LIMITING DISTRIBUTIONS IN BRANCHING PROCESSES WITH TWO TYPES OF PARTICLES

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Abstract. Let us consider a decomposable branching process with two types of particles T_1, T_2 such that particles of type T_2 can only produce particles of types T_1 whereas particles of type T_1 can produce particles of both types. The aim of this paper is to characterize the kind of distribution of the particles of types T_1 and T_2 when the total number n of all particles is fixed. Especially we are interested in the limit case $n \to \infty$. It turns out that depending on the parameters of the process a number of different limiting distributions, e.g. normal or χ^2 distributions, appear.

Key words. branching processes, limiting distributions

1. Introduction. Consider a discrete branching process with $N \geq 2$ different kinds of particles $T_1, T_2, \ldots, T_N$, in which a particle of type T_i only produces particles of types $T_i, T_{i+1}, \ldots, T_N$.

For $i = 1, \ldots, N$ let $(\xi(i,i), \xi(i,i+1), \ldots, \xi(i,N))$ be the random vector of offspring particles produced by a particle of type T_i and let

$$f^i_{m_i, m_{i+1}, \ldots, m_N} = \mathbf{P}\{\xi(i,j) = m_j,\, j = i, i+1, \ldots, N\}.$$

Set also

$$\begin{aligned} F^i(y_i, y_{i+1}, \ldots, y_N) &= \sum_{m_i, m_{i+1}, \ldots, m_N} f^i_{m_i, m_{i+1}, \ldots, m_N} y_i^{m_i} y_{i+1}^{m_{i+1}} \cdots y_N^{m_N} \\ &= \mathbf{E}\, y_i^{\xi(i,i)} \cdots y_N^{\xi(i,N)}. \end{aligned} \tag{1.1}$$

We will always assume that for all $i = 1, 2, \ldots, N$

$$F^i(0, \ldots, 0) = \mathbf{P}\{\xi(i,j) = 0,\, j = i, i+1, \ldots, N\} > 0.$$

Let $\nu(i,j)$, $j = i, i+1, \ldots, N$ be the total number of particles of type T_j born in the process during its evolution if the process was initiated by one particle of type T_i and let

$$\nu(i) = \nu(i,i) + \nu(i,i+1) + \ldots + \nu(i,N).$$

It is known (see, for example, [Se1]) that the tuple of functions

$$\varphi_i(x, z_i, z_{i+1}, \ldots, z_N) = \mathbf{E}\, x^{\nu(i)} z_i^{\nu(i,i)} z_{i+1}^{\nu(i,i+1)} \cdots z_N^{\nu(i,N)}, \quad i = 1, \ldots, N$$

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satisfies the system of equations

$$
\begin{aligned}
\varphi_1(x, z_1, \ldots, z_N) &= x z_1 F^1(\varphi_1(x, z_1, \ldots, z_N), \ldots, \varphi_N(x, z_N)) \\
\varphi_2(x, z_2, \ldots, z_N) &= x z_2 F^2(\varphi_1(x, z_2, \ldots, z_N), \ldots, \varphi_N(x, z_N)) \\
&\vdots \\
\varphi_i(x, z_i, z_{i+1}, \ldots, z_N) &= x z_i F^i(\varphi_i(x, z_i, \ldots, z_N), \ldots, \varphi_N(x, z_N)) \\
&\vdots \\
\varphi_N(x, z_N) &= x z_N F^N(\varphi_N(x, z_N))
\end{aligned} \tag{1.2}
$$

which can be solved iteratively. Let

$$c_{n,k_1,k_2,\ldots,k_N} = \mathbf{P}\{\nu(1) = n; \nu(1,1) = k_1, \ldots, \nu(1,N) = k_N\}$$

and $c_n = \mathbf{P}\{\nu(1) = n\}$.

The aim of this paper is to study the behaviour of the probabilities

$$
\begin{aligned}
p_{n,k_1,k_2,\ldots,k_N} &= \mathbf{P}\{\nu(1,1) = k_1, \ldots, \nu(1,N) = k_N \mid \nu(1) = n\} \\
&= \frac{c_{n,k_1,k_2,\ldots,k_N}}{c_n}.
\end{aligned} \tag{1.3}
$$

as $n \to \infty$ and $k_1, \ldots, k_N$ vary with n in this or that way.

We give a description of possible limiting distributions for $N = 2$ in section 2 and discuss briefly the case of arbitrary $N \geq 2$ in section 3.

Notice also that for the case $N = 1$ the probabilities c_n have been studied in [Se2].

2. Two types of particles. For $N = 2$ the system (1.2) reduces to

$$\varphi_1(x, z_1, z_2) = x z_1 F^1(\varphi_1(x, z_1, z_2), \varphi_2(x z_2)) \tag{2.1}$$

$$\varphi_2(x z_2) = x z_2 F^2(\varphi_2(x z_2)). \tag{2.2}$$

Note that the function φ_2 only depends on the product $x z_2$. Let d_{11}, d_{12}, $d_2 \geq 1$ be defined by

$$
\begin{aligned}
d_{11} &= \gcd\{m_1 \mid \sum_{m_2} f^1_{m_1 m_2} > 0\}, \\
d_{12} &= \gcd\{m_2 \mid \sum_{m_1} f^1_{m_1 m_2} > 0\}, \\
d_2 &= \gcd\{m_2 \mid f^2_{m_2} > 0\}
\end{aligned}
$$

and set

$$
\begin{aligned}
d_{22} &= \gcd\{d_{12}, d_2\}, \\
d &= \gcd\{d_{11}, d_{12}, d_2\}.
\end{aligned}
$$

Obviously, the number of direct descendents of any particle is proportional to d and, therefore, any tree-realization of the process consists of $n \equiv 1 \bmod d$ nodes. Similar arguments show that

$$c_{n,k,n-k} = [x^n z_1^k z_2^{n-k}]\varphi(x, z_1, z_2)$$

can be different from 0 if $k \equiv 1 \bmod d_{11}$ and $n - k \equiv 0 \bmod d_{22}$. In fact we can solve the functional equation (2.2) for $\varphi_2(xz_2)$ by using Lagrange's inversion formula:

$$[(xz_2)^n]\varphi_2(xz_2) = \frac{1}{n}[u^{n-1}](F^2(u))^n.$$

Next we interpret (2.1) as a functional equation for $\varphi_1 = \tilde{\varphi}_1(xz_1, \varphi_2)$, where (xz_1) and φ_2 are considered to be independent complex variables. Again this can be managed formally by Lagrange's inversion formula:

$$[(xz_1)^n \varphi_2^k]\tilde{\varphi}_1(xz_1, \varphi_2) = \frac{1}{n}[v^{n-1}\varphi_2^k](F^1(v, \varphi_2))^n.$$

Finally, insert $\varphi_2(xz_2)$ into $\tilde{\varphi}_1(xz_1, \varphi_2)$ to obtain

$$\varphi_1(x, z_1, z_2) = \tilde{\varphi}_1(xz_1, \varphi_2(xz_2)).$$

Obviously, the formal way of solving system (2.1), (2.2) by Lagrange's inversion formula provides no deeper insight to the asymptotic structure of the solution. Nevertheless we will have to use this description in the following proofs.

Observe also that applying Lagrange's inversion formula gives a more general relation

$$[z^n]\varphi_2^k(z) = \frac{k}{n}[u^{n-k}](F^2(u))^n \tag{2.3}$$

which has the following probabilistic interpretation (see [Se1]). Let $\xi_j(2,2)$ be independent copies of $\xi(2,2)$:

$$\mathbf{E}\, y_2^{\xi_j(2,2)} = F^2(y_2), \quad j = 1, 2, \ldots$$

and let $\nu(k; 2, 2)$ denote the total number of particles ever existed in a branching process initiated a time zero be k particles of type T_2. Then equality (2.3) can be rewritten as

$$\mathbf{P}\{\nu(k; 2, 2) = n\} = \frac{k}{n}\mathbf{P}\{\xi_1(2,2) + \cdots \xi_n(2,2) = n - k\}. \tag{2.4}$$

In order to eliminate trivial situations we will assume that $F^1(\varphi_1, \varphi_2)$ really depends on φ_1 and on φ_2. Furthermore we will distinguish between four cases depending on the vanishing or non-vanishing of $F^1_{y_1^2}(1,1)$ and $F^2_{y_2^2}(1)$.

Our proofs are rather technical and so to see more explicitly what kinds of distributions one can expect to obtain as limiting ones we first consider a two examples.

Our first example is related to the case $F^1_{y_1^2}(1,1) > 0$ and $F^2_{y_2^2}(1) > 0$.

