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ANALYSIS OF GENERATING FUNCTIONS AND PROBABILITIES ON TREES

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ABSTRACT

The aim of this paper is to present analytic tools for the analysis of generating functions which can be applied especially for tree enumeration problems and for distribution problems related to random trees.

As an (extended) example we prove an asymptotic formula for the number of rooted trees (in a simply generated family) of size n with l leaves such that the m-th leaf has height k.

1. INTRODUCTION

The advantage of trees with respect to combinatorial problems is that in many important cases it is possible to translate the counting problem of one or more parameters of trees to the language of generating functions. These generating funcions can either be used to solve the combinatorial counting problem explicitly or to solve it asymptotically and to obtain distributional results.

The purpose of this contribution is to present analytic methods leading to asymptotic formulas and distribution results for specific tree parameters.

In section 2 we will present the notion of simply generated families of trees and the connection to Galton-Watson branching processes. This concept leads to functional equations for generating functions which coefficients count trees with specific tree parameters.

Section 3 is devoted to a description of the main analytic tools in order to obtain asymptotic estimates for the coefficents of generating functions: the saddle point method and singularity analysis.

Finally in section 4 we provide an application of these concepts. We prove

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an asymptotic formula for the number of rooted trees (in a simply generated family) of size n with l leaves such that the m-th leaf has height k. This asymptotic formula also shows that the the height of the m-th leaf is Maxwell distributed (under suitable normalization). Furthermore it can be shown that the corresponding contour process converges to Brownian excursion.

2. TREES AND GENERATING FUNCTIONS

2.1. Simply Generated Trees and Galton-Watson Processes

Let $\mathcal{F}$ denote a simply generated family of (rooted) trees (introduced by Meir and Moon, 1978). This means that there is a (formal) power series

$$\varphi(t) = \varphi_0 + \varphi_1 t + \varphi_2 t^2 + \cdots \tag{1}$$

of non-negative numbers $\varphi_0 > 0$, $\varphi_1 \ge 0$, $\varphi_2 \ge 0, \ldots$, which are used to define a weight $\omega(T)$ for any planted plane (rooted) tree T by

$$\omega(T) = \prod_{i\ge 0} \varphi_i^{D_i(T)}, \tag{2}$$

where $D_i(T)$ denotes the number of nodes in T with out-degree i ($D_0(T)$ is the number of leaves, for example) such that $\mathcal{F}$ consists of all trees with weight $\omega(T) > 0$. Let $\mathcal{F}_n$ be the set of trees $T \in \mathcal{F}$ with $|T| = n$ (i.e. T consists of n nodes). Then the generating function

$$y(x) = \sum_{n\ge 0} y_n x^n \tag{3}$$

of the numbers

$$y_n = \sum_{|T|=n} \omega(T) = \sum_{T\in\mathcal{F}_n} \omega(T) \tag{4}$$

satisfies the (formal) functional equation

$$y = x\varphi(y). \tag{5}$$

Hence $\mathcal{F}$, interpreted as a combinatorial structure can be defined recursively by

$$\mathcal{F} = \circ \times \varphi(\mathcal{F}), \tag{6}$$

where $\circ$ denotes a node and $\varphi(\mathcal{F})$ a substituted structure (compare with Vitter and Flajolet, 1990).

For example, let $\varphi_i = 1$ for all $i \geq 0$, then y_n counts all rooted planted plane trees of size n. From $\varphi(t) = 1/(1-t)$ we get $y = x/(1-y)$ and hence

$$y_n = \frac{1}{n}\binom{2n-2}{n-1}. \tag{7}$$

Recursively, all planted plane trees can be described as a node followed by a sequence of trees:

$$\mathcal{F} = \circ + \begin{array}{c}\circ\\ |\\ \mathcal{F}\end{array} + \begin{array}{c}\circ\\ / \ \backslash\\ \mathcal{F}\ \mathcal{F}\end{array} + \cdots = \circ \times \mathcal{F}^*. \tag{8}$$

For $\varphi(t) = 1 + t^2$ we get binary trees, for $\varphi(t) = 1 + t + t^2$ Motzkin trees and so on. Gererally if $\varphi_i \in \{0, 1\}$ we are always counting special kinds of rooted trees admitting a recursive description. It is easy to extend this concept to count special parameters in these families of trees, e.g. the number of leaves as we will see in a moment.

Alternatively if $\varphi(1) = \sum_{i\geq 0} \varphi_i = 1$ then simply generated families of trees can be considered as realizations of a (Galton-Watson) branching process where $\omega(T)$ denotes the probabilty that T occurs (see Athreya and Ney, 1972). Instead of counting a parameter we directly get the probabilistic distribution of a parameter. However, both concepts are equivalent when we are only interested in discrete parameters in $\mathcal{F}_n$ (e.g. the number of leaves) resp. in the conditional distribution of the parameter of interest in realizations of size n. In fact, if we set $\overline{\varphi}_i = \varphi_i \tau^i/\varphi(\tau)$ for some $\tau > 0$ such that $\varphi(\tau)$ exists, then $\sum_{i\geq 0} \overline{\varphi}_i = 1$ ($\overline{\varphi}_i$ can be considered as the "offspring probability") and the conditional distributions are the same (compare with Kennedy, 1975). The corresponding weight $\overline{\omega}(T)$ is the probability that T occurs.

It should be further noted that the "offspring variance"

$$\sigma^2 = \sigma^2(\tau) = \sum_{i\geq 0} (i-1)^2 \overline{\varphi}_i = \sum_{i\geq 0} \frac{(i-1)^2 \varphi_i \tau^i}{\varphi(\tau)} = \frac{\tau^2 \varphi''(\tau)}{\varphi(\tau)} \tag{9}$$

plays an important role since it appears very often as a normalizing constant in the analysis of the asymptotic behaviour of simply generated families of trees (see Aldous, 1991), especially if τ is chosen in a way that the corresponding branching process becomes critical, i.e. $\sum_{i\geq 0} i\overline{\varphi}_i = 1$ resp $\tau\varphi'(\tau) = \varphi(\tau)$.

