q-ADDITIVE FUNCTIONS AND WELL DISTRIBUTION

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ABSTRACT. J. Coquet [1] proved that the sequence $(x_{s(n)})$ is well distributed modulo 1 if (x_n) is well distributed modulo 1, where $s(n)$ denotes the sum of q-ary digits of n. This theorem is generalized to arbitrary q-additive functions $f(n)$ and quantified in term of the uniform discrepancy $\tilde{D}_N(x_n)$.

1. INTRODUCTION

A real sequence $(x_n)_{n\ge 0}$ is called uniformly distributed modulo 1 (for short: u.d. mod 1) if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \chi_I(\{x_n\}) = \lambda(I)$$

for all intervals $I \subseteq [0,1]$, where χ_I denotes the characteristic function of I, $\{x\} = x - [x]$ denotes the fractional part of x, and λ denotes the Lebesgue measure. Equivalently, a sequence is u.d. mod 1 if the discrepancy

$$D_N(x_n) = \sup_{I\subseteq[0,1]} \left| \frac{1}{N}\sum_{n=0}^{N-1} \chi_I(\{x_n\}) - \lambda(I)\right|$$

satisfies

$$\lim_{N\to\infty} D_N(x_n) = 0.$$

It is clear that every shifted sequence $(x_{n+\nu})_{n\ge 0}$ $(\nu \ge 0)$ is u.d. mod 1 if $(x_n)_{n\ge0}$ is u.d. mod 1, i.e. $\lim_{N\to\infty} D_N(x_{n+\nu}) = 0$ for all $\nu \ge 0$. However, this convergence is not necessarily uniform for $\nu \ge 0$.

A real sequence $(x_n)_{n\ge0}$ is called well distributed modulo 1 (for short w.d. mod 1) if the uniform discrepancy

$$\tilde{D}_N(x_n) = \sup_{\nu\ge 0}\sup_{I\subseteq[0,1]} \left| \frac{1}{N}\sum_{n=0}^{N-1} \chi_I(\{x_{n+\nu}\}) - \lambda(I)\right| = \sup_{\nu\ge0} D_N(x_{n+\nu})$$

satisfies

$$\lim_{N\to\infty} \tilde{D}_N(x_n) = 0.$$

Obviously, every w.d. sequence is u.d. It should be mentioned that the converse is not true, e.g. $x_n = \sqrt{n}$ is u.d. mod 1 but not w.d. mod 1. In fact, this is the *typical* situation. Almost all sequences $(x_n)_{n\ge0} \in [0,1)^{\mathbf{N}}$ are u.d. mod 1 but not w.d. mod 1. (For more details of u.d. resp. w.d. sequences we refer to [10, 8, 4].)

The most prominent u.d. resp. w.d. real sequence is the linear sequence $(\alpha n)_{n\ge0}$ for irrational α. However, there are other interesting w.d. sequences of the form $(\alpha f(n))_{n\ge0}$, where $f(n)$ is an integer valued function, e.g. Coquet [1] showed that $(\alpha s_q(n))_{n\ge0}$ is w.d. mod 1 for irrational

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α, where $s_q(n)$ denotes the sum of digits in the q-ary representation of n. This result can be extended to strongly q-additive functions $f(n)$, which are defined by

$$f(a+qb) = f(a) + f(b) \quad (0 \le a < q,\, b \ge 0),$$

i.e. if n is given by $n = d_0 + d_1 q + \cdots + d_k q^k$ $(0 \le d_j < q)$ and $f(0)(=0), f(1), \ldots, f(q-1)$ are known then

$$f(n) = f(d_0) + f(d_1) + \cdots + f(d_k).$$

Our first result provides an almost optimal bound for the uniform discrepancy of $(\alpha f(n))_{n\ge 1}$. We consider irrationals α of finite approximation type η, i.e. for every $\varepsilon > 0$ there exists a constant $c(\alpha,\varepsilon) > 0$ such that

$$\|h\alpha\| \ge \frac{c(\alpha,\varepsilon)}{h^{\eta+\varepsilon}}$$

for all positive integers h, where $\|x\| = \min(\{x\}, 1-\{x\})$ denotes the nearest distance to integers.