EXAMPLE 1. Assume that $F^2(y_2)$ is an arbitrary function with $\mu_2 = F^2_{y_2}(1) = \mathbf{E}\xi(2,2)$ and $0 < \sigma_2^2 = \mathbf{V}\xi(2,2) < \infty$ and let $F^1(y_1, y_2) = G(y_1 y_2)$ where $G(y)$ satisfies the conditions

$$G(0) > 0, \; G'(1) = \mathbf{E}\,\xi(1,1) = F^1_{y_1}(1,1) = 1 \tag{2.5}$$

$$0 < G''(1) = \mathbf{V}\xi(1,1) = \sigma_1^2 < \infty. \tag{2.6}$$

Applying (2.4) we get

$$\begin{aligned} c_{n,k,n-k} &= \mathbf{P}\{\nu(1) = n; \nu(1,1) = k, \nu(1,2) = n-k\} \\ &= \mathbf{P}\{\nu(1,1) = k\}\mathbf{P}\{\nu(k-1; 2,2) = n-k\} \\ (2.7) \quad &= \mathbf{P}\{\nu(1,1) = k\}\frac{k-1}{n-k}\mathbf{P}\{\xi_1(2,2) + \cdots + \xi_{n-k}(2,2) = n-2k-1\}. \end{aligned}$$

It is known (see [Se1]) that under conditions (2.5) and (2.6)

$$\mathbf{P}\{\nu(1,1) = k\} = \frac{d_{11}}{\sigma_1 k^{3/2}}(1 + o(1)) \tag{2.8}$$

as $k \to \infty$. Hence everything depends on the probabilities $\mathbf{P}\{\xi_1(2,2) + \cdots + \xi_{n-k}(2,2) = n-2k-1\}$ in (2.7). There are three essentially different possibilities:

$$\mu_2 = \mathbf{E}\xi(2,2) < \mathbf{E}\xi(1,1) = 1, \; \mu_2 = 1, \; \mu_2 > 1.$$

We consider the cases separatly.

a) $\mu_2 < 1$: Applying the local limit theorem for $n - 2k + 1 \equiv 0 \bmod d_2$ we obtain

$$\begin{aligned} &\mathbf{P}\{\xi_1(2,2) + \cdots + \xi_{n-k}(2,2) = n-2k-1\} \\ (2.9) \quad &= \frac{d_2}{\sigma_2\sqrt{2\pi(n-k)}} \exp\left(-\frac{x_k^2}{2}\right)(1 + \epsilon(n-k)) \end{aligned}$$

where $x_k = \frac{n-2k+1-\mu_2(n-k)}{\sigma_2\sqrt{n-k}}$ and $\epsilon(n-k) \to 0$ as $n-k \to \infty$ uniformly in x_k from any finite interval. Hence it is not difficult to deduce that

$$c_n = \sum_k c_{n,k,n-k} \sim \frac{b}{n^{3/2}}, \quad n \to \infty$$

for some positive b and that the random variable $\left(\nu(1,1) - \frac{1-\mu_2}{2-\mu_2}n\right) n^{-1/2}$ conditioned on the event $\{\nu(1) = n\}$ is asymptotically normal with parameters 0 and σ^2 as $n \to \infty$. (We do not give here an explicit expression for σ^2 because below we obtain the corresponding formula for the general case.)

b) $\mu_2 = 1$: In this case (2.9) takes the form

$$\begin{aligned}&\mathbf{P}\{\xi_1(2,2) + \cdots + \xi_{n-k}(2,2) = n - 2k - 1\}\\ &\qquad = \frac{d_2}{\sigma_2\sqrt{2\pi(n-k)}}\exp\left(-\frac{(k-1)^2}{2\sigma_2^2(n-k)}\right)(1+\epsilon(n-k)).\end{aligned}$$

Thus,

$$c_{n,k,n-k} \sim \frac{d_{11}d_2}{\sigma_1\sigma_2^{3/2}x^{1/2}n^{7/4}}\exp\left(-\frac{x^2}{2}\right)$$

if $k = x\sigma_2\sqrt{n}(1+o(1))$, $k \equiv 1 \bmod d_{11}$, $n - 2k + 1 \equiv 0 \bmod d_2$.

Now standard arguments show that $c_n \sim bn^{-5/4}$, $b > 0$, as $n \to \infty$ and that

$$\lim_{n\to\infty} \mathbf{P}\{\nu(1,1)/\sigma_2\sqrt{n} \le x \mid \nu(1) = n\} = \frac{2^{3/4}}{\Gamma(\frac{1}{4})}\int_0^x y^{-1/2}e^{-y^2/2}\,dy.$$

c) $\mu_2 > 1$: Assume for simplicity that $F^2(y_2) = r + ty_2^2$ and, therefore, $\mu_2 = 2t > 1$ and $d_2 = 2$. Letting $n = 2g + 1$ and $k \equiv 1 \bmod d_{11}$ we have

$$\begin{aligned}P(g,k) &= \mathbf{P}\{\xi_1(2,2) + \cdots + \xi_{2g-k-1}(2,2) = 2(g-k)\}\\ &= \binom{2g-k-1}{g-k} r^{g-1}t^{g-k}\\ &= (2t)^g(2r)^{g-1}(2t)^{-k}\binom{2g-k-1}{g-k}2^{-(2g-k-1)}.\end{aligned}$$

Hence it follows that there exists a constant b such that

$$P(g,k) \le b(2t)^g(2r)^{g-1}(2t)^{-k}\frac{1}{\sqrt{g}}e^{-k^2/g} \tag{2.10}$$

for all k and

$$P(g,k) = (2t)^g(2r)^{g-1}(2t)^{-k}\frac{1}{\sqrt{\pi g}}e^{-k^2/g}(1+o(1)) \tag{2.11}$$

if $k = o(n^{1/3})$. Combining these estimates with (2.8) we see that

$$c_{n,k,n-k} \sim b_1(2\sqrt{rt})^n n^{-3/2}(2t)^{-k} \tag{2.12}$$

in which $k \equiv 1 \bmod d_{11}$, $n = 2g - 1$, and $k = o(n^{1/3})$, whereas

$$c_n \sim (2t)^{-1}\left(1 - (2t)^{-d_{11}}\right) b_1(2\sqrt{rt})^n n^{-3/2}, \quad n \to \infty. \tag{2.13}$$

Now relations (2.12) and (2.13) yield

$$\lim_{n\to\infty} p_{n,k,n-k} = \left(1-(2t)^{-d_{11}}\right)(2t)^{-k+1}$$

for $k \equiv 1 \bmod d_{11}$ □

The next example deals with the case $F^1_{y_1^2} = 0$, $F^2_{y_2} > 0$.

EXAMPLE 2. Let $F^1(y_1, y_2) = q + py_1y_2$ and $F^2(y_2) = r + ty_2^2$. Assume also that $t < p$. Then it easily follows from (2.7) that for $n = 2g-1$

$$\begin{aligned} c_{n,k,n-k} &= qp^{k-1}\frac{k-1}{2g-k-1}\binom{2g-k-1}{g-k} r^{g-1}t^{g-k} \\ &= qp^{g-1}\gamma^{-(g-1)}\binom{2g-k-1}{g-k}(\gamma r)^{g-1}(tp^{-1})^{g-k}, \end{aligned}$$

where $\gamma > 0$ solves the equation $r\gamma + tp^{-1} = 1$. From this represenation we obtain

$$c_{n,k,n-k} = q(p\gamma^{-1})^{(g-1)}\frac{k-1}{n-k}\mathbf{P}\{\tilde\xi_1(2,2) + \cdots \tilde\xi_{n-k}(2,2) = n-2k+1\},$$

where $\tilde\xi_i(2,2)$, $i = 1, 2, \ldots$, are independent and identically distributed random variables with

$$\mathbf{E}y^{\tilde\xi_i(2,2)} = r\gamma + tp^{-1}y^2.$$

Hence, we obtain as in Example 1 that the random variable $\nu(1,1)$ (conditioned on the event $\nu(1) = n = 2g-1$) is asymptotically normal if $\mathbf{E}\tilde\xi(2,2) = 2tp^{-1} < 1$ or equivalently if $\mathbf{E}\xi(2,2) = 2t < p = \mathbf{E}\xi(1,1)$. Moreover, if $2t > p$ then

$$\lim_{n\to\infty} p_{n,k,n-k} = p_k > 0, \quad k = 1, 2\ldots$$

for some sequence p_k satisfying $\sum_{k\ge 1} p_k = 1$, and finally, if $2t = p$ then

$$\lim_{n\to\infty} \mathbf{P}\{\nu(1,1)n^{-1/2} \le x \,|\, \nu(1) = n\} = 1 - e^{-cx^2}, \quad x > 0. \quad \square$$

As we will show below the case $F^1_{y_1^2}(1,1) > 0$, $F^1_{y_2}(1) = 0$ adds no new kinds of limiting distributions (see Theorem 3) and the remaining case $F^1_{y_1^2}(1,1) = F^1_{y_2}(1) = 0$ is much easier than the others (see Theorem 4).

The following two theorems provide general conditions on $F^1(y_1, y_2)$ and on $F^2(y_2)$ under which the limiting distributions described in Examples 1 and 2 appear. However the proofs are completely different from the reasonings used in the preceding examples. They use singularity analysis of generating functions

THEOREM 1. *Suppose that* $F^1_{y_1}(1,1)$, $F^1_{y_2}(1,1)$, $F^2_{y_2}(1)$, $F^1_{y_1^2}(1,1)$, *and* $F^2_{y_2^2}(1)$ *are all different from* 0. *Furthermore assume that the regions of convergence of* $F^1(y_1,y_2)$ *and* $F^2(y_2)$ *are large enough such that solutions* $y_1=\tau_1>0$, $y_2=\tau_2>0$ *(where* τ_1 *and* τ_2 *are chosen to be minimal) of*

$$\begin{aligned} y_1F^1_{y_1}(y_1,y_2) &= F^1(y_1,y_2) \\ y_2F^2_{y_2}(y_2) &= F^2(y_2) \end{aligned} \tag{2.14}$$

exist in its interiors.

