COMPLETELY UNIFORMLY DISTRIBUTED SEQUENCES OF MATRICES

M. Drmota, R. F. Tichy, and R. Winkler

Technical University of Vienna

Abstract. It is proved that for almost all $s \times s$-matrices A with largest eigenvalue $\lambda(A) > 1$ the sequence of powers $(A^{p(n)})_{n=1}^{\infty}$ is completely uniformly distributed modulo 1, where $p(n)$ are different positive integers. Furthermore a constructive example for a matrix A is given such that the sequence $(A^n)_{n=1}^{\infty}$ is completely uniformly distributed.

1. Introduction

A sequence $(x_n)_{n=1}^{\infty}$ of points in $\mathbf{R}^s$ is called uniformly distributed modulo 1 (for short: u.d.), if for every s-dimensional subinterval I of $U_s = [0,1)^s$ the number $A(I,N,x_n)$ of points $\{x_1\},\dots,\{x_N\}$ contained in I ($\{x\}$ denoting the componentwise fractional part of x) is asymptotically equal to N-times the volume $|I|$ of the interval I, i.e.

$$\lim_{N\to\infty} \frac{A(I,N,x_n)}{N} = |I| \tag{1.1}$$

for every $I \subseteq U_s$. Furthermore (x_n) is called completely uniformly distributed mod 1 (for short: c.d.), if for every poitive integer k the sequence $y_n^{(k)} = (x_n, x_{n+1}, \dots, x_{n+k-1})_{n=1}^{\infty}$ is u.d. in $\mathbf{R}^{s\cdot k}$. As a measure of the distribution behaviour of a sequence (x_n) we consider as usual the discrepancy

$$D_N(x_n) = \sup_{I\subseteq U_s} \left| \frac{A(I,N,x_n)}{N} - |I| \right| . \tag{1.2}$$

It is well known that (x_n) is u.d. if and only if

$$\lim_{N\to\infty} D_N(x_n) = 0 .$$

Similarly, (x_n) is c.d. if for any fixed positive interger k the discrepancy $D_N(k,x_n) := D_N(y_n^{(k)})$ tends to 0. For more details on uniformly distributed sequences we refer to the monographs [2] and [3].

In the present article we are mainly interested in the sequence $A^{p(n)}$ of powers of a real $s \times s$-matrix A, where $p(n)$ is a given sequence of distinct positive integers. Every element of the sequence $A^{p(n)}$ is considered in a natural way as a point x_n in $\mathbf{R}^{s^2}$. Of course, for the uniform distribution of this sequence (x_n) it is necessary that the eigenvalue $\lambda(A)$ of largest absolute value is ≥ 1.

In [5] it is proved that for almost all (in the sense of s^2-dimensional Lebesgue measure) matrices A with $\lambda(A) > 1$ the sequence $(A^{p(n)})$ is u.d., where $p(n)$ is a strictly increasing sequence of positive integers. Furthermore estimates of the discrepancy are obtained in [5] and [7] as an application of the method of Erdös—Gaal—Koksma. We will establish an analogous theorem for general exponents $p(n)$ and complete uniform distribution by a refinement of a recently developed method for the solution a problem of D.E. Knuth (cf. [6], [9]).

Theorem. *Let $k(N)$ be a sequence of positive integers with $k(N) \le (\log N)^\theta$ (for sufficiently large N and some fixed θ with $0 < \theta < 1/2$), let $p(n)$ be a sequence of distinct positive integers, and let η be an arbitrary positive real number. Then for almost all real $s \times s$-matrices A with $\lambda(A) > 1$ there is a constant C (depending on A, η, and the given integral sequences p, k) such that*

$$D_N(k(N), A^{p(n)}) \le C\, N^{-\frac{1}{2}+\eta} \tag{1.3}$$

for all positive integers N.

Remark. Since any constant sequence $k = k(N)$ satisfies the above growth condition almost all sequences $A^{p(n)}$ with $\lambda(A) > 1$ are c.d. Furthermore we note that the theorem can be generalized to positive non necessary integral exponents $p(n)$ satisfying a seperation condition $|p(m) - p(n)| > \delta > 0$ for $m \ne n$.

In section 2 we will establish some auxiliary results for the proof of the above theorem and in section 3 the details of the proof are given. In the final section we will give an explicit construction of a class of $s \times s$-matrices A such that A^n is c.d. The construction is based on a recent method due to M.B. Levine [4].

2. Auxiliary Results

Proposition 1. *(Erdös—Turan—Koksma,* [3,p. 116]*). Let (x_n) be a sequence of points in $\mathbf{R}^s$ and G an arbitrary positive integer. Then*

$$D_N(x_n) \le 2s^2 3^{s+1} \left(\frac{1}{G} + \sum_{0<\|h\|\le G} \frac{1}{R(h)} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \langle h, x_n \rangle} \right| \right) , \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbf{R}^s$, $\|h\|$ is the maximum of the absolute values of the components of the integral lattice point $h = (h_0, \dots , h_{s-1})$, and $R(h) = \prod_{0\le j<s} \max(|h_j|, 1|)$.