Theorem 1.1. *Let α be an irrational of finite approxiation type η and let $f(n)$ be a strongly q-additive function which attains only non-negative integers such that there exists $1 \le b \le q-1$ with $f(b) > 0$. Then for every $\varepsilon > 0$*

$$\tilde{D}_N(\alpha f(n)) \ll \frac{1}{(\log N)^{1/(2\eta)-\varepsilon}}$$

for all $N > 1$, where the constant implied by $\ll$ depends on q, α, ε, and on f.

Remark 1. Theorem 1 is a generalization of results of Tichy and Turnwald [16, 17], where corresponding upper bounds for the usual discrepancy $D_N(\alpha f(n))$ and worse estimates for the uniform discrepancy $\tilde{D}_N(\alpha f(n))$ are derived. In [12] it is mentioned that estimates for $\tilde{D}_N(\alpha f(n))$ can be derived from bounds for $D_N(\alpha f(n))$. However, the formulation of Theorem 4 in [12] is not sufficient to confirm this statement. Its proof needs a sligth modification. In fact, the proof of Theorem 1 uses simlar ideas to that of Theorem 4 in [12] and it is easy to extract the following estimate:

$$\tilde{D}_N(\alpha f(n)) \le \min_{k \le \frac{\log N}{\log q}} \left(\frac{2q^k}{N} + 2D_{q^k}(\alpha f(n)) \right) \ll D_{[\sqrt{N}]}(\alpha f(n)).$$

Remark 2. In [16] it is also shown that if α is not of approximation type η' for any $\eta' < \eta$ then for every $\varepsilon > 0$ and infinitely many N

$$D_N(\alpha f(n)) \ge \frac{1}{(\log N)^{1/(2\eta)+\varepsilon}}.$$

Since $\tilde{D}_N(\alpha f(n)) \ge D_N(\alpha f(n))$ Theorem 1 is almost optimal.
It should be further mentioned that it is also possible to show (see [16, 11]) that for every irrational α there exists a constant $c'(q,\alpha,f) > 0$ such that for all $N \ge 2$

$$D_N(\alpha f(n)) > \frac{c'(q,\alpha,f)}{(\log N)^{1/2}}.$$

By the theorem of THUE-SIEGEL-ROTH every irrational real algebraic number α is of approximation type $\eta = 1$. Hence the exponent $1/2$ in this general lower bound cannot be replaced by a larger exponent.

Note that if $x_n = \alpha n$ then $x_{f(n)} = \alpha f(n)$. Actually this is not only a formal observation but the deeper reason for Theorem 1. Coquet [1] showed that $(x_{s_q(n)})_{n\ge 0}$ is w.d. mod 1 if $(x_n)_{n\ge 0}$ is w.d. mod 1. In [4] this result was (non-trivially) generalized to strongly q-additive functions $f(n)$. Here we provide a quantitative version of this relationship in terms of the uniform discrepancy.

Theorem 1.2. *Suppose that $f(n)$ is strongly q-additive which attains only non-negative integers such that* $\gcd\{0 < j < q \,:\, f(j) > 0\} = 1$. *Then for every $c < 1/\log q$ we have*

$$\tilde{D}_N(x_{f(n)}) \ll \sup_{M \geq c(\log N)^{1/4}} \tilde{D}_M(x_n),$$

i.e. if a sequence $(x_n)_{n\geq 0}$ is w.d. mod 1 then $(x_{f(n)})$ is w.d. mod 1, too.

This paper is organized in the following way. In section 2 we prove Theorem 1 whereas secion 3 is devoted to the proof of Theorem 1.2. In the final section 4 we discuss other types of integer valued functions $f(n)$ with the property that $(\alpha f(n))_{n\geq 0}$ are u.d. mod 1 for irrational α.

2. Proof of Theorem 1

The basic tool for the proof of Theorem 1 is the inequality of Erdős-Turán [6, 7]. As usual we will use the notation $e(x) = e^{2\pi x}$.