If $F^1_{y_1}(\tau_1,\tau_2)<F^2_{y_2}(\tau_2)$ *then the number of particles of type* T_1 *satisfies a discrete limit theorem with finite mean value* E_n *and variance* V_n. *Set* $\zeta_{d_2}=\exp(2\pi i/d_2)$, $\rho_2=1/F^2_{y_2}(\tau_2)$, *and let* $\overline{\tau}_{1,j}$ *denote the solution of*

$$\overline{\tau}_{1,j}=\rho_2\zeta^j_{d_2}F^1(\overline{\tau}_{1,j},\tau_2\zeta^j_{d_2}) \quad (j\ge 0).$$

Furthermore set

$$\begin{aligned} A^{(0)}_n &= \sum_{j=0}^{\frac{d_2}{d}-1}\zeta_{d_2}^{j(1-n)}\frac{\rho_2\zeta^j_{d_2}F^1_{y_2}(\overline{\tau}_{1,j},\tau_2\zeta^j_{d_2})}{1-\rho_2\zeta^j_{d_2}F^1_{y_1}(\overline{\tau}_{1,j},\tau_2\zeta^j_{d_2})}, \\ A^{(1)}_n &= \sum_{j=0}^{\frac{d_2}{d}-1}\zeta_{d_2}^{j(1-n)}\frac{\rho_2\zeta^j_{d_2}F^1_{y_2}(\overline{\tau}_{1,j},\tau_2\zeta^j_{d_2})}{(1-\rho_2\zeta^j_{d_2}F^1_{y_1}(\overline{\tau}_{1,j},\tau_2\zeta^j_{d_2}))^2}, \\ A^{(2)}_n &= \sum_{j=0}^{\frac{d_2}{d}-1}\zeta_{d_2}^{j(1-n)}\frac{2\rho_2^2\zeta^{2j}_{d_2}F^1_{y_1}(\overline{\tau}_{1,j},\tau_2\zeta^j_{d_2})F^1_{y_2}(\overline{\tau}_{1,j},\tau_2\zeta^j_{d_2})}{(1-\rho_2\zeta^j_{d_2}F^1_{y_1}(\overline{\tau}_{1,j},\tau_2\zeta^j_{d_2}))^3}. \end{aligned}$$

Then we have for $n\equiv 1 \bmod d$

$$E_n=\sum_{k\ge 0}kp_{n,k,n-k}=\frac{A^{(1)}_n}{A^{(0)}_n}+\mathcal{O}(n^{-1})$$

and

$$V_n=\sum_{k\ge 0}(k-E_n)^2p_{n,k,n-k}=\frac{A^{(2)}_n}{A^{(0)}_n}+\frac{A^{(1)}_n}{A^{(0)}_n}-\left(\frac{A^{(1)}_n}{A^{(0)}_n}\right)^2+\mathcal{O}(n^{-1})$$

More precisely, for any fixed $k\equiv 1 \bmod d_{11}$ *and* $n\to\infty$ *(*$n\equiv k \bmod d_{22}$*) we have*

$$\begin{aligned} p_{n,k,n-k} &= \frac{d_{22}}{d}\frac{\rho_2^k}{A^{(0)}_n}\sum_{j=0}^{\frac{d_2}{d_{22}}-1}\zeta_{d_2}^{j(1-n+k)}[u^{k-1}]F^1(u,\tau_2\zeta^j_{d_2})^{k-1}F^1_{y_2}(u,\tau_2\zeta^j_{d_2}) \\ &\quad +\mathcal{O}_k(n^{-1}) \end{aligned}$$

Moreover we have ($k \equiv 1 \bmod d_{11}$, $n \equiv k \bmod d_{22}$, $n \to \infty$)

$$
\begin{aligned}
p_{n,k,n-k} &= \frac{d_{11}d_{22}}{d}\frac{k^{-1/2}}{\sqrt{2\pi}}\frac{\rho_2^k}{A_n^{(0)}}F^1_{y_1}(\tau_1,\tau_2)^k\left(\frac{F^1_{y_2}}{F^1F^1_{y_1^2}}(\tau_1,\tau_2)\right)^{1/2}\cdot \\
&\quad \cdot\left[\exp\left(-\frac{k^2}{2n}\left(\frac{F^1_{y_2}(\tau_1,\tau_2)}{F^1(\tau_1,\tau_2)}\right)^2\frac{F^2(\tau_2)}{F^2_{y_2^2}(\tau_2)}\right)+\mathcal{O}\left(\frac{1}{\sqrt{n}}+\frac{\sqrt{n}}{k^2}\right)\right]
\end{aligned}
$$

uniformly for $k = \mathcal{O}(n\log^{-2} n)$.

If $F^1_{y_1}(\tau_1,\tau_2) > F^2_{y_2}(\tau_2)$ *then the number of particles of type* T_1 *satisfies a central limit theorem with mean value* $E_n = \mu n + \mathcal{O}(1)$ *and variance* $V_n = \sigma^2 n + \mathcal{O}(1)$ *for* $n \equiv 1 \bmod d$, *where* $\mu > 0$ *and* $\sigma^2 > 0$ *are defined as follows. Let* $y_1 = \overline{\tau}_1 > 0$, $y_2 = \overline{\tau}_2 > 0$, *and* $x = \overline{\rho} > 0$ *are the (smallest) solutions of*

$$y_1 = xF^1(y_1,y_2), \quad 1 = xF^1_{y_1}(y_1,y_2), \quad y_2 = xF^2(y_2).$$

Then we have

$$\mu = \frac{\overline{\tau}_1(1-\overline{\rho}F^2_{y_2}(\overline{\tau}_2))}{\overline{\tau}_1(1-\overline{\rho}F^2_{y_2}(\overline{\tau}_2))+\overline{\tau}_2\overline{\rho}F^1_{y_2}(\overline{\tau}_1,\overline{\tau}_2)}$$

and

$$\sigma^2 = \mu^2+\mu+\frac{N}{D},$$

where

$$
\begin{aligned}
N &= 2F^1_{y_1^2}(F^2_{y_2})^2\mu\overline{\rho}^4(\overline{\tau}_1F^2_{y_2}-\overline{\tau}_2F^1_{y_2}) \\
&- \overline{\rho}^3(6F^1_{y_1^2}(F^2_{y_2})^2\mu\overline{\tau}_1-\mu^2\overline{\tau}_2^2(F^1_{y_1^2}(F^1_{y_2}F^2_{y_2^2}-F^1_{y_2^2}F^2_{y_2})+F^1_{y_1y_2}F^2_{y_2}) \\
&\quad +F^2_{y_2}(2\mu\overline{\tau}_2(F^1_{y_1^2}F^1_{y_2}(\mu-2)+F^1_{y_1y_2}F^2_{y_2}(\mu-1))-(F^2_{y_2})^2(\mu-1)^2)) \\
&\quad +\overline{\rho}^2(6F^1_{y_1^2}F^2_{y_2}\mu\overline{\tau}_1+\mu^2\overline{\tau}_2^2(F^1_{y_1^2}F^1_{y_2^2}-(F^1_{y_1y_2})^2) \\
&\quad +2\mu\overline{\tau}_2(\mu-1)(F^1_{y_1^2}F^1_{y_2}+2F^1_{y_1y_2}F^2_{y_2})-3(F^2_{y_2})^2(\mu-1)^2) \\
&- \overline{\rho}(2F^1_{y_1^2}\mu\overline{\tau}_1-(\mu-1)(2F^1_{y_1y_2}\mu\overline{\tau}_2-3F^2_{y_2}(\mu-1))) \\
&- (\mu-1)^2
\end{aligned}
$$

and

$$D = \overline{\rho}F^1_{y_1^2}(\overline{\tau}_1(1-\overline{\rho}F^2_{y_2})+\overline{\rho}\overline{\tau}_2F^1_{y_2})(1-\overline{\rho}F^2_{y_2})^2.$$

Moreover we have a local limit theorem of the form

$$
\begin{aligned}
p_{n,k,n-k} &= \frac{d_{11}d_{22}}{d\sqrt{2\pi\sigma^2 n}}\left(\exp\left(-\frac{(k-\mu n)^2}{2\sigma^2 n}\right)+\mathcal{O}\left(n^{-\frac{1}{2}}+k^{-1}\right)\right)\cdot \\
&\quad \cdot\left(1+\mathcal{O}\left(\frac{|k-\mu n|}{n}\right)\right)
\end{aligned}
$$

uniformly for $k \ge 0$ $(k \equiv 1 \bmod d_{11})$ *and* $n \to \infty$ $(n \equiv k \bmod d_{22})$.

Finally, if $F^1_{y_1}(\tau_1,\tau_2) = F^2_{y_2}(\tau_2)$ *then the number of particles of type* T_1 *admits a* χ^2 *limiting distribution (*$\frac{1}{2}$ *degree of freedom) with mean value*

$$E_n = \frac{\sqrt{2}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\frac{F^1(\tau_1,\tau_2)}{F^1_{y_2}(\tau_1,\tau_2)}\left(\frac{F^2_{y_2^2}(\tau_2)}{F^2(\tau_2)}\right)^{\frac{1}{2}}\sqrt{n} + \mathcal{O}(n^{\frac{1}{4}})$$

and variance

$$V_n = \left(1 - \frac{4\Gamma(\frac{3}{4})^2}{\Gamma(\frac{1}{4})^2}\right)\left(\frac{F^1(\tau_1,\tau_2)}{F^1_{y_2}(\tau_1,\tau_2)}\right)^2 \frac{F^2_{y_2^2}(\tau_2)}{2F^2(\tau_2)} n + \mathcal{O}(n^{\frac{3}{4}})$$

for $n \equiv 1 \bmod d$. *Locally we have*

$$\begin{aligned} p_{n,k,n-k} &= \frac{d_{11}d_{22}}{d}\frac{\sqrt{2}}{\Gamma(\frac{1}{4})}k^{-\frac{1}{2}}n^{-\frac{1}{4}}\left(\frac{F^1_{y_2}(\tau_1,\tau_2)}{F^1(\tau_1,\tau_2)}\right)^{\frac{1}{2}}\left(\frac{2F^2(\tau_2)}{F^2_{y_2^2}(\tau_2)}\right)^{\frac{1}{4}} \cdot \\ &\quad \cdot \left[\exp\left(-\frac{k^2}{2n}\left(\frac{F^1_{y_2}(\tau_1,\tau_2)}{F^1(\tau_1,\tau_2)}\right)^2 \frac{F^2(\tau_2)}{F^2_{y_2^2}(\tau_2)}\right) + \mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{n}}{k^2}\right)\right] \end{aligned}$$

uniformly for $k = \mathcal{O}(n\log^{-2} n)$.

