rfloor) = (010010100100101001\ldots)$$

which is the fixed point of the morphism $\sigma(0) = 01$, $\sigma(1) = 0$, starting with 0 (and where $\varphi = (1+\sqrt{5})/2$ denotes the golden mean). Möbius orthogonality for this case, and more generally for Sturmian words, follows from [29, Theorem 5.2] by setting $x_n = \lfloor n\alpha + \beta \rfloor - \lfloor (n-1)\alpha + \beta \rfloor$ or $x_n = \lceil n\alpha + \beta \rceil - \lceil (n-1)\alpha + \beta \rceil$ for some irrational α and real β. We note that Sturmian words are characterized as binary nonperiodic words having minimal factor complexity: there are exactly $k+1$ different factors (contiguous subsequences) of length k [2, Theorem 10.5.2]. Automatic sequences, on the other hand, have sublinear factor complexity [2, Corollary 10.3.2]. Moreover, morphic sequences have at most a quadratic number of factors of length k [2, Corollary 10.4.9]. Sturmian, automatic and morphic sequences are, therefore, deterministic. Hence, by Sarnak's conjecture it is expected that all morphic sequences are orthogonal to the Möbius function.

In contrast to automatic sequences (where we know that all of them are orthogonal to the Möbius function [46]) almost nothing is known for general morphic sequences. Only the logarithmic version is solved (at least for primitive morphic sequences) by Frantzikinakis and Host [24], which works in a much more general setting. For the original version there seem to be very few results: one theorem by Houcein El Abdalaoui, Lemańczyk and De La Rue [19], which shows that uniquely ergodic models of totally ergodic rotations satisfy the Sarnak conjecture (this result covers some morphic sequences like the Fibonacci word), and one result by Ferenczi [20] that settles by different methods the Sarnak conjecture for some other special morphic sequences.

In what follows we comment on the sequence $(-1)^{Z(n)}$, where $Z(n)$ denotes the so-called Zeckendorf sum-of-digits function, that is, the minimal number of Fibonacci numbers needed to represent n as their sum. This sequence can be considered as an analogue of the Thue-Morse sequence $T(n) = (-1)^{s_2(n)}$.

The sequence $(-1)^{Z(n)}$ is morphic, see [8, p. 14] but does not belong to the class covered in [19] or [20]. It is given by the following substitution σ together with the coding π

$$\sigma : \left\{ \begin{array}{lll} a & \mapsto & ab \\ b & \mapsto & c \\ c & \mapsto & cd \\ d & \mapsto & a \end{array} \right\}, \qquad \pi : \left\{ \begin{array}{lll} a & \mapsto & 1 \\ b & \mapsto & -1 \\ c & \mapsto & -1 \\ d & \mapsto & 1 \end{array} \right\},$$

and we are interested in the fixed point starting with a. Therefore,

$$((-1)^{Z(n)})_{n\geq 0}$$
$$= (1, -1, -1, -1, 1 -1, 1, 1, -1, 1, 1, 1, -1, -1, 1, 1, 1, -1, 1, -1, -1 \ldots) .$$

By using a combinatorial approach the authors of [14] proved the following theorem that shows orthogonality not only for the Möbius function $\mu(n)$ but for all multiplicative function of modulus 1.

Theorem 5.1. *Let $Z(n)$ be the Zeckendorf sum-of-digits function and $m(n)$ a multiplicative function of modulus* 1*. Then, we have*

$$\sum_{n<N} (-1)^{Z(n)} m(n) = o(N) \qquad (N \to \infty). \tag{11}$$

The proof of Theorem 5.1 relies on the general principle due to Katai [35] (namely that (9) implies (10)). Thus, it is sufficient to check the condition

$$\sum_{n<N} (-1)^{Z(pn)+Z(qn)} = o(N) \qquad (N \to \infty). \tag{12}$$

The strategy for the proof of (12) is as follows. First a generating function approach is used to analyze (12). The generating functions are used as a formalization for a recurrence relation of sums of the form (12). This implies already that (12) behaves like N^δ for some $\delta \in [0, 1]$ (due to a singularity analysis). It then only remains to show that the terms in (12) change signs at least once, which is done by combinatorial arguments.[9]

It is a challenging problem to settle the Sarnak conjecture for morphic sequences. Whereas automatic sequences can be represented with the help of q-adic digital expansions this property is widely lost for morphic sequences and Diophantine problems appear in the analysis of

[9]This strategy is similar to the one by Tao to prove Möbius orthogonality for the Rudin-Shapiro sequence [53].