2.2. The Number of Leaves

Set $y_l(T) = 1$ if T has exactly l leaves and $y_l(T) = 0$ otherwise and let

$$y(x,u) = \sum_{n,l} y_{ln} x^n u^l \tag{10}$$

be the generating function of the numbers

$$y_{ln} = \sum_{|T|=n} \omega(T) y_l(T). \tag{11}$$

Then it is easy to verify that $y(x,u)$ satisfies the functional equation

$$y(x,u) = \varphi_0 x(u-1) + x\varphi(y(x,u)), \tag{12}$$

where it is quite useful to apply the combinatorial interpretation. Recall (6), saying that simply generated families of trees can be described recursively by

$$\mathcal{F} = \varphi_0 \cdot \circ + \varphi_1 \cdot \begin{array}{c}\circ\\ |\\ \mathcal{F}\end{array} + \varphi_2 \cdot \begin{array}{c}\circ\\ /\ \backslash\\ \mathcal{F}\ \mathcal{F}\end{array} + \cdots. \tag{13}$$

In order to "count" the leaves it is only necessary to mark those nodes in this construction correspondion to leaves:

$$\mathcal{F} = \varphi_0 \cdot \bullet + \varphi_1 \cdot \begin{array}{c}\circ\\ |\\ \mathcal{F}\end{array} + \varphi_2 \cdot \begin{array}{c}\circ\\ /\ \backslash\\ \mathcal{F}\ \mathcal{F}\end{array} + \cdots. \tag{14}$$

Now (14) directly translates to (12).

3. ANALYTIC TOOLS

The basic object is to extract an asymptotic expansion of the coefficients a_n from an power series $A(x) = \sum\limits_{n\ge 0} a_n x^n$. Analytically this can be done by Cauchy's formula

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{A(x)}{x^{n+1}}\, dx, \tag{15}$$

where γ is closed curve with winding number 1 around the origin 0, e.g. a positively oriented circle contained in the region of convergence of the power series $A(x)$.

In what follows we present two methods how analytic properties of $A(x)$ translate to asymptotic expansions of a_n, the saddle point method and singularity analysis.
We will frequently use the notation

$$a_n = [x^n]A(x) \tag{16}$$

for the n-th coefficient of the power series $A(x) = \sum_{n\geq 0} a_n x^n$.

3.1. Saddle Point Method

In many applications (as we will see later) one has to extract the coefficient of a function of the form $g(x)y(x)^k$. Functions of this kind are especially suited for the saddle point method as the following theorem shows. (Similar theorems can be found in Good, 1957, Greene and Knuth, 1982, and Drmota, 1994a)

THEOREM 1. *Let $g_n, y_n \geq 0$ be two sequences of non-negative real numbers such that*

$$d = gcd\{n \mid y_n > 0\} = 1$$

and that the generating functions

$$y(x) = \sum_{n\geq 0} y_n x^n \quad \text{and} \quad g(x) = \sum_{n\geq 0} g_n x^n$$

have positive radius of convergence R resp. R'. Let $0 < a < b$ such that $a > \lim_{r\to 0} \frac{ry'(r)}{y(r)}$ and $b < \min\left(\lim_{r\to R} \frac{ry'(r)}{y(r)}, R'\right)$. Then the coefficient

$$y_{nk} = [x^n]g(x)y(x)^k$$

has the following asymptotic behaviour

$$y_{nk} = \frac{g(\rho)y(\rho)^k}{\rho^n\sqrt{2\pi c(\rho)k}}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)\right) \quad (k \to \infty) \tag{17}$$

uniformly for $\frac{n}{k} \in [a, b]$, where $\rho = \rho_{\frac{n}{k}}$ is (uniquely) defined by

$$\frac{n}{k} = \frac{\rho y'(\rho)}{y(\rho)} \tag{18}$$

and $c(\rho)$ is given by

$$c(\rho) = \frac{\rho y'(\rho)}{y(\rho)} - \left(\frac{\rho y'(\rho)}{y(\rho)}\right)^2 + \frac{\rho^2 y''(\rho)}{y(\rho)} \tag{19}$$

Remark: Note that $\rho_{\frac{n}{k}}$ is the saddle point of the function $y(x)^k x^{-n}$.

Proof. Let $\mu = n/k \in [a,b]$ of equivalently $\rho = \rho_\mu \in [\rho_a, \rho_b]$, where $0 < \rho_a < \rho_b < R$. Then we have by Cauchy's formula

$$\begin{aligned} y_{nk} &= \frac{1}{2\pi i}\int_{|x|=\rho} \frac{g(x)y(x)^k}{x^{n+1}}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\rho e^{it})y(\rho e^{it})^k \rho^{-n} e^{-int}\,dt. \end{aligned} \tag{20}$$

Since $y_n \ge 0$ and $d = \gcd\{n \,|\, y_n > 0\} = 1$ we have $|y(x)| = y(|x|)$ if and only if x is real and non-negative. By continuity there exists some $\delta > 0$ and some α with $0 < \alpha < 1$ such that $|y(\rho e^{it})| \le \alpha y(\rho)$ uniformly for $\rho \in [\rho_a, \rho_b]$ and $\delta \le |t| \le \pi$ and that

$$\left|\log(y(\rho e^{it})/y(\rho)) - i\mu + \frac{t^2}{2}c(\rho)\right| = \mathcal{O}(t^3) < \frac{c(\rho)t^2}{6}$$

uniformly for $\rho \in [\rho_a, \rho_b]$ and $|t| \le \delta$. Hence we get for any positive $\varepsilon < \frac{1}{2}$

$$\begin{aligned} y_{nk} &= \frac{1}{2\pi}\int_{-\delta}^{\delta} g(\rho e^{it})y(\rho e^{it})^k \rho^{-n} e^{-int}\,dt + \mathcal{O}\left(g(\rho)y(\rho)^k\rho^{-n}\alpha^k\right) \\ &= \frac{g(\rho)y(\rho)^k}{2\pi\rho^n}\left(\int_{-\delta}^{\delta} \exp\left(-\frac{c(\rho)k}{2}t^2 + \mathcal{O}(kt^3)\right)(1+\mathcal{O}(t))\,dt + \mathcal{O}\left(\alpha^k\right)\right) \\ &= \frac{g(\rho)y(\rho)^k}{2\pi\rho^n}\left(\frac{1}{\sqrt{c(\rho)k}}\int_{-k^\varepsilon}^{k^\varepsilon} \exp\left(-\frac{u^2}{2}\right)\left(1+\mathcal{O}\left(\frac{|u|^3+|u|}{\sqrt{k}}\right)\right)du \right. \\ &\qquad \left. +\mathcal{O}\left(\delta e^{-k^{2\varepsilon}/6}\right) + \mathcal{O}\left(\alpha^k\right)\right), \end{aligned}$$

where we have estimated the integrals for $\delta \le |t| \le \pi$ and $k^{-1/2+\varepsilon}/\sigma \le |t| \le \delta$ and have substituted $u = \sqrt{\sigma^2 k}t$. This finally gives (17) by estimating the remainder integrals for $|u| \le k^\varepsilon$ and expandind the Gaussian integral to $\int_{-\infty}^{\infty}$. $\square$

Of course the above saddle point method is very robust, e.g. it is not necessary to know that a representation of the kind $g(x)y(x)^k$ is valid for all x. It is sufficient to know that the function in question behaves like $g(x)y(x)^k$ near the real axis and that it is sufficiently small for x bounded away from the real axis. In fact will apply the above ideas in such more general situations.