Remark. Note that the dicrete limiting distribution has a periodic structure, i.e. $A^{(0,1,2)}_{n+d_2} = A^{(0,1,2)}_n$ $(n \equiv 1 \bmod d)$ and the limiting values

$$\lim_{n\to\infty, n\equiv k \bmod d_{22}} p_{n,k,n-k}$$

exist. Especially, if $d_2|d_{11}$ and $d_2|d_{12}$ then we have $d = d_2$ and we get a much easier description than above:

$$\begin{aligned} E_n &= \frac{1}{1-\rho_2 F^1_{y_1}(\overline{\tau}_{1,0},\tau_2)} + \mathcal{O}(n^{-1}), \\ V_n &= \frac{\rho_2 F^1_{y_1}(\overline{\tau}_{1,0},\tau_2)}{(1-\rho_2 F^1_{y_1}(\overline{\tau}_{1,0},\tau_2))^2} + \mathcal{O}(n^{-1}), \\ p_{n,k,n-k} &= \frac{1-\rho_2 F^1_{y_1}(\overline{\tau}_{1,0},\tau_2)}{\rho_2 F^1_{y_1}(\overline{\tau}_{1,0},\tau_2)}\rho_2^k [u^k]F^1(u,\tau_2)F^1_{y_2}(u,\tau_2) + \mathcal{O}_k(n^{-1}). \end{aligned}$$

where $k \equiv 1 \bmod d_{11}$, $n-k \equiv 0 \bmod d_{22}$, and $n \equiv 1 \bmod d$. □

Next we consider the case $F^1_{y_1^2}(1,1) = 0$, $F^1_{y_2^2}(1) \ne 0$.

Theorem 2. *Let* $F^1(y_1,y_2) = b_1(y_2) + y_1 b_2(y_2)$, $F^2(y_2) \ne r + ty_2$, *and suppose that there exists a solution* $y_2 = \tau_2 > 0$ *of* $y_2 F^2_{y_2}(y_2) = F^2(y_2)$ *(which is chosen to be minimal).*

If $F^1_{y_1}(\cdot,\tau_2) = b_2(\tau_2) < F^2_{y_2}(\tau_2)$, *then the number of particles of type* T_1 *satisfies a discrete limit theorem with finite mean value* E_n *and variance*

V_n. Set $\zeta_{d_2} = \exp(2\pi i/d_2)$ and $\rho_2 = 1/F^2_{y_2}(\tau_2)$. Furthermore set

$$
\begin{aligned}
B_n^{(0)} &= \sum_{j=0}^{\frac{d_2}{d_{22}}-1} \zeta_{d_2}^{j(1-n)} \left(\frac{\rho_2 \zeta_{d_2}^j b_1'(\tau_2\zeta_{d_2}^j)}{1-\rho_2\zeta_{d_2}^j b_2(\tau_2\zeta_{d_2}^j)} + \frac{\rho_2^2 \zeta_{d_2}^{2j} b_1(\tau_2\zeta_{d_2}^j) b_2'(\tau_2\zeta_{d_2}^j)}{(1-\rho_2\zeta_{d_2}^j b_2(\tau_2\zeta_{d_2}^j))^2} \right), \\
B_n^{(1)} &= \sum_{j=0}^{\frac{d_2}{d_{22}}-1} \zeta_{d_2}^{j(1-n)} \left(\frac{\rho_2 \zeta_{d_2}^j b_1'(\tau_2\zeta_{d_2}^j)}{(1-\rho_2\zeta_{d_2}^j b_2(\tau_2\zeta_{d_2}^j))^2} + \frac{2\rho_2^2 \zeta_{d_2}^{2j} b_1(\tau_2\zeta_{d_2}^j) b_2'(\tau_2\zeta_{d_2}^j)}{(1-\rho_2\zeta_{d_2}^j b_2(\tau_2\zeta_{d_2}^j))^3} \right) \\
B_n^{(2)} &= \sum_{j=0}^{\frac{d_2}{d_{22}}-1} \zeta_{d_2}^{j(1-n)} \left(\frac{2\rho_2^2 \zeta_{d_2}^{2j} (b_1'(\tau_2\zeta_{d_2}^j) b_2(\tau_2\zeta_{d_2}^j) b_1(\tau_2\zeta_{d_2}^j) b_2'(\tau_2\zeta_{d_2}^j))}{(1-\rho_2\zeta_{d_2}^j F^1_{y_1}(0,\tau_2\zeta_{d_2}^j))^3} \right. \\
&\qquad \left. + \frac{6\rho_2^3 \zeta_{d_2}^{3j} b_1(\tau_2\zeta_{d_2}^j) b_2(\tau_2\zeta_{d_2}^j) b_2'(\tau_2\zeta_{d_2}^j)}{(1-\rho_2\zeta_{d_2}^j b_2(\tau_2\zeta_{d_2}^j))^4} \right)
\end{aligned}
$$

Then we have for $n \to \infty$

$$
E_n = \frac{B_n^{(1)}}{B_n^{(0)}} + \mathcal{O}(n^{-1})
$$

and

$$
V_n = \frac{B_n^{(2)}}{B_n^{(0)}} + \frac{B_n^{(1)}}{B_n^{(0)}} - \left(\frac{B_n^{(1)}}{B_n^{(0)}} \right)^2 + \mathcal{O}(n^{-1}).
$$

More precisly, for any fixed $k \ge 0$ and $n \to \infty$ ($n \equiv k \bmod d_{22}$) we have

$$
\begin{aligned}
p_{n,k,n-k} &= d_{22} \frac{\rho_2^k}{B_n^{(0)}} \sum_{j=0}^{\frac{d_2}{d}-1} \zeta_{d_2}^{j(1-n+k)} b_1(\tau_2\zeta_{d_2}^j) b_2(\tau_2\zeta_{d_2}^j)^{k-2} \cdot \\
&\qquad \cdot (b_1'(\tau_2\zeta_{d_2}^j) + (k-1) b_2'(\tau_2\zeta_{d_2}^j)) + \mathcal{O}_k(n^{-1}).
\end{aligned}
$$

Moreover, we have ($n \to \infty$, $n \equiv k \bmod d_{22}$)

$$
\begin{aligned}
p_{n,k,n-k} &= d_{22} \frac{\rho_2^k}{B_n^{(0)}} \sum_{j=0}^{\frac{d_2}{d_{22}}-1} \zeta_{d_2}^{j(1-n+k)} b_1(\tau_2\zeta_{d_2}^j) b_2(\tau_2\zeta_{d_2}^j)^{k-2} \cdot \\
&\qquad \cdot (b_1'(\tau_2\zeta_{d_2}^j) + (k-1) b_2'(\tau_2\zeta_{d_2}^j)) \cdot \\
&\qquad \cdot \left[\exp\left(-\frac{k^2}{2n} \frac{F^2(\tau_2\zeta_{d_2}^j)}{F^2_{y_2^2}(\tau_2\zeta_{d_2}^j)} \frac{b_2'(\tau_2\zeta_{d_2}^j)}{b_2(\tau_2\zeta_{d_2}^j)} \right) + \mathcal{O}(n^{-\frac12}) \right] \\
&= d_{22} \frac{\rho_2^k}{B_n^{(0)}} k b_1(\tau_2) b_2'(\tau_2) b_2(\tau_2)^{k-2} \cdot \\
&\qquad \cdot \left[\exp\left(-\frac{k^2}{2n} \frac{F^2(\tau_2)}{F^2_{y_2^2}(\tau_2)} \frac{b_2'(\tau_2)}{b_2(\tau_2)} \right) + \mathcal{O}(n^{-\frac12} + \alpha^k) \right]
\end{aligned}
$$

uniformly for $k = \mathcal{O}(n \log^{-2} n)$ *and some constant* $\alpha < 1$, *where the last equation only holds if* $b_2'(\tau_2) \neq 0$.

If $b_2(\tau_2) > F_{y_2}^2(\tau_2)$ *then the number of particles of type* T_1 *satisfy a central limit theorem with parameters calulated the same as in Theorem 1.*

Finally, suppose that $b_2(\tau_2) = F_{y_2}^2(\tau_2)$. *If* $b_2'(\tau_2) \neq 0$ *then the particles of type* T_1 *admit a Rayleigh limiting distribution with mean value*

$$E_n = \left(\frac{2F_{y_2^2}^2(\tau_2)}{F^2(\tau_2)} \frac{b_2(\tau_2)}{b_2'(\tau_2)} \right)^{\frac{1}{2}} \sqrt{n} + \mathcal{O}(1)$$

and variance

$$V_n = \left(1 - \frac{\pi}{4}\right) \frac{2F_{y_2^2}^2(\tau_2)}{F^2(\tau_2)} \frac{b_2(\tau_2)}{b_2'(\tau_2)} n + \mathcal{O}(\sqrt{n})$$

for $n \to \infty$. *Furthermore we have for* $n \equiv k \bmod d_{22}$

$$\begin{aligned} p_{n,k,n-k} &= d_{22} \frac{k}{n} \frac{F^2(\tau_2)}{F_{y_2 y_2}^2(\tau_2)} \frac{b_2'(\tau_2)}{b_2(\tau_2)} \cdot \\ &\quad \cdot \left[\exp\left(-\frac{k^2}{2n} \frac{F^2(\tau_2)}{F_{y_2 y_2}^2(\tau_2)} \frac{b_2'(\tau_2)}{b_2(\tau_2)} \right) + \mathcal{O}(n^{-\frac{1}{2}} + \alpha^k) \right] \end{aligned}$$

uniformly for $k = \mathcal{O}(n \log^{-2} n)$ *and* $n \to \infty$. *If* $b_2(\tau_2) = \text{const.}$ *then the particles of type* T_2 *satisfy a discrete limit theorem with mean value*

$$\overline{E}_n = \sum_{k \geq 0} (n-k) p_{n,k,n-k} = \frac{b_1'(\tau_2)\tau_2}{b_1(\tau_2)} + \mathcal{O}(n^{-\frac{1}{2}})$$