exponential sums. It is, however, very likely that a combination of the techniques of [46] and [14] might cover so-called Fibonacci-automatic sequences. A sequence is Fibonacci-automatic, if it is the output sequence of a finite state automaton, where the input sequence to such an automaton is the Zeckendorf expansion of n. A major new tool of [46] is a proper decomposition of a general automatic sequence into a so-called *synchronizing* and into an *invertible* part (c.f. the decomposition $f = g' + g''$ in Section 6.0.2). The problem of a general automatic sequence can be – more or less – reduced to the analysis of synchronizing automatic sequences (which are highly structured) and invertible automatic sequences (which are close to random in some aspects). Since the structure of morphic sequences is a proper generalization of automatic sequences it is natural to ask whether the above mentioned decomposition for automatic sequences generalizes to morphic sequences. However, this question is completely open.

Finally we want to mention that there are few PNTs that are known for morphic sequences, namely for Sturmian words [25] (this also follows from the fact that Sturmian words are nilsequences and [29]) and also for the sequence $(-1)^{Z(n)}$:

$$\sum_{n\le N} \Lambda(n)(-1)^{Z(n)} = o(N).$$

This has been established in a recent paper [15].

6. Appendix – Proof of Theorem 2.2

The proof of Theorem 2.2 relies on several properties of dynamical systems and a quite general orthogonality theorem for automatic sequences [39]. For the sake of brevity we only consider products of two automatic sequences. The generalization for $d > 2$ products is immediate.

6.0.1. *Background on dynamical systems.* When $a : \mathbb{N} \to \mathbb{C}$ is automatic (with a substitution η) and $\mathcal{A} = a(\mathbb{N})$ is the "alphabet" of a, then we let $X_a \subset \mathcal{A}^{\mathbb{Z}}$ denote the subshift generated by a and since it is highly related to η, we will also denote it by X_η [10]. We hence obtain a topological dynamical system (X_a, S), where S stands for the left shift, $S\big((x_n)_{n\in\mathbb{Z}}\big) = (x_{n+1})_{n\in\mathbb{Z}}$. The benefit of working with a substitution η is that it acts naturally on the corresponding subshift, but we do

[10] The set X_a consists of all sequences x in $\mathcal{A}^{\mathbb{Z}}$ such that each subword of x appears in a.

not go into any detail here. We recall that a is primitive[11] if and only if (X_η, S) is minimal. In fact, this condition is further equivalent to the assertion that (X_a, S) is strictly ergodic, in which case the unique invariant measure is denoted by μ_η. Such subshifts originating from primitive automatic sequences have been thoroughly studied (see for example [49]).

Joinings and disjointness. By a *dynamical system* we mean $(X, \mathcal{B}, \mu, S)$, where $(\mathrm{X}, \mathcal{B}, \mu)$ is a probability standard Borel space and $S : X \to X$ is an a.e. bijection which is bimeasurable and measure-preserving. If no confusion arises, we will speak about S itself and call it an *automorphism.*[12]

Remark. Each homeomorphism S of a compact metric space X determines many dynamical systems $(X, \mathcal{B}(X), \mu, S)$ with $\mu \in M(X, S)$, where $M(X, S)$ stands for the set of probability Borel measures on X. Recall that by Krylov-Bogoljubov theorem, $M(X, S) \neq \emptyset$, and moreover, $M(X, S)$ endowed with the weak topology becomes a compact metric space. The set $M(X, S)$ has a natural structure of a convex set (in fact, it is a Choquet simplex) and its extremal points are precisely the ergodic measures. We say that the topological system is *uniquely ergodic* if (X, S) has only one invariant measure.

Given another system $(Y, \mathcal{C}, \nu, T)$, we may speak about the set $J(S, T)$ of joinings of automorphisms S and T. Namely, $\kappa \in J(S, T)$ if κ is an $S \times T$-invariant probability measure on $\mathcal{B} \otimes \mathcal{C}$ with the projections μ and ν on X and Y, respectively. Note that the projection maps $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ settle factor maps between the dynamical system

$$(X \times Y, \mathcal{B} \otimes \mathcal{C}, \kappa, S \times T) \text{ and } (X, \mathcal{B}, \mu, S),\ (Y, \mathcal{C}, \nu, T), \text{ respectively.}$$

The automorphisms S and T are called *disjoint* (or Furstenberg disjoint) if the only joining of S and T is product measure $\mu \otimes \nu$, i.e. $J(S, T) = \{\mu \otimes \nu\}$. We will then write $S \perp T$. Note that if $S \perp T$ then at least one of these automorphisms must be ergodic. If both of them are ergodic, then the set of ergodic joinings $J^e(S, T)$ is nonempty (in fact, an ergodic decomposition of a joining consists of ergodic self-joinings).