Lemma 2.1. *For any choice of real numbers $x_0, x_2, \ldots, x_{N-1}$ and for every positive integer H*

$$D_N(x_n) \leq \frac{2}{H+1} + 2\sum_{h=1}^{H} \left(\frac{1}{\pi h} + \frac{1}{H+1}\right) \left|\frac{1}{N}\sum_{n=0}^{N-1} e(hx_n)\right|. \tag{2.1}$$

Remark . Note that the inequality of Erdős-Turán implies Weyl's criterion which says that if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} e(hx_n) = 0 \tag{2.2}$$

holds for every positive integer h then $(x_n)_{n\geq 0}$ is u.d. mod 1. Suppose that (2.2) is satisfied for all positive integers h then (2.1) provides

$$\limsup_{N\to\infty} D_N(x_n) \leq \frac{1}{H+1}$$

for every positive integer H. Consequently $\lim\limits_{N\to\infty} D_N(x_n) = 0$.

In order to apply Lemma 2.1 we will deal with exponential sums. The following lemma is due to Tichy and Turnwald [16]. (For the reader's convenience we repeat the proof.)

Lemma 2.2. *Let $B = \max\limits_{1\leq b<q} |f(b)|$, where f is any integer valued function defined on $\{0, 1, \ldots, q\}$ with $f(0) = 0$. Then*

$$\left|\sum_{j=0}^{q-1} e(\alpha f(j))\right| \leq q - 2\pi\|B\alpha\|^2.$$

Proof. First observe that

$$\begin{aligned}\left|\sum_{j=0}^{q} e(\alpha f(j))\right| &\leq |1 + e(B\alpha)| + q - 2 \\ &= 2|\cos(\pi B\alpha)| + q - 2 = 2\cos(\pi\|B\alpha\|) + q - 2.\end{aligned}$$

Furthermore we have

$$\cos x = 1 - \int_0^x \sin t\, dt \leq 1 - \int_0^x \frac{2}{\pi}\, t\, dt = 1 - \frac{x^2}{\pi}$$

for $|x| \leq \pi/2$. This proves Lemma (2.2). □

Corollary . *Suppose that strongly f is a q-additive function which attains only non-negative integers and set $B = \max_{1\le b<q} f(b)$. Then for every real l*

$$\left|\sum_{j=0}^{q^k-1} e(h\alpha(f(n)+l))\right| = \left|\sum_{j=0}^{q} e(\alpha f(j))\right|^k \le (q - 4\|hB\alpha\|^2)^k.$$

Now we are able to prove Theorem 1.

Proof. (Theorem 1) Suppose that $q^k \le N$ and for every $\nu \ge 0$ define m_1, m_2 by $(m_1-1)q^k \le \nu < m_1 q^k$ and by $(m_2-1)q^k \le \nu + N - 1 < m_2 q^k$. Then

$$\begin{aligned}\left|\sum_{n=0}^{N-1} e(\alpha f(n+\nu))\right| &\le 2q^k + \sum_{t=m_1}^{m_2-1}\left|\sum_{n=tq^k}^{(t+1)q^k-1} e(\alpha f(n))\right| \\ &\le 2q^k + \sum_{t=m_1}^{m_2-1}\left|\sum_{j=0}^{q^k-1} e(\alpha(f(t)+f(j))\right| \\ &\le 2q^k + (m_2-m_1)(q-4\|hB\alpha\|^2)^k \\ &\le 2q^k + N\exp\left(-k\frac{4\|hB\alpha\|^2}{q}\right).\end{aligned}$$

Now set $k = [(\log\sqrt{N})/(\log q)] + 1 \ge (\log N)/(2\log q)$ and use the assumption that α is of approximation type η, i.e. for every ε with $0<\varepsilon<1/(4\eta)$ there exists a constant $c_0>0$ such that $\|hB\alpha\| \ge c_0 h^{-\eta-\varepsilon}$ holds for every positive integer h, to obtain

$$\frac{1}{N}\left|\sum_{n=0}^{N-1} e(\alpha f(n+\nu))\right| \le \frac{2q}{\sqrt{N}} + \exp\left(-\frac{2c_0^2 h^{-2\eta-2\varepsilon}}{q\log q}\log N\right)$$

uniformly for all $\nu \ge 0$. Thus, the inequality of Erdős-Turán (Lemma 2.1) yields

$$\tilde{D}_N(\alpha f(n)) \ll \frac{1}{H} + \frac{\log H}{\sqrt{N}} + \log H \exp\left(-\frac{2c_0^2 H^{-2\eta-2\varepsilon}}{q\log q}\log N\right).$$