and variance

$$\overline{V}_n = \sum_{k \geq 0} (n - k - \overline{E}_n) p_{n,k,n-k} = \frac{b_1''(\tau_2)\tau_2^2}{b_1(\tau_2)} + \overline{E}_n - \overline{E}_n^2 + \mathcal{O}(n^{-\frac{1}{2}})$$

for $n \to \infty$. *Moreover we have for any fixed difference* $m = n - k \equiv 0 \bmod d_{22}$ *and* $n \to \infty$

$$p_{n,n-m,m} = \frac{\tau_2^m}{b_1(\tau_2)} [u^m] b_1(u) + \mathcal{O}_m(n^{-\frac{1}{2}})$$

and if $m = n - k \equiv 0 \bmod d_{22}$ *tends to* ∞

$$\begin{aligned} p_{n,n-m,m} &= \frac{d_{22} b_2(\tau_2)}{b_1(\tau_2)} m^{-\frac{3}{2}} \sum_{j=0}^{\frac{d_2}{d_{22}}-1} \zeta_{d_2}^{j(1-m)} \left(\frac{2F^2(\tau_2 \zeta_{d_2}^j)}{F_{y_2^2}^2(\tau_2 \zeta_{d_2}^j)} \right)^{\frac{1}{2}} \cdot \\ &\quad \cdot \left(1 + \mathcal{O}(m^{-1} + n^{-\frac{1}{2}})\right). \end{aligned}$$

REMARK. As in Theorem 1 there is a periodic structure in the discrete limit case if $d_2 > d_{22}$ which obviously simplifies if $d_2 = d_{22}$. $\square$

The next two theorems deal with the case $F^2_{y_2^2}(1) = 0$.

THEOREM 3. *Suppose that* $F^2(y_2) = r + ty_2$, $rt > 0$, *and either* $F^1_{y_1^2}(1,1) \neq 0$ *or* $F^1_{y_1^2}(1,1) = 0$ *and* $F^1_{y_1y_2}(1,1) \neq 0$. *Then the number of particles of type* T_1 *satisfies a central limit theorem with parameters calculated the same as in Theorem 1.*

In order to formulate Theorem 4 we need the following definition of a slowly varying function (see [FO]).

A function $L(y)$ is called slowly varying at ∞ if
1. there exists a positive number y_0 and and anlge $\varphi \in (0, \frac{\pi}{2})$ such that $L(y) \neq 0$ and is analytic in the domain $\{y \in \mathbf{C} \mid -(\pi - \varphi) \leq \arg(y - y_0) \leq \pi - \varphi\}$ and
2. there exists a function $\epsilon(y)$, defined for all $y \geq 0$ with $\lim_{y\to\infty} \epsilon(y) = 0$ such that for all $\vartheta \in [-(\pi - \varphi), \pi - \varphi]$ and $y \geq y_0$ we have

$$\left|\frac{L(ye^{i\vartheta})}{L(y)} - 1\right| < \epsilon(y) \quad \text{and} \quad \left|\frac{L(y\log^2 y)}{L(y)} - 1\right| < \epsilon(y).$$

We will write $H(z) \in R_\alpha(\gamma)$, $\alpha > 0$ if there is a domain $\Delta = \Delta(\eta, \varphi) = \{z \in \mathbf{C} > |z| < \gamma + \eta, |\arg(z - \gamma)| > \varphi\}$, where η and φ are positive, such that $H(x)$ is analytic in Δ with the sole exception $x = \gamma \in (0, \infty)$ and that

$$H(x) \sim \frac{1}{(1 - x/\gamma)^\alpha} L\left(\frac{1}{1 - x/\gamma}\right), \quad x \to \gamma, x \in \Delta \setminus \{\gamma\},$$

where $L(y)$ is slowly varying a ∞ in the above sense.

THEOREM 4. *Suppose that* $F^1(y_1, y_2) = g(y_2) + py_1$ *and* $F^2(y_2) = r + ty_2$, *and let* γ *denote the radius of convergence of*

$$H(x) = g\left(\frac{rx}{1 - tx}\right) = \sum_{n \geq 0} h_n x^n.$$

Then one of the following statements is valid.

If $\gamma > p^{-1}$ *then*

$$\lim_{n\to\infty} p_{n,n-m,m} = \frac{h_m p^{-m}}{H(p^{-1})} \tag{2.15}$$

for any fixed $m \geq 0$.

If $\gamma = p^{-1}$ *and* $H(x) \in R_\alpha(\gamma)$ *then*

$$p_{n,k,n-k} \sim \frac{1}{\alpha n}\left(1 - \frac{k}{n}\right)^{\alpha - 1} \tag{2.16}$$

uniformly in the range $n - k \geq n/(\log n)^2$ *as* $n \to \infty$.

If $\gamma < p^{-1}$ *and* $H(x) \in R_\alpha(\gamma)$ *then*

$$\lim_{n\to\infty} p_{n,k,n-k} = (\gamma p)^{k-1}(1-\gamma p) \tag{2.17}$$

for any fixed $k \geq 1$.

3. Proofs. A basic tool for the proofs of Theorems 1 and 2 is the following property of solutions of analytic functional equations.

Lemma 1 ([Dr3]). *Suppose that* $F(x, \mathbf{u}, y)$ *(*$\mathbf{u} = (u_1, \ldots, u_K)$*) is a power series with real non-negative coefficents in* $K+2$ *variables such that* $F(0,\ldots,0) = 0$ *and that* $(x_0, 1, \ldots, 1, y_0)$ *is contained in the range of convergence, where* $x = x_0$, $y = y_0$ *are the smallest positive solutions of*

$$y = F(x, 1, \ldots, 1, y), \quad 1 = F_y(x, 1, \ldots, 1, y).$$

Then there exists some real $\varepsilon > 0$ *such that there is a unique solution* $x = f(\mathbf{u})$ *(*$y = y(f(\mathbf{u}), \mathbf{u})$*) of*

$$y = F(x, \mathbf{u}, y), \quad 1 = F_y(x, \mathbf{u}, y) \tag{3.1}$$

for $|u_i - 1| < \varepsilon$ *(*$i = 1, \ldots, K$*) satisfying* $f(1, \ldots, 1) = x_0$. *If*

$$F_x(f(\mathbf{u}), \mathbf{u}, y(f(\mathbf{u}), \mathbf{u})) \neq 0$$

and

$$F_{y^2}(f(\mathbf{u}), \mathbf{u}, y(f(\mathbf{u}), \mathbf{u})) \neq 0$$

then $y(x, \mathbf{u})$ *has the local representation*

$$y(x, \mathbf{u}) = g(x, \mathbf{u}) - h(x, \mathbf{u})\sqrt{1 - \frac{x}{f(\mathbf{u})}} \tag{3.2}$$

for $|x - f(\mathbf{u})| < \varepsilon$, $\arg\left(1 - \frac{x}{f(\mathbf{u})}\right) \neq \pi$, *where* $g(x, \mathbf{u})$ *and* $h(x, \mathbf{u})$ *are analytic in this range and the latter satisfies*

$$h(f(\mathbf{u}), \mathbf{u}) = \sqrt{\frac{2f(\mathbf{u})F_x(f(\mathbf{u}), \mathbf{u}, y(f(\mathbf{u}), \mathbf{u}))}{F_{y^2}(f(\mathbf{u}), \mathbf{u}, y(f(\mathbf{u}), \mathbf{u}))}}. \tag{3.3}$$

If $|F_y(x, 1 \ldots, 1, y(x, 1 \ldots, 1))| < F_y(|x|, 1 \ldots, 1, y(|x|, 1 \ldots, 1))$ *for* $x \neq |x|$ *then* $x = f(\mathbf{u})$ *is the only singularity of* $y(x, \mathbf{u})$ *for* $|x| \leq x_0 + \varepsilon$, $\arg\left(1 - \frac{x}{f(\mathbf{u})}\right) \neq \pi$.

Lemma 2 ([Dr2]). *Let* C *be a Hankel contour starting from* $e^{2\pi i}\infty$, *passing around* $z = 0$ *and terminating at* ∞. *Then we have for any complex number* γ

$$\frac{1}{2\pi i}\int_C e^{-\gamma\sqrt{-z}-z}\, dz = \frac{\gamma}{2\sqrt{\pi}} e^{-\gamma^2/4}.$$

Proof of Theorem 1. The assumptions on $F^1(y_1,y_2)$ and on $F^2(y_2)$ ensure that the singularities of $\tilde\varphi_1(x,\varphi_2)$ and $\varphi_2(x)$ are of a special kind. By Lemma 1 $\varphi_2(x)$ has radius of convergence $\rho_2 = 1/F^2_{y_2}(\tau_2)$ and a local representation of the kind

$$\varphi_2(x) = g(x) - h(x)\sqrt{1-\frac{x}{\rho_2}}, \tag{3.4}$$

where $g(x)$ and $h(x)$ are analytic around $x=\rho_2$ and satisfy $g(\rho_2)=\tau_2$ and

$$h(\rho_2) = \left(\frac{2F^2(\tau_2)}{F^2_{y_2^2}(\tau_2)}\right)^{\frac12}.$$

Since

$$\varphi_2(x\zeta_{d_2}^l) = \zeta_{d_2}^l\varphi_2(x)$$

there are similar expansions around $\rho_2\zeta_{d_2}^l$ $(l=1,\ldots,d_2-1)$. Furthermore these are the only singularities on the circle of convergence $|x|=\rho_2$. In the same manner it follows that the radius of convergence of $\tilde\varphi_1(x,\tau_2)$ is given by $\rho_1 = 1/F_{y_1}(\tau_1,\tau_2)$ and there is a representation of the kind

$$\tilde\varphi_1(x,\tau_2) = g_1(x,\tau_2) - h_1(x,\tau_2)\sqrt{1-\frac{x}{f(\tau_2)}}.$$

By Lemma 1 this kind of representation even holds if φ_2 varies locally around $\varphi_2=\tau_2$, i.e.

$$\tilde\varphi_1(x,\varphi_2) = g_1(x,\varphi_2) - h_1(x,\varphi_2)\sqrt{1-\frac{x}{f(\varphi_2)}}, \tag{3.5}$$

where $g_1(x,\varphi_2)$, $h_1(x,\varphi_2)$, and $f(\varphi_2)$ are analytic functions. It turns out that these algebraic singularities determine the asymptotic behaviour of the coefficients of $\varphi_1(x,z_1,1)$, but it depends on the values $F^1_{y_1}(\tau_1,\tau_2)$, $F^2_{y_2}(\tau_2)$ whether the singularity of $\varphi_2(x)$ or that of $\tilde\varphi_1(x,\varphi_2)$ comes into account.