3.2. Singularity Analysis

Another method for obtaining asymptotic expansions for the coeffients is called singularity analysis. It is essentially due to Flajolet and Odlyzko, 1990, and is especially suited for functions with an algebraic singularity. A typical theorem reads like the following one.

THEOREM 2. *Let $y(x)$ be analytic in a region $|x| < R + \varepsilon$, $|\arg(x - R)| > \phi$ ($\varepsilon > 0$, $0 \le \phi < \frac{\pi}{2}$) such that*

$$y(x) = \mathcal{O}\left(\frac{1}{(1 - x/R)^\alpha}\right) \tag{21}$$

for some real α. Then

$$y_n = [x^n]y(x) = \mathcal{O}\left(R^{-n} n^{\alpha - 1}\right) \qquad (n \to \infty). \tag{22}$$

Proof. (Sketch) Again we start with Cauchy's formula

$$y_n = \frac{1}{2\pi i} \int_\gamma \frac{y(x)}{x^{n+1}}\, dx,$$

where we use the path of integration $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma$ with

$$\begin{aligned}
\gamma_1 &= \left\{ x : |x - R| = \frac{1}{n},\ \arg(x - R) \ge \phi \right\}, \\
\gamma_2 &= \left\{ x = R + s e^{i\phi} : \frac{1}{n} \le s \le \frac{\varepsilon}{2} \right\}, \\
\gamma_3 &= \overline{\gamma_2}, \\
\gamma_4 &= \left\{ x : |x| = \left| R + (\varepsilon/2) e^{i\phi} \right|,\ |\arg x| \ge \arg\left(R + (\varepsilon/2) e^{i\phi}\right) \right\}.
\end{aligned}$$

By (21) it immediately follows that

$$\int_{\gamma_1 \cup \gamma_2 \cup \gamma_3} \frac{y(x)}{x^{n+1}}\, dx = \mathcal{O}\left(R^{-n} n^{\alpha - 1}\right)$$

and that
$$\int_{\gamma_4} \frac{y(x)}{x^{n+1}}\,dx = \mathcal{O}\left(R^{-n}e^{-\delta n}\right)$$
for some $\delta = \delta(\varepsilon) > 0$. □

It is well known that
$$(1-x)^{-\alpha} = \sum_{n\ge 0} (-1)^n \binom{-\alpha}{n} x^n \tag{23}$$
and that
$$\binom{-\alpha}{n} = (-1)^n \frac{n^{\alpha-1}}{\Gamma(\alpha)} + \mathcal{O}\left(n^{\alpha-2}\right)) \tag{24}$$
Hence, if we know that a function $y(x)$ has a local expansion of the kind
$$y(x) = c_0 - c_1\sqrt{1-x} + c_2(1-x) + \mathcal{O}\left((1-x)^{3/2}\right) \tag{25}$$
in a region of the form $|x| < 1+\varepsilon$, $|\arg(x-1)| > \phi$, then we immediately get
$$y_n = [x^n]y(x) = \frac{c_1}{2\sqrt{\pi}n^{3/2}} + \mathcal{O}(n^{-5/2}). \tag{26}$$

Finally we want to sketch a direct proof of (24) with help of similar arguments as in the proof of Theorem 2. By Cauchy's formula we have
$$(-1)^n \binom{-\alpha}{n} = \frac{1}{2\pi i}\int_\gamma (1-x)^{-\alpha} x^{-n-1}\,dx,$$
where we use the path of integration $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ with
$$\begin{aligned}
\gamma_1 &= \left\{x : |x-1| = \frac{1}{n}, \Re(x) \le 1\right\},\\
\gamma_2 &= \left\{x : \Im(x) = 1/n,\ 1 < \Re(x) \le 1+\varepsilon\right\},\\
\gamma_3 &= \overline{\gamma_2},\\
\gamma_4 &= \left\{x : |x| = |1+\varepsilon/2 + i/n|,\ |\arg x| \ge \arg\left(1+\varepsilon/2+i/n\right)\right\}.
\end{aligned}$$

By using the substitution $x = 1+\frac{t}{n}$ for $x \in \gamma_1 \cup \gamma_2 \cup \gamma_3$ (and trivial estimates) we directly get
$$(-1)^n \binom{-\alpha}{n} = n^{\alpha-1}\frac{1}{2\pi i}\int_{\gamma_H} (-t)^{-\alpha} e^{-t}\,dt + \mathcal{O}(n^{\alpha-2}), \tag{27}$$

where γ_H denotes the Hankel contour

$$\gamma_H = \{t : |t| = 1,\ \Re(t) \le 0\} \cup \{t : \Im(t) = \pm 1,\ \Re(t) > 0\}. \tag{28}$$

Hence, by Hankel's representation of the reciprocal of the Gamma function we obtain (24).

In what follows we will apply both ideas, firstly estimates around a singularity (as in the proof of Theorem 2) and secondly approximation around the singularity (as in the proof of (24)). Following Flajolet and Odlyzko, 1990, this approach is called Singularity Analysis.

3.3. The Number of Leaves

As a first application we show how we can use the above two concepts in order to obtain asymptotic expansions for y_{ln}, the "number" of trees of size n with l leaves in simply generated families of trees. In section 2 it was shown that the generating function of these numbers satisfies the functional equation (12)

$$y(x, z) = \varphi_0 x(u-1) + x\varphi(y(x, u)).$$

The first step is to extract the analytic behaviour around its singularities. This can be done by the following theorem (compare with Drmota, 1997)

THEOREM 3. *Let us suppose that $F(x, y, u)$ is an analytic function in x, u, and y such that $F(0, y, u) \equiv 0$, that $F(x, 0, u) \not\equiv 0$, and that all Taylor coefficients of F around 0 are real and non-negative. Then the unique solution $y = y(x, u)$ of the functional equation*

$$y = F(x, y, u) \tag{29}$$

with $y(0, u) = 0$ is analytic around 0 and has non-negative Taylor coefficients around 0.