Finally, if we set $H = [(\log N)^{1/(2\eta)-\varepsilon}]$ we get

$$\begin{aligned}\tilde{D}_N(\alpha f(n)) &\ll \frac{1}{(\log N)^{1/(2\eta)-\varepsilon}} + \log\log N \exp\left(-c_1(\log N)^{-\varepsilon/\eta+2\varepsilon\eta+2\varepsilon^2}\right) \\ &\ll \frac{1}{(\log N)^{1/(2\eta)-\varepsilon}},\end{aligned}$$

where the constant implies by $\ll$ depends on q, α, ε, and f. □

3. Proof of Theorem 1.2

The main ingredience for the proof of Theorem 1.2 is the following lemma due to Odlyzko and Richmond [13].

Lemma 3.1. *Let $b_0, b_1, \dots, b_d$ be a finite sequence of non-negative numbers with $b_0>0$, $b_d>0$, and*

$$\gcd\{j : b_j \ne 0\} = 1.$$

Let a_{nk} be defined by

$$\sum_{n\ge 0} a_{nk}x^n = \left(b_0 + b_1 x + \cdots + b_d x^d\right)^k.$$

Then for every $\delta>0$ there exists a $k_0(\delta)$ such that for every $k \ge k_0(\delta)$

$$a_{nk}^2 \ge a_{n-1,k}a_{n+1,k}, \qquad \delta k \le n \le (d-\delta)k. \tag{3.1}$$

Note that the gcd-condition is no real restriction and that (3.1) implies unimodality of the sequence a_{nk}, $k \ge k_0(\delta)$, $\delta k \le n \le (d-\delta)k$, i.e. there exists an n_0 such that a_{nk} is increasing for $n < n_0$ and decreasing for $n > n_0$.

It should be further noticed that this Lemma is strongly related to the central limit theorem for a sum of independent discrete random variables. Set $b = b_0 + b_1 + \cdots + b_d$. Then

$$\frac{a_{nk}}{b^k} = \mathbf{P}[X_1 + X_2 + \cdots + X_k = n],$$

where X_j, $1 \le j \le k$ are independent discrete random variables with

$$\mathbf{P}[X_j = n] = \frac{b_n}{b}.$$

It is well known (see Petrov [14]) that there is a local limit theorem of the form

$$a_{nk} = \frac{b^k}{\sqrt{2\pi k\sigma^2}} \left(\exp\left(-\frac{(n-k\mu)^2}{2k\sigma^2} \right) + \mathcal{O}(k^{-1/2}) \right), \tag{3.2}$$

with

$$\mu = \frac{1}{b}\sum_{j=0}^{d} j b_j \qquad \sigma^2 = \frac{1}{b}\sum_{j=0}^{d} (j-\mu)^2 b_j$$

and exponential tail estimates of the form

$$\sum_{|n-k\mu| \ge x\sqrt{k\sigma^2}} a_{nk} \le e^{-cx^2} q^k$$

for some $c > 0$. Especially, the following properties are satisfied

$$\max_{n \ge 0} a_{nk} = \mathcal{O}\left(\frac{q^k}{\sqrt{k}} \right) \tag{3.3}$$

and for (sufficiently small) $\delta > 0$

$$\sum_{n \le k\delta} a_{nk} + \sum_{n \ge (d-\delta)k} a_{nk} \le q'(\delta)^k, \tag{3.4}$$

where $q'(\delta) < q$.

With help of Lemma 3.1 and using these properties we are able to prove the following lemma.