There will be three different cases. In any case the general idea of the present proof is the following one. First, we will calculate the asymptotic behaviour of the mean value E_n and the variance V_n in order to get a feeling which kind of range for k in relation to n is significant. Since

$$E_n = \sum_{k\ge0} kp_{n,k,n-k} = \frac{[x^n]\frac{\partial}{\partial z_1}\varphi(x,1,1)}{[x^n]\varphi(x,1,1)}$$

and

$$V_n = \sum_{k\ge0}(k-E_n)^2 p_{n,k,n-k} = \frac{[x^n]\frac{\partial^2}{\partial z_1^2}\varphi(x,1,1)}{[x^n]\varphi(x,1,1)} + E_n - E_n^2,$$

we just have to evaluate $[x^n]\varphi(x,1,1)$, $[x^n]\frac{\partial}{\partial z_1}\varphi(x,1,1)$, and $[x^n]\frac{\partial^2}{\partial z_1^2}\varphi(x,1,1)$. Then we will use (more of less) explicit expressions for

$$f_k(\varphi_2) = [x^k]\tilde{\varphi}_1(x,\varphi_2)$$

to determine the behaviour of

$$[x^n z_1^k]\varphi_1(x,z_1,1) = [x^n]x^k f_k(\varphi_2(x)).$$

and to obtain

$$p_{n,k,n-k} = \frac{[x^n z_1^k]\varphi_1(x,z_1,1)}{[x^n]\varphi(x,1,1)}.$$

First, suppose that $F^1_{y_1}(\tau_1,\tau_2) < F^2_{y_2}(\tau_2)$. Here we have $\rho_1 > \rho_2$ and $\varphi_1(x,z_1,z_2)$ can be representated by

$$\varphi_1(x,z_1,1) = \tilde{\varphi}_1(xz_1, g(x) - h(x)\sqrt{1-x/\rho_2}),$$

where $\tilde{\varphi}_1(x,\varphi_2)$ is regular at $x=\rho_2$, $\varphi_1=\tau_2$. Now, $\varphi_1(x\zeta^l_{d_2},1,1)$ has the local expansion

$$\begin{aligned}\varphi_1(x\zeta^l_{d_2},1,1) &= \varphi_1(\rho_2\zeta^l_{d_2},1,1) - \zeta^l_{d_2}h(\rho_2)\frac{\partial}{\partial\varphi_2}\tilde{\varphi}_1(\rho_2\zeta^l_{d_2},\tau_2\zeta^l_{d_2})\sqrt{1-x/\rho_2}\\ &+ C_j(1-x/\rho_2) + \mathcal{O}\left((1-x/\rho)^{3/2}\right).\end{aligned}$$

Hence, using the equality

$$\frac{\partial}{\partial\varphi_2}\tilde{\varphi}_1(\rho_2\zeta^l_{d_2},\tau_2\zeta^l_{d_2}) = \frac{\rho_2\zeta^l_{d_2}F^1_{y_2}(\overline{\tau}_{1,l},\tau_2\zeta^l_{d_2})}{1-\rho_2\zeta^l_{d_2}F^1_{y_1}(\overline{\tau}_{1,l},\tau_2\zeta^l_{d_2})}$$

and the property

$$\frac{\partial}{\partial\varphi_2}\tilde{\varphi}_1\left(a\zeta_{d_2}^{md_2/d}, b\zeta_{d_2}^{md_2/d}\right) = \frac{\partial}{\partial\varphi_2}\tilde{\varphi}_1(a,b).$$

we get by [FO]

$$\begin{aligned}&[x^n]\varphi_1(x,1,1)\\ &= \frac{n^{-3/2}\rho_2^{-n}}{2\sqrt{\pi}}h(\rho_2)\sum_{l=0}^{d_2-1}\zeta_{d_2}^{l(1-n)}\frac{\partial}{\partial\varphi_2}\tilde{\varphi}_1(\rho_2\zeta^l_{d_2},\tau_2\zeta^l_{d_2})\left(1+\mathcal{O}(n^{-1})\right)\\ &= d\,\frac{n^{-3/2}\rho_2^{-n}}{2\sqrt{\pi}}h(\rho_2)A_n^{(0)}\left(1+\mathcal{O}(n^{-1})\right)\end{aligned}$$

for $n \equiv 1 \bmod d$. By the same mehod applying the equalities

$$\begin{aligned}\frac{\partial^2}{\partial x\partial\varphi_2}\tilde{\varphi}_1(\rho_2\zeta^l_{d_2},\tau_2\zeta^l_{d_2}) &= \frac{F^1_{y_2}(\overline{\tau}_{1,l},\tau_2\zeta^l_{d_2})}{(1-\rho_2\zeta^l_{d_2}F^1_{y_1}(\overline{\tau}_{1,l},\tau_2\zeta^l_{d_2}))^2}\\ \frac{\partial^3}{\partial x^2\partial\varphi_2}\tilde{\varphi}_1(\rho_2\zeta^l_{d_2},\tau_2\zeta^l_{d_2}) &= \frac{2F^1_{y_1}(\overline{\tau}_{1,l},\tau_2\zeta^l_{d_2})F^1_{y_2}(\overline{\tau}_{1,l},\tau_2\zeta^l_{d_2})}{(1-\rho_2\zeta^l_{d_2}F^1_{y_1}(\overline{\tau}_{1,l},\tau_2\zeta^l_{d_2}))^3}\end{aligned}$$

we obtain

$$[x^n]\frac{\partial}{\partial z_1}\varphi(x,1,1) = d\,\frac{n^{-3/2}\rho_2^{-n}}{2\sqrt{\pi}}A_n^{(1)}\left(1+\mathcal{O}(n^{-1})\right)$$

and

$$[x^n]\frac{\partial^2}{\partial z_1^2}\varphi(x,1,1) = d\,\frac{n^{-3/2}\rho_2^{-n}}{2\sqrt{\pi}}A_n^{(2)}\left(1+\mathcal{O}(n^{-1})\right)$$

for $n \equiv 1 \bmod d$. Combining these relations we get the needed relations for E_n and V_n.

Now, let $k \equiv 1 \bmod d_{11}$ be fixed and set $f_k(\varphi_2) = [x^k]\tilde{\varphi}_1(x,\varphi_2)$. As above we get

$$\begin{aligned}
&[x^n z_1^k]\varphi_1(x,z_1,1)\\
&\quad = \frac{n^{-3/2}\rho_2^{-n+k}}{2\sqrt{\pi}}h(\rho_2)\sum_{l=0}^{d_2-1}\zeta_{d_2}^{l(1-n+k)}f_k'(\tau_2\zeta_{d_2}^l)\left(1+\mathcal{O}_k(n^{-1})\right)\\
&\quad = d_{22}\,\frac{n^{-3/2}\rho_2^{-n+k}}{2\sqrt{\pi}}h(\rho_2)\sum_{j=0}^{\frac{d_2}{d_{22}}-1}\zeta_{d_2}^{j(1-n+k)}f_k'(\tau_2\zeta_{d_2}^j)\left(1+\mathcal{O}_k(n^{-1})\right)
\end{aligned}$$

for $n \equiv k \bmod d_{22}$. Since

$$f_k'(\tau_2\zeta_{d_2}^j) = [u^{k-1}]F^1(u,\tau_2\zeta_{d_2}^j)^{k-1}F_{y_2}^1(u,\tau_2\zeta_{d_2}^j)$$

we obtain the proposed asymptotic formula for $p_{n,k,n-k}$ for any fixed $k \equiv 1 \bmod d_{11}$ and $n\to\infty$ ($n \equiv k \bmod d_{22}$).