[11]The substitution η is called primitive if there exists $n \in \mathbb{N}$ such that for all $a, b \in \mathcal{B}, \eta^n(a)$ contains b.

[12]In what follows we will also use notation $S \in \mathrm{Aut}\,(\mathrm{X}, \mathcal{B}, \mu)$, where $\mathrm{Aut}\,(\mathrm{X}, \mathcal{B}, \mu)$ stands for the Polish group of all automorphisms of $(\mathrm{X}, \mathcal{B}, \mu)$. The topology is given by the strong operator topology of the corresponding unitary operators U_S, $U_S f := f \circ S$ on $L^2\,(\mathrm{X}, \mathcal{B}, \mu)$.

Quasi-disjointness. A useful generalization of disjointness is the so called quasi-disjointness introduced by Berg [3]. We recall that a *Kronecker factor* of an ergodic system is its largest factor which is isomorphic to a rotation on a compact abelian group (a notation that goes back to Furstenberg).

Remark. The Kronecker factor can be constructed using the eigenvalues and eigenfunctions of the system (see for example [3]). Here λ is said to be an eigenvalue of S if there exists $f \in L^2(\mathrm{X}, \mathcal{B}, \mu)$ where $f \neq 0$ and $f \circ S = \lambda f$ μ-a.e.

Let $K(S,T)$ be the maximum (common) Kronecker factor of $(X, \mathcal{B}, \mu, S)$ and $(Y, \mathcal{C}, \nu, T)$. Thus, there exist factor maps, $\alpha : (X, \mathcal{B}, \mu, S) \to K(S,T)$ and $\beta : (Y, \mathcal{C}, \nu, T) \to K(S,T)$. We say that $(X, \mathcal{B}, \mu, S)$ and $(Y, \mathcal{C}, \nu, T)$ are *quasi-disjoint* if for almost every $k \in K(S,T)$ there is exactly one joining of the systems giving full measure to $\gamma^{-1}(k)$. This measure is called μ_k and it is ergodic. Here, we used the function $\gamma : (X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \times \nu, S \times T) \to K(S,T), \gamma(x,y) = \alpha(x) - \beta(y)$.

Group and isometric extensions. Given $(X, \mathcal{B}, \mu, S)$ an ergodic dynamical system, consider a measurable $\varphi : X \to G$, where G is a compact metric group. Let m_G denote the Haar measure of G. The automorphism $S_\varphi : X \times G \to X \times G$,

$$S_\varphi(x,g) = (Sx, \varphi(x)g)$$

is called a *compact group extension* of S. We obtain the dynamical system $(X \times G, \mathcal{B} \otimes \mathcal{B}(G), \mu \otimes m_G, S_\varphi)$ which need not be ergodic (for example, it is not ergodic when $\varphi(x) = \xi(x)\xi(Sx)^{-1}$ for a measurable $\xi : X \to G$, i.e. when φ is a coboundary).

Group extensions work particularly well in the context of quasi-disjointness.

Theorem 6.1 ([3])**.** *Let $(X, \mathcal{B}, \mu, S), (Y, \mathcal{C}, \nu, T)$ and $(Z, \mathcal{D}, \rho, U)$ be ergodic dynamical systems, and suppose S is a compact group extension of T. If T and U are quasi-disjoint, then S and U are quasi-disjoint.*

Odometers. Odometers are given by inverse limits of cyclic groups. Therefore, let $n_t \in \mathbb{N}$ for $t \in \mathbb{N}$ such that $n_t | n_{t+1}$. Then we define

$$X := \operatorname{liminv} \mathbb{Z}/n_t\mathbb{Z}$$

with the rotation S by 1 on each coordinate. An example of that is the *dyadic odometer*, where $n_t = 2^t$. Then (X,S) is uniquely ergodic (with the unique measure being the Haar measure m_X of X). Then S^r is ergodic (uniquely ergodic) iff $(r, n_t) = 1$ for each $t \geq 1$. In this case S^r and S are isomorphic. It easily follows that whenever $p \neq q$

are prime numbers not dividing any n_t then each $\rho \in J^e(S^p, S^q)$ is a graph joining (of an isomorphism of S^p and S^q). Two odometers S and T (T is given by $m_t|m_{t+1}$) are disjoint iff they do not have common non-trivial eigenvalues, that is, $(n_t, m_t) = 1$ for each $t \geq 1$. It follows for example that dyadic and triadic odometers are disjoint.