Lemma 3.2. *Suppose that $f(n)$ is strongly q-additive which attain only non-negative integers such that* $\gcd\{0 < j < q : f(j) > 0\} = 1$. *Then for every (sufficiently small) $\delta > 0$ we have for all $M \ge 1$, for all $k \ge 0$ with $q^k \le N$, and for all (sufficiently large) $N \ge N_0(\delta)$*

$$\tilde{D}_N(x_{f(n)}) \le 2 \sup_{L \ge M} \tilde{D}_L(x_n) + 2\frac{q^k}{N} + \left(\frac{q'(\delta)}{q} \right)^k + \mathcal{O}\left(\frac{M}{\sqrt{k}} \right).$$

Proof. Set $b_n(I) = \chi_I(\{x_n\}) - \lambda(I)$, $\varepsilon(M) = \sup\limits_{L \ge M} \tilde{D}_L(x_n) \le 1$, and $a_{nk} = |\{j < q^k : f(j) = n\}|$. Then we have

$$\sum_{j=0}^{q^k} \chi_I(\{x_{f(j)+l}\}) - q^k \lambda(I) = \sum_{j=0}^{q^k} b_{f(j)+l}(I) = \sum_{n \ge 0} a_{nk} b_{n+l}(I).$$

First let us consider the sum $\sum_{n=n_0}^{(d-\delta)k}$, where n_0 is defined by $a_{n_0 k} = \max\limits_{n \ge 0} a_{nk}$, $d = \max\{j < q : f(j) > 0\}$, and $\delta > 0$ is chosen in a way that $(d-\delta)k$ is a positive integer. By partial summation we obtain

$$\sum_{n=n_0}^{(d-\delta)k} a_{nk} b_{n+l}(I) = a_{(d-\delta)k} \sum_{n=n_0}^{(d-\delta)k} b_{n+l}(I) + \sum_{n=n_0}^{(d-\delta)k-1} (a_{nk} - a_{n+1,k}) \sum_{j=n_0}^{n} b_{j+l}(I).$$

If $k \ge k_0(\delta)$ then $a_{nk} > a_{n+1,k}$ for $n_0 \le n \le (d-\delta)k$. Furthermore, since

$$\left|\sum_{j=J}^{J+M-1} b_j(I)\right| \le M\varepsilon(M)$$

for all $M \ge 0$ and all intervals $I \subseteq [0,1]$ we get

$$\begin{aligned}
\left|\sum_{n=n_0}^{(d-\delta)k} a_{nk} b_{n+l}(I)\right| &\le a_{(d-\delta)k}((d-\delta)k - n_0 + 1)\varepsilon((d-\delta)k - n_0 + 1) \\
&+ \sum_{n=n_0}^{(d-\delta)k-1} (a_{nk} - a_{n+1,k})(n - n_0 + 1)\varepsilon(n - n_0 + 1) \\
&\le \varepsilon(M)\left(a_{(d-\delta)k}((d-\delta)k - n_0 + 1) + \sum_{n=n_0}^{(d-\delta)k-1} (a_{nk} - a_{n+1,k})(n - n_0 + 1)\right) \\
&+ \sum_{n=n_0}^{n_0+M-1} (a_{nk} - a_{n+1,k})(n - n_0 + 1) \\
&= \varepsilon(M) \sum_{n=n_0}^{(d-\delta)k} a_{nk} + \sum_{n=n_0}^{n_0+M} a_{nk} - M a_{n_0+M} \\
&\le \varepsilon(M) q^k + M a_{n_0 k}.
\end{aligned}$$

A similar estimate holds for the sum $\sum_{n=\delta k}^{n_0-1}$. Thus

$$\left|\sum_{j=0}^{q^k} \chi_I(\{x_{f(j)+l}\}) - q^k \lambda(I)\right| \le 2\varepsilon(M) q^k + 2M a_{n_0 k} + q'(\delta)^k$$

holds for all $l \ge 0$.

Finally, suppose that $q^k \le N$ and for every $\nu \ge 0$ define m_1, m_2 by $(m_1 - 1)q^k \le \nu < m_1 q^k$ and by $(m_2 - 1)q^k \le \nu + N - 1 < m_2 q^k$. Then

$$\begin{aligned}
\left|\sum_{n=0}^{N-1} b_{f(n+\nu)}(I)\right| &\le 2q^k + \sum_{t=m_1}^{m_2-1} \left|\sum_{n=tq^k}^{(t+1)q^k-1} b_{f(n)}(I)\right| \\
&\le 2q^k + \sum_{t=m_1}^{m_2-1} \left|\sum_{j=0}^{q^k-1} b_{f(t)+f(j)}(I)\right| \\
&\le 2q^k + (m_2 - m_1)(2\varepsilon(M) q^k + 2M a_{n_0 k} + q'(\delta)^k) \\
&\le 2q^k + N\left(2\varepsilon(M) + \left(\frac{q'(\delta)}{q}\right)^k + \mathcal{O}\left(\frac{M}{\sqrt{k}}\right)\right).
\end{aligned}$$

This completes the proof of Lemma 3.2. $\square$

It is now easy to show that Lemma 3.2 implies Theorem 1.2.