Furthermore, if we assume that the region of convergence of $F(x, y, u)$ is large enough such that there exist non-negative solutions $x = x_0$ and $y = y_0$ of the system of equations

$$\begin{aligned} y &= F(x, y, u_0), \\ 1 &= F_y(x, y, u_0). \end{aligned}$$

with $F_x(x_0, y_0, u_0) \ne 0$ and $F_{yy}(x_0, y_0, u_0) \ne 0$ and some $u_0 > 0$ then there exist functions $f(u)$, $g(x, u)$, $h(x, u)$ which are analytic around $x = x_0$, $u = u_0$ such that $y(x, u)$ is analytic for $|x| < x_0$ and $|u| \le_0$, and has a representation of the form

$$y(x, u) = g(x, u) - h(x, u)\sqrt{1 - \frac{x}{f(u)}} \tag{30}$$

locally around $x = x_0$, $u = u_0$. We have $f(u_0) = x_0$, $g(f(u),u) = y(f(u),u)$, and

$$h(f(u),u) = \sqrt{\frac{2f(u)F_x(f(u),g(f(u),u),u)}{F_{yy}(f(u),g(f(u),u),u)}}.$$

If u is real and close to u_0 then $f(u)$ is the radius of convergence of the power series $x \mapsto y(x,u)$. Moreover, if $\arg(x - f(u)) \neq 0$ then (30) provides a local analytic continuation of $y(x,u)$.

Proof. By Banach's fixed point theorem there exists a unique (analytic) solution $y = y(x,u)$ with non-negative Taylor coefficients of $y = F(x,y,u)$ with $y(0,u) = 0$ since the functional mapping

$$y(x,u) \mapsto F(x,y(x,u),u)$$

is a contraction for small x.

For a moment let $u' \leq u_0$ be fixed. Let $f_0(u')$ denote the radius of convergence of $y(x,u')$. It is well known that $x_0 = f_0(u')$ is a singular value of $y(x,u')$. The mapping

$$x \mapsto F_y(x,y(x,u'),u')$$

is strictly increasing for real and non-negative x as long as $y(x,u')$ is regular. Note that $F_y(0,y(0,u'),u') = 0$. As long as $F_y(x,y(x,u'),u') < 1$ it follows from the implicit function theorem that $y(x,u')$ is regular even in a neighbourhood of x. Hence there exists a finite limit point x_0 such that $\lim_{x \to x_0 -} y(x,u') = y_0$ is finite and satisfies $F_y(x_0,y_0,u') = 1$. If $y(x,u_0)$ were regular at $x = x_0$ then

$$y_x(x_0,u') = F_x(x_0,y(x,u'),u') + F_y(x_0,y(x_0,u'),u')y_x(x_0,u')$$

would imply $F_x(x_0,y(x_0,u'),u') = 0$ which is surely not true. Thus $y(x,u')$ is singular at $x = x_0$ (i.e. $x_0 = f_0(u')$ is the radius of convergence) and $y(x_0,u')$ is finite.

Now, let us consider the equation $y - F(x,y,u) = 0$ around $x = x_0$, $u = u_0$, $y = y_0$. We have $1 - F_y(x_0,y_0,u_0) = 0$ but $-F_{yy}(x_0,y_0,u_0) \neq 0$. Hence by the Weierstrass preparation theorem (see Kaup and Kaup, 1983) there exist functions $H(x,y,u)$, $p(x,u)$, $q(x,u)$ which are analytic around $x = x_0$, $u = u_0$, $y = y_0$ and satisfy $H(x_0,y_0,u_0) \neq 1$, $p(x_0,u_0) = q(x_0,u_0) = 0$ and

$$y - F(x,y,u) = H(x,y,u)((y-y_0)^2 + p(x,u)(y-y_0) + q(x,u))$$

locally around $x = x_0$, $u = u_0$, $y = y_0$. Since $F_x(x_0,y_0,u_0) \neq 0$ we also have $q_x(x_0,u_0) \neq 0$. This means that any analytic function $y = y(x,u)$ which satisfies $y = F(x, y, u)$ in a subset of a neighbourhood of $x = x_0$, $u = u_0$ with x_0, u_0 on its boundary is given by

$$y(x,u) = y_0 - \frac{p(x,u)}{2} \pm \sqrt{\frac{p(x,u)^2}{4} - q(x,u)}.$$

Since $p(x_0, u_0) = 0$ and $q_x(x_0, u_0) \neq 0$ we have

$$\frac{\partial}{\partial x}\left(\frac{p(x,u_0)^2}{4} - q(x,u_0)\right)_{x=x_0} \neq 0,$$

too. Again by the Weierstrass preparation theorem there exist functions $K(x,u)$ and $r(u)$ which are analytic around $x = x_0$, $u = u_0$ such that $K(x_0, u_0) \neq 0$, $r(u_0) = 0$, and

$$\frac{p(x,u)^2}{4} - q(x,u) = K(x,y)((x - x_0) + r(u))$$

locally around $x = x_0$, $u = u_0$. This finally leads to a local representation of $y = y(x,u)$ of the kind

$$y(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{f(u)}}, \tag{31}$$

in which $g(x,u)$, $h(x,u)$, and $f(u)$ are analytic around $x = x_0$, $u = u_0$ and satisfy $g(x_0, u_0) = y_0$, $h(x_0, u_0) < 0$, and $f(u_0) = x_0$.

Above we showed that for positive real u' the radius of convergence $x_0 = f_0(u')$ of $y(x, u')$ can be extracted from the system of equations $y = F(x, y, u')$, $1 = F_y(x, y, u')$. Thus $f_0(u') = f(u')$ for positive real u_0. Similarly we can show that $x = f_0(u)$ is a singular point of $y(x,u)$ if u is close to the reals, where $x = f_0(u)$ (and $y = g_0(u)$) are the solutions of the system of equations

$$\begin{aligned} y &= F(x, y, u_0), \\ 1 &= F_y(x, y, u_0). \end{aligned} \tag{32}$$

Thus $f_0(u) = f(u)$ even in this case. Therefore we can calculate the derivatives of $f(u)$ by implicit differentiation.