Proposition 2. *(*[3,p. 15, Lemma 2.1.]*) Let $w(x)$ be a continuously differentiable function on $[\alpha, \beta]$ with monotonic derivative and $|w'(x)| \ge K > 0$ for all $x \in [\alpha, \beta]$. Then*

$$\left| \int_\alpha^\beta e(u(x))\, dx \right| \le \frac{1}{K} , \tag{2.2}$$

where $e(t) := e^{2\pi i t}$.

Proposition 3. *Let* $F(x) = \delta_1 e^{\gamma_1 x} + \cdots + \delta_s e^{\gamma_s x}$, $0 < \alpha \le x \le \beta$, $s \ge 2$, $\sum_{j=2}^{s} |\delta_j| \le H$, $\gamma_1 > \cdots > \gamma_s > 0$ *(*γ_i *integral),* $\psi(N)$ *an arbitrary monotonic function tending to infinity, and* $M = [\gamma_1 \log H \, \log \psi(N)]$. *Then*

$$|F^{(M)}(x)| \ge \frac{|\delta_1|}{2} e^{\alpha} H^{\log \psi(N)}$$

for sufficiently large N.

Proof. From

$$\frac{\gamma_j}{\gamma_1} \le 1 - \frac{1}{\gamma_1} ,$$

$$\left(1 - \frac{1}{\gamma_1}\right)^M \le e^{-M/\gamma_1} \le e^{-\log H \cdot \log \psi(N)} = H^{-\log \psi(N)} ,$$

and

$$\gamma_1^M \ge \gamma_1^{\gamma_1 \log H \, \log \psi(N) - 1} \ge H^{\gamma_1 \log \gamma_1 \, \log \psi(N)} \ge H^{\log \psi(N)}$$

we get

$$\begin{aligned} |F^{(M)}(x)| &= \left| \sum_{i=1}^{s} \delta_i \, \gamma_i^M \, e^{\gamma_i x} \right| \\ &\ge \gamma_1^M \, e^{\gamma_1 x} \left(|\delta_1| - \left| \sum_{i=1}^{s} \delta_i \, \left(\frac{\gamma_i}{\gamma_1}\right)^M \, e^{(\gamma_i - \gamma_1)x} \right| \right) \\ &\ge \gamma_1^M \, e^{\gamma_1 x} \left(|\delta_1| - H \left(1 - \frac{1}{\gamma_1}\right)^M e^{-x} \right) \\ &\ge \frac{|\delta_1|}{2} e^{\alpha} H^{\log \psi(N)} , \end{aligned}$$

whence the proposition follows.

Proposition 4. *(*[9,Proposition 4]*) Let* F *be* M*-times continuously differentiable on* $[\alpha, \beta]$ *and* $|F^{(M)}(x)| \ge A > 0$ *for all* $x \in [\alpha, \beta]$. *Then for every* $\epsilon > 0$ *there exists a family of at most* $M(M+1)/2 + 1$ *intervals* $I_j = I_j(\epsilon)$ *such that the Lebesgue measure of their union is bounded by*

$$\mu\left(\bigcup_j I_j\right) \ge \beta - \alpha - 2M(M+1)\epsilon$$

and that all derivatives $F^{(M-k)}(x)$ *(*$\le k \le M$*) are monotonic and are bounded by*

$$|F^{(M-k)}(x)| \ge A \, \epsilon^k$$

on the intervals I_j.

3. Proof of the Theorem

For $k = k(N) \le (\log N)^\theta$ and an integral lattice point $h = (h_l^{i,j})_{\substack{1\le i,j\le s\\ 1\le l\le k}}$ we consider the exponential sum

$$\frac{1}{N}\sum_{n=1}^{N} e(\tilde{u}_n^{(N)}(h,A)) \tag{3.1}$$

with

$$\tilde{u}_n^{(N)}(h,A) = \sum_{\substack{1\le i,j\le s\\ 1\le l\le k}} h_l^{i,j} \cdot a_{i,j}^{(p(n+l-1))} \ , \tag{3.2}$$

where the powers of the matrix A are given by $A^m = (a_{i,j}^{(m)})_{1\le i,j\le s}$. It is sufficient to consider only matrices with different eigenvalues (cf. [5]): $\lambda_i \in \mathbf{R}$, $i = 1,\dots,\rho$, $\lambda_\kappa(\cos\varphi_\kappa \pm i\,\sin\varphi_\kappa) \in \mathbf{C}$, $\kappa = 1,\dots,\tau$ with $\rho + 2\tau = s$. Then A has a representation of the form $A = X^{-1}Y\,X$, where

$$Y = \begin{pmatrix} \lambda_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & \lambda_\rho & 0 & \dots & 0 \\ 0 & \dots & 0 & \Lambda_{\rho+1} & \dots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \Lambda_{\rho+\tau} \end{pmatrix} , \tag{3.3}$$

$$\Lambda_{\rho+\kappa} = \lambda_{\rho+\kappa} \cdot \begin{pmatrix} \cos\varphi_\kappa & \sin\varphi_\kappa \\ -\sin\varphi_\kappa & \cos\varphi_\kappa \end{pmatrix}$$

and transformation matrices

$$X = (x_{i,j})_{1\le i,j\le s} \quad , \quad X^{-1} = (\overline{x}_{i,j})_{1\le i,j\le s} \ . \tag{3.4}$$

One of the eigenvalues λ has modulus $|\lambda| > 1$. Therfore we have to distinguish two cases: $\lambda = \lambda_1$ and $\lambda = \lambda_{\rho+1}(\cos\varphi_1 \pm i\,\sin\varphi_1)$.