Proof. (Theorem 1.2) Fix any sufficiently small $\delta > 0$ and any $c < 1/\log q$ and set $k = c\log N$ and $M = k^{1/4}$. We have

$$\frac{q^k}{N} = N^{c\log q - 1} = \mathcal{O}\left(\frac{1}{M}\right),$$
$$\left(\frac{q'(\delta)}{q}\right)^k = N^{-c\log(q/q'(\delta))}\mathcal{O}\left(\frac{1}{M}\right),$$
$$\frac{M}{\sqrt{k}} = \frac{1}{M}.$$

Since $\varepsilon(M) \ge \tilde{D}_M(x_n) \ge D_M(x_n) \ge \frac{1}{M}$ we finally obtain

$$\tilde{D}_N(x_{f(n)}) \le 2\varepsilon(M) + \mathcal{O}\left(\frac{1}{M}\right) \ll \varepsilon(M),$$

which proves Theorem 1.2. □

4. Uniform Distribution of Sequences $(\alpha f(n))$

By inspecting the proof of Theorem 1.2 it turns out that the essential ingredience was a distribution result of the numbers $a_{nk} = \{j < q^n : f(j) = n\}$. We will now try to generalize this idea in order to provide more general integer valued sequences $f(n)$ such that sequences of the kind $(\alpha f(n))_{n\ge 0}$ are u.d. mod 1. The only disadvantage of this approach is that it seems to be impossible to prove also well distribution in this generality.

Theorem 4.1. *Let $(f(n))_{n\ge 0}$ be a sequence of non-negative integers such that the numbers*

$$c_{mN} = |\{n < N : f(n) = m\}|$$

satisfy

$$\sum_{m\ge 0} |c_{m+1,N} - c_{mN}| = o(N) \qquad (N \to \infty). \tag{4.1}$$

Then the sequence $(\alpha f(n))_{n\ge 0}$ is u.d. mod 1 if and only if α is irrational.

More precisely, if α is of approximation type η and

$$\sum_{m\ge 0} |c_{m+1,N} - c_{mN}| \le N\psi(N)$$

then for every $\varepsilon > 0$

$$D_N(\alpha f(n)) \ll \psi(N)^{(1/\eta)+\varepsilon}. \tag{4.2}$$

Proof. Obviously, if α is rational and $f(n)$ is an integer sequence then the sequence $(\alpha f(n))_{n\ge 0}$ is surely not u.d. mod 1 since the fractional parts $\{\alpha f(n)\}$ attain only finitely many values.

If α is irrational then

$$\left|\sum_{n=0}^{N-1} e(h\alpha n)\right| \le \frac{N}{|\sin(\pi h\alpha)|}.$$

Hence by Abel summation

$$\begin{aligned}\left|\sum_{n=0}^{N-1} e(h\alpha f(n))\right| &= \left|\sum_{m\ge 0} c_{mN} e(h\alpha m)\right| \\ &\le \sum_{m\ge 0} |c_{m+1,N} - c_{mN}| \frac{N}{|\sin(\pi h\alpha)|} \\ &= o(N).\end{aligned}$$

Thus Weyl's criterion implies that $(\alpha f(n))_{n\ge 0}$ is u.d. mod 1.