Furthermore it is clear that $g(f(u), u) = g_0(f(u)) = y(f(u), u)$. In order to calculate $h(f(u), u)$ we use Taylor's theorem for $F(x, u, y)$ around $x = f(u)$ and $y = y(f(u), u)$ and by comparing the coefficients of $(x - f(u))$ we immediately obtain

$$h(f(u),u) = \sqrt{\frac{2f(u)F_x(f(u),u,g_0(u))}{F_{yy}(f(u),u,g_0(u))}}. \quad \square$$

COROLLARY. *Suppose that $F(x,y,u)$ satisfies the same assumptions as in Theorem 3. Then*

$$\begin{aligned} y(x,u) &= g(f(u),u) - \sqrt{\frac{2f(u)F_x(f(u),g(f(u),u),u)}{F_{yy}(f(u),g(f(u),u),u)}}\sqrt{1-\frac{x}{f(u)}} \quad (33)\\ &+ g_x(f(u),u)(x-f(u)) + O(|1-x/f(u)|^{3/2}), \end{aligned}$$

Proof. Since $h(x,u) = h(f(u),u) + O(|1-x/f(u)|)$, (33) immediately follows from Theorem 3. □

We now apply this concept to the functional equation

$$y = \varphi_0 x(u-1) + x\varphi(y) = F(x,y,u).$$

Let $\tau = \tau(u)$ be the solution of

$$\tau\varphi'(\tau) = \varphi_0(u-1) + \varphi(\tau) \tag{34}$$

then $f(u) = 1/\varphi'(\tau(u))$ and $g(f(u),u) = y(f(u),u) = \tau(u)$. Hence we get

$$\begin{aligned} y(x,u) &= a(x,u) - \sqrt{\frac{2(\varphi_0(u-1)+\varphi(\tau(u)))}{\varphi''(\tau(u))}}\sqrt{1-x\varphi'(\tau(u))}\\ &+ \mathcal{O}(|1-x\varphi'(\tau(u))|^{3/2}), \end{aligned} \tag{35}$$

where $a(x,u)$ is a linear function in x and analytic in u. Hence, if we apply singularity analysis to this function, i.e. Theorem 2, we directly obtain

$$[x^n]y(x,u) = \frac{\varphi'(\tau(u))^n}{2\sqrt{\pi}n^{3/2}}\sqrt{\frac{2(\varphi_0(u-1)+\varphi(\tau(u)))}{\varphi''(\tau))}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right).$$

where u is considered as a "parameter". If we consider the main term

$$\frac{1}{2\sqrt{\pi}n^{3/2}}\cdot\sqrt{\frac{2(\varphi_0(u-1)+\varphi(\tau(u)))}{\varphi''(\tau(u))}}\,\varphi'(\tau(u))^n$$

it exactly fits to the shape of Theorem 1. In fact, if we assume that

$$d = \gcd\{i \ge 1 : \varphi_i \ne 0\} = 1$$

then the same proof as the proof of Theorem 1 shows that y_{ln} is asymptotically given by

$$y_{ln} = \frac{u_{\frac{l}{n}}^{-l}\varphi'(\tau(u_{\frac{l}{n}}))^n}{2\sqrt{2c(u_{\frac{l}{n}})\pi n^2}}\sqrt{\frac{2(\varphi_0(u_{\frac{l}{n}}-1)+\varphi(\tau(u_{\frac{l}{n}})))}{\varphi''(\tau(u))}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \tag{36}$$

uniformly for $l, n \to \infty$ as $u' \le u_{\frac{l}{n}} \le u''$, where $u = u_{\frac{l}{n}}$ is determined by

$$\frac{\varphi_0 u}{\varphi_0(u-1)+\varphi(\tau(u))} = \frac{l}{n} \tag{37}$$

and $c(u)$ is given by

$$\begin{aligned} c(u) \quad &= \quad \frac{\varphi_0 u}{\varphi_0(u-1)+\varphi(\tau(u))} - \left(\frac{\varphi_0 u}{\varphi_0(u-1)+\varphi(\tau(u))}\right)^2 \\ &\quad - \frac{\varphi_0^2 u^2}{\varphi'(\tau(u))\varphi''(\tau(u))}. \end{aligned} \tag{38}$$

Of course $0 < u' < u''$ have to be chosen in a way that $\tau(u'') < R$, where R denotes the radius of convergence of $\varphi(t)$.

4. THE SHAPE OF TREES

4.1. The Height of Leaves

As mentioned in the Introduction we will now consider the number of trees of size n with l leaves such that the m-th leaf has height k. The height of a leaf in a rooted tree is the distance to the root. Since $\mathcal{F}$ consists only of planted plane trees it is possible to number the leaves in a natural way (i.e. from left to right). Let $a_{kml}(T) = 1$ if T has exactly l leaves and if the m-th leaf of of T has height k and $a_{kml}(T) = 0$ otherwise. Our next result provides an asymptotic formula for

$$a_{kmln} = \sum_{|T|=n} \omega(T) a_{kml}(T). \tag{39}$$

(Obviously a_{kmnl} can be interpreted combinatorically or probabilistically.)

THEOREM 4. *Suppose that $\varphi(t)$ has positive radius of convergence $R > 0$ and $d = \gcd\{i \ge 1 : \varphi_i > 0\} = 1$. Let $\sigma^2(\tau)$ be defined by (9), $c(u)$ by (38),*

$u_{\frac{l}{n}}$ by (37) and $\tau(u)$ by (34). Then we have

$$
\begin{aligned}
&a_{kmln} \\
&= \frac{\tau(u_{\frac{l}{n}})\,\sigma^2(\tau(u_{\frac{l}{n}}))\,u_{\frac{l}{n}}^{-l}\,\varphi'(\tau(u_{\frac{l}{n}}))^n\,k^2}{8\sqrt{2}\,n\,c(u_{\frac{l}{n}})\,\pi^{3/2}\left(m\frac{\varphi_0(u_{\frac{l}{n}}-1)+\varphi(\tau(u_{l_n})}{\varphi_0 u_{\frac{l}{n}}}\left(n-m\frac{\varphi_0(u_{\frac{l}{n}}-1)+\varphi(\tau(u_{\frac{l}{n}})}{\varphi_0 u_{\frac{l}{n}}}\right)\right)^{\frac{3}{2}}} \\
&\times \exp\left(-\frac{\sigma^2(\tau(u_{\frac{l}{n}}))\,k^2\,n}{8m\frac{\varphi_0(u_{\frac{l}{n}}-1)+\varphi(\tau(u_{\frac{l}{n}}))}{\varphi_0 u_{\frac{l}{n}}}\left(n-m\frac{\varphi_0(u_{\frac{l}{n}}-1)+\varphi(\tau(u_{\frac{l}{n}}))}{\varphi_0 u_{\frac{l}{n}}}\right)}\right) \qquad (40) \\
&\quad +\mathcal{O}\left(u_{\frac{l}{n}}^{-l}\,\varphi'(\tau(u_{l_n}))^n\,\frac{k}{n^{7/2}}\right)
\end{aligned}
$$

uniformly for $\varepsilon \le \frac{m}{n} \le \frac{\varphi_0 u_{\frac{l}{n}}}{\varphi_0(u_{\frac{l}{n}}-1)+\varphi(\tau(u_{\frac{l}{n}}))} - \varepsilon$, $u' \le u_{\frac{l}{n}} \le u''$ and $C_1\sqrt{n} \le k \le C_2\sqrt{n}$, where $\varepsilon > 0$, $0 < C_1 < C_2$ and $0 < u' < u''$ are arbitrary but fixed with the only restriction that $\tau(u'') < R$.