Next, vary φ_2 locally around $\tau_2\zeta_{d_2}^l$ and set $\lambda_l(\varphi_2) = 1/F_{y_1}^1(\overline{\varphi}_1,\varphi_2)$, where $\overline{\varphi}_1 = \overline{\varphi}_1(\varphi_2)$ is the solution of

$$\overline{\varphi}_1 F_{y_1}^1(\overline{\varphi}_1,\varphi_2) = F^1(\overline{\varphi}_1,\varphi_2)$$

such that $\lambda_l(\varphi_2)$ has minimal modulus. (Obviously, $\lambda_0(\varphi_2) = f_1(\varphi_2)$.) Furthermore, let $\tau_{1,l} = \lambda_l(\tau_2\zeta_{d_2}^l)$ and set $\rho_{1,j} = 1/F_{y_1}^1(\tau_{1,j},\tau_2\zeta_{d_2}^j)$. Observe that

$$\lambda_l'(\tau_2\zeta_{d_2}^l) = -\frac{\rho_{1,l}F_{y_2}^1(\tau_{1,l},\tau_2\zeta_{d_2}^j)}{F^1(\tau_{1,l},\tau_2\zeta_{d_2}^j)}.$$

Hence we can apply [FO] to the local representations

$$\tilde{\varphi}_1(x,\varphi_2) = g_{1,l}(x,\varphi_2) - h_{1,l}(x,\varphi_2)\sqrt{1-\frac{x}{\lambda_l(\varphi_2)}}$$

to obtain

$$f_k(\varphi_2) = d_{11}\frac{k^{-3/2}}{2\sqrt{\pi}}h_{1,l}(\lambda_l(\varphi_2),\varphi_2)\lambda_l(\varphi_2)^{-k}(1+\mathcal{O}(k^{-1})),$$

where $k \equiv 1 \bmod d_{11}$ and

$$h_{1,l}(\tau_{1,l}, \tau_2\zeta_{d_2}^l) = \left(2\frac{F^1(\tau_{1,l}, \tau_2\zeta_{d_2}^l)}{F^1_{y_1^2}(\tau_{1,l}, \tau_2\zeta_{d_2}^l)}\right)^{1/2}.$$

It should be mentioned that we have used the fact that there exists a real number $R > \tau_1$ such that the equation $\overline{\varphi}_1 F^1_{y_1}(\overline{\varphi}_1, \varphi_2) = F^1(\overline{\varphi}_1, \varphi_2)$ has exactly d_{11} solutions satisfying $|\overline{\varphi}_1| \le R$ if $|\varphi_2| = \tau_2$. Since this statement is true for $\varphi_2 = \tau_2$ it suffices to find some $R > \tau_1$ such that $\overline{\varphi}_1 F^1_{y_1}(\overline{\varphi}_1, \varphi_2) \neq F^1(\overline{\varphi}_1, \varphi_2)$ if $|\overline{\varphi}_1| = R$ and $|\varphi_2| = \tau_2$, and this can easily be established.

Since $\varphi_2(x)$ admits the local representation (3.4) we can approximate $\lambda_l(\varphi_2)^{-k}$ by

$$\lambda_l(\varphi_2)^{-k} = \rho_{1,l}^{-k} \exp\left(k\frac{\lambda_l'(\tau_2\zeta_{d_2}^l)}{\rho_{1,l}} h(\rho_2)\zeta_{d_2}^l \sqrt{1 - \frac{x}{\rho_2}}\right) \cdot (1 + \mathcal{O}(k|x - \rho_2|)).$$

Now we will use Cauchy's formula and singularity analysis to get an asymptotic expansion for $p_{n,k,n-k}$. For this purpose we will use small parts of a Hankel contour around the singularities $\rho_2\zeta_{d_2}^l$ $(l = 0, \ldots, d_2 - 1)$. (We will just give a sketch. For more detail see [Dr2].) As above it is sufficient to discuss the singularities $\rho_2\zeta_{d_2}^j$ for $j = 0, \ldots, \frac{d_2}{d_{22}} - 1$ if $n \equiv k \bmod d_{22}$ and to multiply the resulting integral by d_{22}. Using the substitution $x = \rho_2\left(1 + \frac{t}{n}\right)$ we immediately obtain

$$\begin{aligned}
[x^n]x^k f_k(\varphi_2(x)) &= d_{11}d_{22}\,\frac{k^{-1/2}n^{-3/2}}{4\pi}\rho_2^{k-n}\,\cdot \\
&\quad \cdot \sum_{j=0}^{\frac{d_2}{d_{22}}-1} \zeta_{d_2}^{j(k+1-n)} h(\rho_2)\frac{h_{1,l}F^1_{y_2}}{F^1}(\tau_{1,j}, \tau_2\zeta_{d_2}^j)\rho_{1,j}^{-k}\,\cdot \\
&\quad \cdot \exp\left(-\frac{k^2}{4n}\left(h(\rho_2)\zeta_{d_2}^j\frac{F^1_{y_2}}{F^1}(\tau_{1,j}, \tau_2\zeta_{d_2}^j)\right)^2\right) \\
&+ \mathcal{O}\left(k^{-\frac{1}{2}}n^{-2} + k^{-\frac{5}{2}}n^{-1}\right).
\end{aligned}$$

where we have applied Lemma 2 for

$$\gamma = \gamma_l = \frac{k}{\sqrt{n}}h(\rho_2)\zeta_{d_2}^j\frac{F^1_{y_2}}{F^1}(\tau_{1,j}, \tau_2\zeta_{d_2}^j)$$

and have used quite easy exponentially small estimates for the remaining parts of integrals. Finally, we only have to use the fact that $\rho_{1,j} > \rho_{1,0}$ if $j \not\equiv 0 \bmod \frac{d_2}{d_{22}}$ which implies that only the first term in the above sum is significant. This proves the first part of Theorem 1.

Next let $F^1_{y_1}(\tau_1, \tau_2) > F^2_{y_2}(\tau_2)$. Here $\tilde{\varphi}_1(x, \varphi_2)$ is singular at $x = \overline{\rho}$, $\varphi_2 = \overline{\tau}_2$ and $\varphi_2(x)$ is regular at $x = \overline{\rho}$ and satisfies $\varphi_2(\overline{\rho}) = \overline{\tau}_2$. Hence

$\varphi_1(x,1,1)$ is singular at $x = \overline{\rho}$. Trivially we have $\varphi_1(x\exp(j\frac{2\pi i}{d}),1,1) = \exp(j\frac{2\pi i}{d})\varphi_1(x,1,1)$ $(j=0,\ldots,d-1)$. Thus there are at least d singularities $\overline{\rho}\exp(j\frac{2\pi i}{d})$ $(j=0,\ldots,d-1)$ on the circle of convergence $|x| = \overline{\rho}$. However, it is quite easy to see that these are the only singularities on $|x| = \overline{\rho}$.

From (3.5) it follows that $\varphi_1(x,z_1,1,)$ has the local representation

$$\varphi_1(x,z_1,1) = g_1(xz_1,\varphi_2(x)) - h_1(xz_1,\varphi_2(x))\sqrt{1-\frac{xz_1}{f(\varphi_2(x))}},$$

where $f(\overline{\tau}_2) = \overline{\rho}$ and $f'(\overline{\tau}_2) < 0$. Since $x = \overline{\rho}$ is a single zero of $f(\varphi_2(x)) - x$ it follows by Weierstrass' preparation theorem that there exist two analytic functions $u(x,z_1)$, $v(z_1)$ satisfying $u(\overline{\rho},1) \neq 0$, $v(1) = \overline{\rho}$ such that

$$f(\varphi_2(x)) - xz_1 = u(x,z_1)(v(z_1) - x).$$

locally around $x = \overline{\rho}$, $z_1 = 1$. Hence we locally get

$$\varphi_1(x,z_1,1) = g_2(x,z_1) - h_2(x,z_1)\sqrt{1-\frac{x}{v(z_1)}},$$

where $x = x(z_1)$, $y_1 = y_1(z_1)$, $y_2 = y_2(z_1)$ are the solutions of

$$y_1 = xz_1F^1(y_1,y_2), \quad 1 = xz_1F^1_{y_1}(y_1,y_2), \quad y_2 = xF^2(y_2).$$

Now the situation is quite similar to that discussed in [Dr1]. By differentiating with respect to z_1 and applying [FO] we immediately get the asymptotic expansions for E_n and V_n. The constants μ, σ^2 are given by

$$\mu = -\frac{v'(1)}{v(1)}$$

and by

$$\sigma^2 = \mu^2 + \mu - \frac{v''(1)}{v(1)}$$

and can be calculated easily by implicit differentiation. For example, we get

$$[x^n]\varphi_1(x,1,1) = d\frac{n^{-\frac{3}{2}}}{2\sqrt{\pi}}\overline{\rho}^{-n}h_1(\overline{\rho},\overline{\tau}_2)\mu^{-\frac{1}{2}}(1+\mathcal{O}(n^{-1}))$$

for $n \equiv 1 \bmod d$.

In order to get the local law we will use Cauchy's formula to obtain an asymptotic expansion for $[x^n]x^kf_k(\varphi_2(x))$. For this purpose we will use a saddle point method around the points $x = \overline{\rho}\exp(j\frac{2\pi i}{d_{22}})$. This procedure is justified since $\overline{\rho}$ is a saddle point of the functions $x^{k-n}f_k(\varphi_2(x))$, $f_k(\varphi_2(x\exp(j\frac{2\pi i}{d_{22}})) = f_k(\varphi_2(x))$, and $|f_k(\varphi_2(x))| < f_k(\varphi_2(|x|))$ if $\arg x \neq$

$j\frac{2\pi}{d_{22}}$. First, we will consider a small part of a path of integration passing $x=\overline{\rho}$, more precisely let

$$\gamma = \{x \,|\, |x-\overline{\rho}| \le n^{-\frac{1}{2}+\varepsilon}, |f_1(\varphi_2(x))| = \overline{\tau}_1\}.$$

Furthermore, we have

$$f_k(\varphi_2) = d_{11}\frac{k^{-\frac{3}{2}}}{2\sqrt{\pi}}h_1(f(\varphi_2),\varphi_2)f(\varphi_2)^{-k}(1+\mathcal{O}(k^{-1})).$$

Set $k=\mu n + r$. As in [Dr2] it can be shown that

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma x^k f_k(\varphi_2(x))\,\frac{dx}{x^{n+1}} &= d_{11}\frac{k^{-\frac{1}{2}}n^{-\frac{3}{2}}}{2\sqrt{2\pi}\sigma}h_1(\overline{\tau}_1,\overline{\tau}_2)\cdot \\ &\quad \cdot\left(\exp\left(-\frac{r^2}{2\sigma^2 n}\right)+\mathcal{O}(n^{-\frac{1}{2}}+k^{-1})\right).\end{aligned}$$

Similar expansions can be obtainded around $x=\overline{\rho}\exp(j\frac{2\pi i}{d_{22}})$. Furthermore the remaining integral is exponentially small and the local law follows.