It is classical that if a constant length substitution η (with length λ) is primitive, then the Kronecker factor of (X_η, μ_η, S) is $(H_\lambda \times \mathbb{Z}/h\mathbb{Z}, m_{H_\lambda} \otimes m_{\mathbb{Z}/h\mathbb{Z}}, R \times \tau_h)$, where $h \in \mathbb{N}$ denotes the so called height of η[13], H_λ is the λ-odometer, m_G denotes the Haar-measure of the compact group G and τ_h is the addition by 1 modulo h. In particular, the associated dynamical systems to a λ_1-automatic and a λ_2-automatic sequence, where $(\lambda_1, \lambda_2) = 1$ are always quasi-disjoint by Theorem 6.1 and the discussion above. Moreover, the common Kronecker factor is finite (its order is a divisor of the product of the heights of the corresponding substitutions).

6.0.2. *Orthogonality of automatic sequences with aperiodic multiplicative sequences.* The following theorem by the second and third author [39] was already hinted on earlier.

Theorem 6.2. *Every primitive automatic sequence $a(n)$ is orthogonal to any bounded, aperiodic and multiplicative sequence $m : \mathbb{N} \to \mathbb{C}$, i.e.*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} a(n)m(n) = 0.$$

We now explain some of the key ideas to prove Theorem 6.2 as they are very useful to prove Theorem 2.2.

Let $a(n)$ be a primitive λ-automatic sequence. One of the main results of [39] is that there exists a primitive substitutions Θ (in [39] called $\hat{\Theta}$) such that $a(n)$ is observed by (X_Θ, S) and has useful properties. In particular, (X_Θ, S) is a finite group extension (with group G) of (X_θ, S) (in [39] called $\tilde{\theta}$) which represents measure-theoretically the λ-odometer, $(H_\lambda, m_{H_\lambda}, R)$. It is well-known that the Kronecker factor of (X_Θ, S) is given by $(H_\lambda \times \mathbb{Z}/h\mathbb{Z}, m_{H_\lambda} \otimes m_{\mathbb{Z}/h\mathbb{Z}}, R \times \tau_h)$, where h denotes the height of Θ.

Moreover, any $f \in C(X_\Theta, S)$ can be decomposed as $f = g' + g''$, where g is continuous and orthogonal to the L^2-space of the Kronecker-factor of (X_Θ, S) and g'' is Weyl rationally almost periodic.

[13]Details can be found for example in [49].

One important step in [39] is to show that for $p, q \in \mathbb{P}$ large enough and any $x \in X_\Theta$,

(13)
$$\frac{1}{N}\sum_{n\le N} g'(S^{pn}x)\overline{g'(S^{qn}x)} = \frac{1}{N}\sum_{n\le N} g'\otimes\overline{g'}\left((S^p\times S^q)^n(x,x)\right)\to 0.$$

The proof of (13) splits into two parts. First, one shows that any accumulation point of $\left(\frac{1}{N}\sum_{n\le N}\delta_{(S^p\times S^q)^n(x,x)}\right)$ is generic for the relatively independent extension ρ of (X_Θ, S^p) and (X_Θ, S^q) of the underlying graph joining W of $(R\times\tau_h)^p$ and $(R\times\tau_h)^q$[14]. Second, one shows that

$$\int_{X_\Theta\times X_\Theta} g'\otimes\overline{g'}d\rho = 0,$$

which follows directly from the fact that g' is orthogonal to the L^2-space of the Kronecker factor of (X_Θ, S). Even more, one can easily show that $g'\otimes\overline{g'}$ is orthogonal to the L^2-space of the Kronecker factor of $(X_\Theta\times X_\Theta, \rho, S^p\times S^q)$, which is again isomorphic to $(H_\lambda\times\mathbb{Z}/h\mathbb{Z}, m_{H_\lambda}\otimes m_{\mathbb{Z}/h\mathbb{Z}}, R\times\tau_h)$ via [3, Corollary 1.2]. Moreover, we recall that $(X_\Theta\times X_\Theta, \rho, S^p\times S^q)$ is ergodic (as noted in [39]) and isomorphic to a finite group extension of the λ-odometer via [38, Lemma 8].

6.0.3. *Proof of Theorem 2.2.* We recall that the dynamical systems corresponding to λ_1- and λ_2-automatic sequences are quasi-disjoint, as long as $\gcd(\lambda_1,\lambda_2)=1$. We sketch here an argument, how this can be used to extend Theorem 6.2 to the product of automatic sequences.