Corollary. *Let $0 < c, d < 1$ and $x > 0$. Suppose that $l = dn + o(n)$, $m = cl + o(l)$, and $k = x\sqrt{n} + o(\sqrt{n})$ $(n \to \infty)$. Then we have*

$$
\sqrt{n}\frac{a_{kmnl}}{y_{ln}} = \frac{\sigma^3(\tau(u_d))}{4\sqrt{2\pi}}\frac{x^2}{(c(1-c))^{\frac{3}{2}}}\exp\left(-\frac{\sigma^2(\tau(u_d))}{8}\frac{x^2}{c(1-c)}\right) + o(1). \qquad (41)
$$

as $(n \to \infty)$.

This means that the height of the $c \cdot l$-th leaf is trees of size n with $l = d \cdot n$ leaves is asymptotically Maxwell distributed with mean value

$$
\sim \frac{8}{\sqrt{2\pi}\sigma(\tau(u_d))}(c(1-c))^{\frac{1}{2}}\sqrt{n}
$$

Results of this kind have already appeared in the literature. In the case of no restriction to the number of leaves Gutjahr and Pflug, 1992, obtained this distribution result for binary trees. They used a direct approach with explict formulas and Stirling's asymptotic formula. Later Drmota, 1994b, could generalize Gutjahr and Pflug's result to simply generated families of trees. Abramov used a generalized scheme of allococation to derive a special case of Theorem 4 for planted plane trees ($\varphi(t) = 1/(1-t)$) for $l = \frac{n}{2} + \mathcal{O}(\sqrt{n})$. Now let

$$
A_k(x, u, v) = \sum_{m,n} a_{kmln} x^n u^m v^l \qquad (42)
$$

be the generating function of those leaves with height k.

LEMMA 5. *Let $y(x,z)$ be defined in (10). Then we have*

$$A_k(x,u,v) = \varphi_0 uvx^{k+1}\left(\frac{\varphi(y(x,uv)) - \varphi(y(x,v))}{y(x,uv) - y(x,v)}\right)^k. \tag{43}$$

Proof. We will proceed by induction. Trivially

$$A_0(x,u,v) = \varphi_0 xuv. \tag{44}$$

Now let $A_k(x,u,v)$ be given. Then we have (recall the combinatorial interpreation (13))

$$\begin{aligned} A_{k+1}(x,u,v) &= x\varphi_1 A_k(x,u,v) + x\varphi_2\left(A_k(x,u,v)y(x,v) + y(x,uv)A_k(x,u,v)\right) \\ &\quad + x\varphi_3(A_k(x,u,v)y(x,v)^2 + y(x,uv)A_k(x,u,v)y(x,v) \\ &\qquad + y(x,uv)^2 A_k(x,u,v)) + \cdots \\ &= xA_k(x,u)\sum_{i\ge 1}\varphi_i \sum_{j=0}^{i-1} y(x,v)^j y(x,uv)^{i-1-j} \\ &= xA_k(x,u,v)\sum_{i\ge 1}\varphi_i \frac{y(x,uv)^i - y(x,v)^i}{y(x,uv) - y(x,v)} \\ &= xA_k(x,u,v)\frac{\varphi(y(x,uv)) - \varphi(y(x,v))}{y(x,uv) - y(x,v)} \end{aligned} \tag{45}$$

which proves (43). $\square$

LEMMA 6. *Let γ_H be a Hankel contour*

$$\gamma_H = \{t : |t| = 1,\ \Re(t) \le 0\} \cup \{t : \Im(t) = \pm 1,\ \Re(t) > 0\}.$$

Then we have for $\lambda, \mu > 0$

$$\frac{1}{2\pi i}\int_{\gamma_H} e^{-\lambda\sqrt{-t}-\mu t}\, dt = \frac{\lambda\mu^{-\frac{3}{2}}}{2\sqrt{\pi}}\exp\left(-\frac{\lambda^2}{4\mu}\right). \tag{46}$$

Proof. (46) directly follows from the substitutions $t = v^2$, $v = \frac{x}{\sqrt{\mu}} + i\frac{\lambda}{2\mu}$ and from

$$\int_{-\infty+ic}^{\infty+ic} e^{-x^2}\, dx = \sqrt{\pi}. \quad \square$$

Now we will prove the asymptotic expansion (40) of Theorem 4.

Proof. We will first derive an asymptotic representation of

$$a_{kmn}(v) = \sum_{l\ge 0} a_{kmln} v^l,$$

where v is considered a complex parameter (close to $u_{\frac{l}{n}}$). By Cauchy's formula

$$a_{kmn}(v) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{A_k(x,u,v)}{x^{n+1}u^{m+1}}\, dx\, du, \tag{47}$$

where the paths of integration $\Gamma_1 = \Gamma_{11} \cup \Gamma_{12} \cup \Gamma_{13} \cup \Gamma_{14}$ and $\Gamma_2 = \Gamma_{21} \cup \Gamma_{22} \cup \Gamma_{23} \cup \Gamma_{24}$ are of the following kind ($x_0 = f(v) = 1/\varphi'(\tau(v))$):

$$\begin{aligned}
\Gamma_{11} &= \left\{ x = x_0\left(1+\frac{t}{n}\right) \,\middle|\, \Re t < 0, |t| = 1 \right\} \\
\Gamma_{12} &= \left\{ x = x_0\left(1+\frac{t}{n}\right) \,\middle|\, 0 \le \Re t \le \log^2 n, \Im t = 1 \right\} \\
\Gamma_{13} &= \overline{\Gamma_{12}} \\
\Gamma_{14} &= \left\{ x \,\middle|\, |x| = \left|1+\frac{\log^2 n + i}{n}\right|, \arg\left(1+\frac{\log^2 n + i}{n}\right) \le |\arg x| \le \pi \right\}
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_{21} &= \left\{ u = 1+\frac{s}{m} \,\middle|\, \Re s < -R(t), |s+R(t)| = 1 \right\} \\
\Gamma_{22} &= \left\{ u = 1+\frac{s}{m} \,\middle|\, -R(t) \le \Re s \le \log^2 m, \Im s = 1 \right\} \\
\Gamma_{23} &= \overline{\Gamma_{22}} \\
\Gamma_{24} &= \left\{ u \,\middle|\, |u| = \left|1+\frac{\log^2 m + i}{m}\right|, \arg\left(1+\frac{\log^2 m + i}{m}\right) \le |\arg u| \le \pi \right\},
\end{aligned}$$