Finally let $F^1_{y_1}(\tau_1,\tau_2) = F^2_{y_2}(\tau_2)$. Here we have

$$\varphi_1(x,z_1,1) = g_1(xz_1,\varphi_2(x)) - h_1(xz_1,\varphi_2(x))\sqrt{1-\frac{xz_1}{f(\varphi_2(x))}},$$

where $f(\tau_2)=\rho$ and $\varphi_2(x)$ is singular at $x=\rho$ (compare with (3.4)). Hence $\varphi_1(x,1,1)$ has radius of convergence ρ and a singularity of the kind

$$\varphi_1(x,1,1) = \tau_1 - h_1(\rho,\tau_2)\sqrt{A}\left(1-\frac{x}{\rho}\right)^{\frac{1}{4}} + \mathcal{O}\left(\left(1-\frac{x}{\rho}\right)^{\frac{1}{2}}\right),$$

where

$$A = -\frac{f'(\tau_2)}{\rho}h(\rho) = \sqrt{\frac{2F^2(\tau_2)}{F^2_{y_2}(\tau_2)}\frac{F^1_{y_2}(\tau_1,\tau_2)}{F^1(\tau_1,\tau_2)}}.$$

As above it is easy to show that $\varphi_1(x,1,1)$ has only d singularities $\rho\exp(j\frac{2\pi i}{d})$ $(j=0,\ldots,d-1)$ of this kind on the circle of convergence $|x|=\rho$. The other singularities $x=\rho\zeta^j_{d_2}$ $(j\not\equiv 0 \bmod \frac{d_2}{2})$ on the circle $|x|=\rho$ are of squareroot type. Hence we get by [FO]

$$[x^n]\varphi_1(x,1,1) = d\,\frac{\rho^{-n}}{4\Gamma(\frac{3}{4})}n^{-\frac{5}{4}}h_1(\rho,\tau_2)A(1+\mathcal{O}(n^{-\frac{1}{4}}))$$

for $n\equiv 1 \bmod d$. Similarly we obtain the asymptotic expansion

$$[x^n]\frac{\partial}{\partial z_1}\varphi_1(x,1,1) = d\,\frac{\rho^{-n}}{2\Gamma(\frac{1}{4})}n^{-\frac{3}{4}}\frac{h(\rho,\tau_2)}{\sqrt{A}}(1+\mathcal{O}(n^{-\frac{1}{2}}))$$

for $n \equiv 1 \bmod d$. Hence the asymptotic expansion for E_n follows. By the same way we get the asymptotic expansion for V_n.

In order to get the local law we will proceed as in the discrete limit case. As above we get

$$\begin{aligned}[x^n]x^k f_k(\varphi_2(x)) &= d_{11}d_{22}\frac{k^{-\frac{1}{2}}n^{-\frac{3}{2}}}{4\pi}A\exp\left(-\frac{A^2k^2}{4n}\right) \\ &+ \mathcal{O}(k^{-\frac{1}{2}}n^{-2}+k^{-\frac{5}{2}}n^{-1}).\end{aligned}$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. From the conditions of the theorem it follows that $\varphi_2(x)$ has an algebraic singularty of the kind (3.4) and $\tilde{\varphi}_1(xz_1,\varphi_2)$ is given by

$$\tilde{\varphi}_1(xz_1,\varphi_2) = \frac{xz_1b_1(\varphi_2)}{1-xz_1b_2(\varphi_2)} = b_1(\varphi_2)\sum_{k\ge1}(xz_1)^k b_2(\varphi_2)^{k-1}$$

or equivalently we have

$$f_k(\varphi_2) = b_1(\varphi_2)(b_2\varphi_2))^{k-1}.$$

First suppose that $b_2(\tau_2) < F^2_{y_2}(\tau_2)$. This means that $\rho_2 < 1/b_2(\tau_2)$. Hence, $\varphi_1(x,1,1)$ has algebraic singulaties at $x=\rho_2\zeta_{d_2}^j$ $(j=0,\ldots,d_2-1)$. So we are in a similar situation as in the proof of Theorem 1. The only difference is that the explicit expression for $\tilde{\varphi}(xz_1,\varphi_2)$ can be used to calculate $\frac{\partial}{\partial\varphi_2}\tilde{\varphi}_1(\rho_2,\tau_2)$ etc. more directly.

Next let $b_2(\tau_2) > F^2_{y_2}(\tau_2)$. Here the radius of convergence of $\varphi_1(x,1,1)$ is smaller that ρ_2. Hence $\varphi_1(x,z_1,1)$ has a representation of the kind

$$\varphi_1(x,z_1,1) = \frac{h_3(x,z_1)}{w(z_1)-x}, \tag{3.6}$$

which implies the desired asymptotic normality.

Finally the case $b_2(\tau_2) = F^2_{y_2}(\tau_2)$ can be treated as in the proof of Theorem 1 since $f_k(\varphi_2)$ has a similar shape. The only difference is that a factor of the kind $k^{-\frac{3}{2}}$ does not appear. This is the reason why a Rayleigh limiting distribution instead of a χ^2 limiting distribution appears. □

Proof of Theorem 3. If $F^2(y_2) = r+ty_2$ then $\varphi_2(x) = \frac{rx}{1-tx}$. Hence, $F^2(x)$ has a polar singularity at $x=\rho_2=1/t$, especially $\varphi_2(x)$ is not bounded around $x=\rho$. Combining this fact with thte conditions of the theorem we see that the radius of convergence of $\varphi_1(x,1,1)=\tilde{\varphi}_1(x,\varphi_2(x))$ is always smaller than ρ. More precisely, if $F^1_{y_1^2}(1,1)\ne0$ then $\varphi_1(x,z_1,1)$ can be locally represented by

$$\varphi_1(x,z_1,1) = g_2(x,z_1) - h_2(x,z_1)\sqrt{1-\frac{x}{v(z_1)}},$$

where $v(1) < \rho_2$, and if $F^1_{y_1^2}(1,1) = 0$ and $F^1_{y_1 y_2}(1,1) \neq 0$ then (3.6) is valid and we are in the same situation as in the case $b_2(\tau_2) > F^2_{y_2}(\tau_2)$ of Theorem 2. In both cases we get asymptotic normality. □

Proof of Theorem 4. If $F^1(y_1, y_2) = g(y_2) + py_1$ and $F^2(y_2) = r + ty_2$ then

$$\varphi_2(x) = \frac{rx}{1 - tx}$$

and

$$\varphi_1(x, z_1, 1) = \frac{xz_1}{1 - pxz_1} g(\varphi_2(x)) = \frac{xz_1}{1 - pxz_1} H(x).$$

Hence

$$c_{n,k,n-k} = p^{k-1} h_{n-k}.$$

If $\gamma > p^{-1}$ then

$$c_n \sim H(p^{-1}) p^{n-1}, \quad n \to \infty$$

which gives (2.15).

If $\gamma = p^{-1}$ then $H(x) \in R_\alpha(\gamma)$ yields (see [FO])

$$h_k \sim \frac{k^{\alpha-1} L(k)}{\gamma^k \Gamma(\alpha)}, \quad k \to \infty.$$

Thus

$$c_{n,k,n-k} \sim \frac{(n-k)^{\alpha-1} L(n-k)}{\gamma^{n-1} \Gamma(\alpha)}, \quad n - k \to \infty$$

and

$$c_n \sim \frac{n^\alpha L(n)}{\gamma^{n-1} \Gamma(\alpha+1)}, \quad n \to \infty.$$

Hence (2.16) follows if $n - k \geq n/(\log n)^2$.

Finally, if $\gamma < p^{-1}$ then

$$c_{n,k,n-k} \sim p^{k-1} \frac{(n-k)^{\alpha-1} L(n-k)}{\gamma^{n-k} \Gamma(\alpha)}, \quad n - k \to \infty$$

and

$$c_n \sim \frac{\gamma}{1 - \gamma p} \frac{n^{\alpha-1} L(n)}{\gamma^n \Gamma(\alpha)}, \quad n \to \infty.$$

Thus (2.17) holds for any fixed $k \geq 1$. This completes the proof of Theorem 4. □

4. The general case. In this section we will give an outline how the general case of N different particles can be handled.

In principal this can be managed as the case $N=2$. First, use a proper representation of $\varphi_1 = \tilde\varphi_1(x,\varphi_2,\ldots,\varphi_N)$ which is the solution of

$$\varphi_1 = xF^1(\varphi_1,\varphi_2,\ldots,\varphi_N)$$

to obtain an expression for

$$f_{k_1}(\varphi_2,\ldots,\varphi_N) = [x^{k_1}]\tilde\varphi_1(x,\varphi_2,\ldots,\varphi_N).$$

Next, use a representation of $\varphi_2 = \tilde\varphi_2(x,\varphi_3,\ldots,\varphi_N)$ to get an expression for

$$f_{k_1k_2}(\varphi_3,\ldots,\varphi_N) = [x^{k_2}]f_{k_1}(\tilde\varphi_2(x,\varphi_3,\ldots,\varphi_N),\varphi_3,\ldots,\varphi_N)$$

and so on. Inductively you will finally get an expression for

$$f_{k_1k_2\cdots k_{N-1}}(\varphi_N) = [x^{k_{N-1}}]f_{k_1k_2\cdots k_{N-2}}(\tilde\varphi_{N-1}(x,\varphi_N),\varphi_N)$$

which can be used to obtain an asymptotic expansion for

$$c_{n,k_1,\ldots,k_{N-1},n-k_1-\cdots-k_{N-1}} = [x^n]x^{k_1+\cdots+k_{N-1}}f_{k_1k_2\cdots k_{N-1}}(\varphi_N(x)),$$

where $\varphi_N(x)$ is the solution of $\varphi_N = xF^N(\varphi_N)$.

In this way you will always derive a local limit law and depending on the choice of parameters you will get several different multivariate limiting distributions, e.g. a central limit law.

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