Since $|\sin(\pi h\alpha)| \le \pi\|\alpha h\|$ we also obtain

$$\frac{1}{N}\left|\sum_{n=0}^{N-1} e^{h\alpha f(n)}\right| \le \frac{\psi(N)}{\pi\|\alpha h\|}.$$

Furthermore, if α is of approximation type η then we have (see [10, p. 123])

$$\sum_{h=1}^{H} \frac{1}{h\|\alpha h\|} \ll H^{\eta-1-\varepsilon}$$

for every $\varepsilon > 0$. Hence Lemma 2.1 implies

$$D_N(\alpha f(n)) \ll \frac{1}{H} + \psi(N) H^{\eta-1-\varepsilon}.$$

By choosing $H = [\psi(N)^{-1/(\eta-\varepsilon)}]$ the discrepancy estimate (4.2) follows. $\square$

There are lots of examples of integer sequences $f(n)$ which satisfy (4.1). We will say that a non-negative integer sequence $f(n)$ satisfies a local central limit theorem if there exist sequences μ_N and σ_N with $\lim_{N\to\infty} \sigma_N = \infty$ such that for every real interval $[a,b]$ we have

$$c_{mN} = \frac{N}{\sqrt{2\pi\sigma_N^2}}\left(\exp\left(\frac{(m-\mu_N)^2}{2\sigma_N^2}\right) + o(1)\right) \tag{4.3}$$

uniformly for all integers $m \in [\mu_N + a\sigma_N, \mu_N + b\sigma_N]$ as $N \to \infty$, where

$$c_{mN} = |\{n \le N \,:\, f(n) = m\}|\,.$$

Lemma 4.1. *Suppose that a non-negative integer sequence $f(n)$ satisfies a local central limit theorem. Then*

$$\sum_{m\ge 0} |c_{m+1,N} - c_{mN}| = o(N) \qquad (N \to \infty).$$

holds with $c_{mN} = |\{n < N : f(n) = m\}|$.

Proof. Let $\varepsilon > 0$ be given and let $T(\varepsilon)$ be defined by

$$\frac{1}{\sqrt{2\pi}}\int_{-T(\varepsilon)}^{T(\varepsilon)} e^{-t^2/2}\,dt = 1-\varepsilon.$$

Then (4.3) implies

$$\sum_{|m-\mu_N|\le T(\varepsilon)\sigma_N} c_{mN} = N(1-\varepsilon) + o(N).$$

Hence for sufficiently large N we obtain

$$\sum_{|m-\mu_N|\le T(\varepsilon)\sigma_N} c_{mN} < 2\varepsilon N.$$

Furthermore, we have

$$\sum_{|m-\mu_N|\le T(\varepsilon)\sigma_N} |c_{m+1,N} - c_{m,N}| \le \frac{2N}{\sqrt{2\pi\sigma_N^2}} + o(N) = o(N).$$

This completes the proof of Lemma 4.1. $\square$

Remark . In [15] it is shown that the binary digitial sum $s_2(n)$ satisfies a local central limit theorem of the above form. In fact, the same is true for any strongly q-additive function $f(n)$. Hence Theorem 4.1 and Lemma 4.1 provide a weak version of Theorem 1. In a forthcoming paper [3] local limit theorems for the sum-of-digits-function for more general digital expansions are presented.

As an application of the above method we will reprove that the sequence $(\alpha\omega(n))_{n\ge 1}$ is u.d. mod 1, where $\omega(n)$ denotes the number of different prime factors of n. (This property has been already observed by Erdős [5] and proved by Delange [2] without discrepancy bounds.) It is well known that $\omega(n)$ satisfies a central limit theorem with mean value $\mu_N = \log\log N + C_1 + \mathcal{O}\left((\log N)^{-1}\right)$ and variance $\sigma_N^2 = \log\log N + C_2 + \mathcal{O}\left((\log N)^{-1}\right)$ (Theorem of Erdős-Kac). Corresponding local limit theorems are usually stated in a slightly different form as in (4.3). Nevertheless the following proof of Lemma 4.2 is similar to that of Lemma 4.1. We will use the following asymptotic formula (see [9]):

$$c_{mN} = \frac{N}{\sqrt{2\pi\log\log N}} \exp\left(-\frac{x_{mN}^2}{2} + \mathcal{O}\left(\frac{x_{mN}^3}{\log\log N}\right)\right)\left(1+\mathcal{O}\left(\frac{|x_{mN}|}{\sqrt{\log\log N}}\right)\right) \tag{4.4}$$

uniformly for $x_{mN} = (m-\mu_N)/\sigma_N = o(\sqrt{\log\log N})$.