Proof of Theorem 2.2. First we write $a_i(n) = f_i(S_i^n x_i)$, for some $f_i \in C(X_{\Theta_i}, S)$ and $x_i\in X_{\Theta_i}$. Next, we decompose $f_i = g_i' + g_i''$ via the discussion in Section 6.0.2 Thus, we are left with showing orthogonality to bounded, aperiodic, multiplicative functions for $g_1'\cdot g_2', g_1'\cdot g_2'', g_1''\cdot g_2'$ and $g_1''\cdot g_2''$. We start by treating the last case. We find that $g_1''\cdot g_2''$ is Weyl rationally almost periodic, as it is the product of two Weyl rationally almost periodic sequence. Thus, it is orthogonal to any aperiodic function.

Next we will treat the case $g_1'\cdot g_2'$, for which we aim to use (again) the general principle due to Katai [35] (namely that (9) implies (10)).

[14]If $W: H_\lambda\to H_\lambda$ settles an isomorphism of R^p and R^q, then it yields an ergodic joining $\lambda_W(A\times B) := m_X(A\cap W^{-1}B)$; its relatively independent extension $\widehat{\lambda}_W$ is defined by

$$\widehat{\lambda}_W(\widetilde{A}\times\widetilde{B}) = \int_X E(\widetilde{A}|X)(x)E(\widetilde{B}|X)(Wx)\,dm_X(x).$$

Thus, we are interested in showing for $p, q \in P$ large enough,

$$\sum_{n\leq N} g_1'(S_1^{pn}x_1)g_2'(S_2^{pn}x_2)\overline{g_1'(S_1^{qn}x_1)g_2'(S_2^{qn}x_2)} = o(N). \tag{14}$$

We can rewrite the left-hand side of (14) as

$$\sum_{n\leq N} \left(g_1'(S_1^{pn}x_1)\overline{g_1'(S_1^{qn}x_1)}\right) \cdot \left(g_2'(S_2^{pn}x_2)\overline{g_2'(S_2^{qn}x_2)}\right).$$

To treat this expression, we will first show that any accumulation point of

$$\left(\frac{1}{N}\sum_{n\leq N} \delta_{(S_1^p\times S_1^q\times S_2^p\times S_2^q)^n(x_1,x_1,x_2,x_2)}\right) \tag{15}$$

gives a measure $\hat{\rho}$ with

$$\int_{X_{\Theta_1}\times X_{\Theta_1}\times X_{\Theta_2}\times X_{\Theta_2}} (g_1'\times \overline{g_1'}\times g_2'\times \overline{g_2'})d\hat{\rho} = 0. \tag{16}$$

We recall that $\delta_{(S_i^p\times S_i^q)^n(x_i,x_i)}$ is generic for $(X_{\Theta_i}\times X_{\Theta_i}, \rho_i, S_i^p\times S_i^q)$. Moreover, the two systems $(X_{\Theta_i}\times X_{\Theta_i}, \rho_i, S_i^p\times S_i^q)$ for $i=1,2$ are quasi disjoint, as they are group extensions of the λ_i-odometer (via Theorem 6.1), which are disjoint (since $\gcd(\lambda_1,\lambda_2)=1$). We can even give a complete description of the common Kronecker factor: we denote by $h_1' := \lim_{n\to\infty}\gcd(h_1,\lambda_2^n)$ and $h_2' := \lim_{n\to\infty}\gcd(h_2,\lambda_1^n)$. Then, the common Kronecker factor is isomorphic to the finite group $(\mathbb{Z}/(h_1'h_2'\mathbb{Z}), m_{\mathbb{Z}/(h_1'h_2'\mathbb{Z})}, \tau_{h_1'h_2'})$. A simple computation also shows that γ is invariant under $(S_1^p\times S_1^q)\times(S_2^p\times S_2^q)$. Thus, the theory of quasi-disjointness shows that $\hat{\rho}$ is the relatively independent extension of the two systems $(X_{\Theta_i}\times X_{\Theta_i}, \rho_i, S_i^p\times S_i^q)$, $i=1,2$ over the common Kronecker factor. It just remains to note that (16) holds since $g_1'\times\overline{g_1'}$ is orthogonal to the L^2-space of the Kronecker factor of $(X_{\Theta_1}\times X_{\Theta_1}, \rho_1, S_1^p\times S_1^q)$.

The proofs for $g_1'\cdot g_2''$ and $g_1''\cdot g_2'$ work analogously. □

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