where $R(t) = \max\left\{0, \frac{m}{n}\Re\left(\frac{\varphi_0(v-1)\varphi(\tau(v))}{\varphi_0 v} t\right)\right\}$. Our first aim is to show that

$$\begin{aligned}
a_{kmn}(v) &= \varphi_0 v \frac{x_0^{1-n}}{nm} \frac{1}{2\pi i} \int_{\gamma_H} e^{-\lambda_1\sqrt{-t}-\mu_1 t}\, dt\, \frac{1}{2\pi i} \int_{\gamma_H} e^{-\lambda_2\sqrt{-w}-\mu_2 w}\, dw \\
&+ \mathcal{O}\left(\frac{x_0^{-n}}{mn}\left(\frac{k}{n}+\frac{k}{m}\right)\right),
\end{aligned} \tag{48}$$

where

$$\lambda_1 = \frac{\sigma(\tau(v))k}{\sqrt{2n}}, \qquad \mu_1 = 1 - \frac{m}{n}\frac{\varphi_0(v-1)+\varphi(\tau(v))}{\varphi_0 v}$$

and

$$\lambda_2 = \frac{\sigma(\tau(v))k}{\sqrt{2m}}\sqrt{\frac{\varphi_0 v}{\varphi_0(v-1)+\varphi(\tau(v))}}, \qquad \mu_2 = 1.$$

Obviously, (48) and (46) yield

$$\begin{aligned} a_{kmn}(v) &= \frac{v\tau(v)\sigma^2(\tau(v))\varphi'(\tau(v))^n k^2}{8\pi\left(m\frac{\varphi_0(v-1)+\varphi(\tau(v))}{\varphi_0 v}\left(n-m\frac{\varphi_0(v-1)+\varphi(\tau(v))}{\varphi_0 v}\right)\right)^{\frac{3}{2}}} \\ &\quad \times \exp\left(-\frac{\sigma^2(\tau(v))k^2 n}{8m\frac{\varphi_0(v-1)+\varphi(\tau(v))}{\varphi_0 v}\left(n-m\frac{\varphi_0(v-1)+\varphi(\tau)(v)}{\varphi_0 v}\right)}\right) \\ &\quad +\mathcal{O}\left(\varphi'(\tau(v))^n\frac{k}{n^3}\right). \end{aligned} \tag{49}$$

Now suppose that $c_1 = m/n$ and $c_2 = k/\sqrt{n}$ are contained in a (small) compact interval (contained in the positive reals) then the main term of $a_{kmn}(v)$ has the shape

$$n^{-3/2}\cdot g(c_1,c_2,v)\,\varphi'(\tau(v))^n,$$

where $g(c_1,c_2,v)$ depends on c_1, c_2 continuously. Hence we can apply a saddle point method similar to the proof of Theorem 1 and we directly obtain (40). It remains to prove (48) First we will approximate $A_k(x,u,v)x^{-n}u^{-m}$ for $x \in \Gamma_{11}\cup\Gamma_{12}\cup\Gamma_{13}$ and $u \in \Gamma_{21}\cup\Gamma_{22}\cup\Gamma_{23}$. From (54) we get

$$y(x,v)-\tau(v) = -\sqrt{\frac{2(\varphi_0(v-1)+\varphi(\tau(v))}{\varphi''(\tau)}}\sqrt{-\frac{t}{n}}+\mathcal{O}\left(\frac{|t|}{n}\right), \tag{50}$$

$$\begin{aligned} y(x,uv)-\tau(v) &= -\sqrt{\frac{2(\varphi_0(v-1)+\varphi(\tau(v))}{\varphi''(\tau)}}\sqrt{-\frac{t}{n}-\frac{\varphi_0 v}{\varphi_0(v-1)+\varphi(\tau(v))}\frac{s}{m}} \\ &\quad +\mathcal{O}\left(\frac{|s|}{m}+\frac{|t|}{n}\right) \end{aligned} \tag{51}$$

and

$$\begin{aligned} &\left(x\frac{\varphi(y(x,uv))-\varphi(y(x,v))}{y(x,uv)-y(x,v)}\right)^k \\ &= \left(1-\frac{\sigma(\tau(v))}{\sqrt{2}}\left(\sqrt{-\frac{t}{n}}+\sqrt{-\frac{t}{n}-\frac{\varphi_0 v}{\varphi_0(v-1)+\varphi(\tau(v))}\frac{s}{m}}\right)+\mathcal{O}\left(\frac{|s|}{m}+\frac{|t|}{n}\right)\right)^k \end{aligned}$$

$$= \exp\left(-\frac{\sigma(\tau(v))}{\sqrt{2}}\left(\sqrt{-\frac{k^2}{n}t}+\sqrt{-\frac{k^2}{n}t-\frac{\varphi_0 v}{\varphi_0(v-1)+\varphi(\tau(v))}\frac{k^2}{m}s}\right)\right)\cdot$$
$$\cdot\left(1+\mathcal{O}\left(\frac{k|t|}{n}+\frac{k|s|}{m}\right)\right), \tag{52}$$

where we have used

$$\frac{\varphi(\tau+t_1)-\varphi(\tau+t_2)}{(\tau+t_1)-(\tau+t_2)} = \varphi'(\tau)+\frac{\varphi''(\tau)}{2}(t_1+t_2)+\mathcal{O}(|t_1|^2+|t_2|^2)$$

for sufficiently small t_1, t_2. Using the substitutions $x = x_0(1+t/n)$, $u = 1+s/m$ and setting

$$\begin{aligned} C_1 &= \{t \,|\, |t|=1, \Re t<0\}\cup\{t\,|\,0\le \Re t\le \log^2 n, \Im t=\pm 1\},\\ C_2 &= \{s\,|\,|s+R(t)|=1, \Re s<-R(t)\}\\ &\quad \cup\{s\,|\,-R(t)\le \Re s\le \log^2 m, \Im s=\pm 1\}\end{aligned}$$