Lemma 4.2. *Set*

$$c_{mN} = |\{n < N : \omega(n) = m\}|,$$

where $\omega(n)$ denotes the number of different prime factors of n. Then

$$\sum_{m\ge 0} |c_{m+1,N} - c_{m,N}| \ll \frac{N}{\sqrt{\log\log N}}.$$

Proof. For the sake of shortness we use the abbrevations $\log_2(x) = \log\log x$ and $\log_3(x) = \log\log\log x$. By (4.4) we obtain for $m \le 2\sqrt{\log_3(N)}$

$$c_{mN} = \frac{N}{\sqrt{2\pi\log_2 N}} \exp\left(-\frac{(m-C_1-\log_2 N)^2}{\log_2 N}\right)\left(1+\mathcal{O}\left(\frac{|m-\log_2 N|}{\log_2 N}\right)+\mathcal{O}\left(\frac{\log_3 N}{\log_2 N}\right)\right).$$

Thus

$$\sum_{|m-\log_2 N|\le 2\sqrt{\log_3 N}} c_{mN} = N + \mathcal{O}\left(\frac{N}{\sqrt{\log_2 N}}\right),$$

which implies

$$\sum_{|m-\log_2 N|> 2\sqrt{\log_3 N}} c_{mN} = \mathcal{O}\left(\frac{N}{\sqrt{\log_2 N}}\right).$$

Furthermore we have

$$\begin{aligned}\sum_{|m-\log_2 N|\le 2\sqrt{\log_3 N}} |c_{m+1,N} - c_{mN}| &\le \frac{2N}{\sqrt{2\pi\log_2 N}} + \mathcal{O}\left(\frac{N}{\sqrt{\log_2 N}}\right)\\ &= \mathcal{O}\left(\frac{N}{\sqrt{\log_2 N}}\right)\end{aligned}$$

which completes the proof of Lemma 4.2. □

Corollary . *The sequence $(\alpha\omega(n))_{n\ge 1}$ is u.d. mod 1 if and only if α is irrational. Furthermore, if α is of approximation type η then*

$$D_N(\alpha\omega(n)) \ll \frac{1}{(\log\log N)^{1/(2\eta)-\varepsilon}}$$

for every $\varepsilon > 0$.

If we want to obtain w.d. seqences of the form $(\alpha f(n))_{n\ge 0}$ then we have to assume a more restrictive condition (4.5).

Theorem 4.2. *Let* $(f(n))_{n\ge 0}$ *be a sequence of non-negative integers such that the numbers*

$$c_{mN} = |\{n < N : f(n) = m\}|$$

satisfy

$$\sup_{\nu\ge 0} \sum_{m\ge 0} |c_{m+1,N+\nu} - c_{m,N+\nu} - c_{m+1,\nu} + c_{m,\nu}| = o(N) \qquad (N\to\infty). \tag{4.5}$$

Then the sequence $(\alpha f(n))_{n\ge 0}$ *is w.d. mod 1 if and only if* α *is irrational.*

More precisely, if α *is of approximation type* η *and*

$$\sup_{\nu\ge 0} \sum_{m\ge 0} |c_{m+1,N+\nu} - c_{m,N+\nu} - c_{m+1,\nu} + c_{m,\nu}| \le N\psi(N)$$

then for every $\varepsilon > 0$

$$\tilde{D}_N(\alpha f(n)) \ll \psi(N)^{(1/\eta)+\varepsilon}. \tag{4.6}$$

Proof. The proof runs along the same lines as that of Theorem 4.1. The only difference is that

$$c_{m,N+\nu} - c_{m\nu} = |\{\nu \le n < N+\nu : f(n) = m\}$$

has to be used instead of $c_{mN} = |\{n < N : f(n) = m\}|$. By assumption it is clear that all estimates are uniform for $\nu \ge 0$. □

Remark . It seems to be a non-trivial problem to decide whether $(\alpha\omega(n))_{n\ge 1}$ is w.d. mod 1 or not. The present local limit law (4.4) is surely not sufficient to prove well distribution of $(\alpha\omega(n))_{n\ge 1}$.

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