we immediately get

$$\begin{aligned} a_{kmn}(v) &= \varphi_0 v\frac{x_0^{1-n}}{nm}\frac{1}{(2\pi i)^2}\int_{C_1}\int_{C_2} e^{-s-t}e^{-\sqrt{-\lambda_1^2 t}-\sqrt{-\lambda_1^2 t-\lambda_2^2 s}}\,ds\,dt\\ &\quad +\mathcal{O}\left(\frac{x_0^{-n}}{mn}\left(\frac{k}{n}+\frac{k}{m}\right)\right), \end{aligned}\tag{53}$$

where we have applied

$$\begin{aligned} x^{-n} &= x_0^{-n}e^{-t}\left(1+\mathcal{O}\left(\frac{|t|^2}{n}\right)\right),\\ u^{-m} &= e^{-s}\left(1+\mathcal{O}\left(\frac{|s|^2}{n}\right)\right), \end{aligned}$$

$$\begin{aligned} &\int_{C_1}\int_{C_2}\left|e^{-s-t}e^{-\sqrt{-\lambda_1^2 t}-\sqrt{-\lambda_1^2 t-\lambda_2^2 s}}\right||t^L||s^M|\,|ds|\,|dt|\\ &\qquad \le \int_{C_1}\int_{C_2}\left|e^{-s-t}\right||t^L||s^M|\,|ds|\,|dt| = \mathcal{O}_{L,M}(1) \end{aligned}$$

for $L, M \ge 0$, and

$$\begin{aligned} \sup_{x\in\Gamma_1, u\in\Gamma_{24}} |A(x,u,v)x^{-n}u^{-m}| &= \mathcal{O}(x_0^{-n}m^{-L})\\ \sup_{x\in\Gamma_{14}, u\in\Gamma_2} |A(x,u,v)x^{-n}u^{-m}| &= \mathcal{O}(x_0^{-n}n^{-L}) \end{aligned}$$

for any fixed $L > 0$. Now we use the substitution

$$w = s + \frac{\varphi_0(v-1) + \varphi(\tau(v))}{\varphi_0 v} \frac{m}{n} t. \tag{54}$$

to get (48) where we have to bear in mind that

$$\int_{\gamma_H \cap \{t \mid \Re t \geq K\}} e^{-\lambda\sqrt{-t} - \mu t}\, dt = \mathcal{O}\left(\mu^{-1} e^{-K\mu}\right)$$

for $K > 0$. □

4.2. The Contour of Trees

Theorem 4 is of course only the first step towards a deeper theorem. Let $h_T(m)$ denote the heigth of the m-th leaf of a (rooted) tree. Furthermore, let $\mathcal{F}_{n,l}$ denote the set of trees of size n with l leaves contained in a simply generated family of trees. Then the weights $\omega(T)$ induce a probability distribution on $\mathcal{F}_{n,l}$ and $h_T(m)$ becomes a random variable which we will denote by $H_{n,l}(m)$. (In terms of Galton-Watson branching processes this is just the height of the m-th leaf restricted on trees of size n with l leaves.) The linear interpolation

$$H_{n,l}(x) = ([x] + 1 - x)H_{n,l}([x]) + (x - [x])H_{n,l}([x] + 1)$$

is the contour of those trees and the scaled process

$$X_{n,l}(t) = \frac{\sigma(\tau(u_{\frac{l}{n}}))}{2\sqrt{n}} H_{n,l}(tl), \quad 0 \leq t \leq 1, \tag{55}$$

is the normalized contour process.

Theorem 4 (and its corollary) essentially show that the one dimensional distributions of this contour process are Maxwell distributions. This motivates the following theorem.

THEOREM 7. *Suppose that $\varphi(t)$ has positive radius of convergence R and $d = \gcd\{i \geq 1 : \varphi_i > 0\} = 1$. Then the normalized contour process $X_{n,l}(t)$ converges weakly to Brownian excursion $W^+(t)$ of duration* 1

$$(X_{n,l}; 0 \leq t \leq 1) \to (W^+(t); 0 \leq t \leq 1) \tag{56}$$

in $C[0,1]$ as $n, l \to \infty$ and $u_{\frac{l}{n}} \in [u', u'']$, where we to assume that $\tau(u'') < R$ and $u' > 0$.

The proof of this Theorem 7 requires rather technical calculations (similar to the above ones but much more involved) in order to prove convergence of the finite dimensional distributions and proper estimates for a verification of tightness. Gittenberger has worked out a similar theorem in the case of no restriction to the number of leaves.

REFERENCES

Abramov, A., On the limit distribution of the height of leaves in a plane rooted tree, manuscript.

Aldous, D. J. (1991), The Continuum Random Tree II: An Overview, *Proc. Durham Symp. Stochastic Analysis 1990*, Cambridge University Press, pp. 23–70.

Athreya, K. B. and Ney, P. E. (1972), Branching Processes, Springer, Berlin-Heidelberg-- New York.

Drmota, M. (1994a), A bivariate asymptotic expansion of coefficients of powers of generating functions, *Europ. J. Combinatorics* **15**, pp. 139–152.

Drmota, M. (1994b), The height distribution of leaves in rooted trees, *Discrete Math. Appl.* **4**, pp. 45–58,

Drmota, M. (1997), Systems of functional equations, *Random Structures and Algorithms* **10**, pp. 103–124.

Flajolet, P., and Odlyzko, A. M. (1990), Singularity analysis of generating functions, *SIAM J. Discrete Math.* **3**, pp. 216–240.

Gittenberger, B., On the contour of random trees, *SIAM J. Discrete Math.*, to appear.

Good, I. J. (1957), Saddle-Point Methods for the Multinomial Distribution, *Ann. Math. Statistics* **28**, pp. 861–881.

Greene, D. H., and Knuth, D. E. (1982), Mathematics for the Analysis of Algorithms (2^{nd} edition), Birkhäuser, Boston-Basel-Stuttgart.

Kaup, L., and Kaup, B. (1983), Holomorphic Functions of Several Variables, Studies in Mathematics, de Gruyter, Berlin.

Kennedy, D. P. (1975), The Galton-Watson process condidioned on the total progeny, *J. Appl. Prob.* **12**, pp. 800–806.

Meir, A. and Moon, J. W. (1978), On the altitude of nodes in random trees, *Canad. J. Math.* **30**, pp. 997–1015.

Vitter, J. S. and Flajolet, P. (1990), Average case analysis of algorithms and data structures, In: Handbook of Theoretical Computer Sciences (J. van, Leeuwen ed.), ch. 9, Elvesier.