In the first case we have to show for $[a,b] \subseteq [1,\infty)$ (cf. [5]) that *for almost all (regular) transformation matrices X and fixed $\lambda_2,\dots,\lambda_{\rho+\tau}$, $\varphi_1,\dots,\varphi_\tau$ the theorem is true for almost all $\lambda = \lambda_1 \in [a,b]$.*

In the second case we must prove (cf. [7]) that *for almost all (regular) transformation matrices X and fixed $\lambda_1,\dots,\lambda_\rho,\lambda_{\rho+2},\dots,\lambda_{\rho+\tau}$, $\varphi_2,\dots,\varphi_\tau$ the theorem is true for almost all $\varphi = \varphi_1 \in [0,2\pi]$ and for almost all $\lambda = \lambda_{\rho+1} \in [a,b]$.*

For simplicity we use the following notation:

for $1 \le i \le \rho$ let I_i be defined as Y from (3.3) with $\lambda_i = 1$ and all other entries of I_i are zero, furthermore we set $P_i(X) = X^{-1}I_iX$;

for $\rho+1 \le i \le \rho+\tau$ let J_i be defined as Y from (3.3) with

$$\Lambda_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and all other entries of J_i are zero, furthermore we set $Q_i(X) = X^{-1} J_i X$.
Hence we obtain

$$\begin{aligned} A^{p(n)} = X^{-1} Y^{p(n)} X = \sum_{i=1}^{\rho} \lambda_i^{p(n)} P_i(X) \\ + \sum_{i=\rho+1}^{\rho+\tau} \lambda_i^{p(n)} (P_i(X) \cos(p(n)\varphi_{i-\rho}) + Q_i(X) \sin(p(n)\varphi_{i-\rho})). \end{aligned} \tag{3.5}$$

For an integer-valued $s \times s$-matrix G we set

$$\begin{aligned} \alpha_i(G, X) = \mathrm{tr}(P_i(X) G^*) \quad \text{for} \quad 1 \le i \le \rho + \tau \\ \beta_i(G, X) = \mathrm{tr}(Q_i(X) G^*) \quad \text{for} \quad \rho + 1 \le i \le \rho + \tau \ , \end{aligned} \tag{3.6}$$

where G^* denotes the transponed of the matrix G. From (3.2), (3.5) and (3.6) we immediately derive

$$\tilde{u}_n^{(N)}(h, A) = \sum_{i=1}^{\rho} \sum_{l=1}^{k} \alpha_i(h_l, X) \lambda_i^{p(n+l-1)} \tag{3.7}$$

$$+ \sum_{i=\rho+1}^{\rho+\tau} \sum_{l=1}^{k} \lambda_i^{p(n+l-1)} (\alpha_i(h_l, X) \cos(p(n+l-1)\varphi_{i-\rho})$$

$$+ \beta_i(h_l, X) \sin(p(n+l-1)\varphi_{i-\rho})) \ ,$$

where $h_l = (h_l^{i,j})_{1 \le i,j \le s}$ is considered as $s \times s$-matrix. In the following $\|G\|$ denotes the maximum of the absolute values of the entries of the matrix G.

Lemma 3.1. *a) For almost all $X \in \mathbf{R}^{s^2}$ with $\det X \neq 0$ there is a constant $C(X)$ such that for all integer-valued matrices $G \neq 0$*

$$|\alpha_i(G, X)| \ge C(X) \, \|G\|^{-5s^2} \ (1 \le i \le \rho) \tag{3.8}$$

is satisfied.
b) Let be X so, that (3.8) is valid, then for almost all φ there is a constant $C(\varphi)$ such that

$$|\alpha_i(G, X) \cos(p(n)\varphi) + \beta_i(G, X) \sin(p(n)\varphi)| > C(\varphi) N^{-3s^2-3}$$

$$(\rho + 1 \le i \le \rho + \tau)$$

for all positive integers N, all G with $0 \le \|G\| \le 2\sqrt{N} \log N \log \log N$ and $n \le N$.

Proof. a) cf.[5]
b)Using the periodicity of the trigonometric functions and setting
$\mathbf{y}_n(\varphi) = (\sin n\varphi, \cos n\varphi)$ we get

$$\mu\left(\{\varphi \in [0, 2\pi] \,|\, |\langle \mathbf{x}, \mathbf{y}_n(\varphi) \rangle| \le \varepsilon\}\right) = \mu\left(\{\varphi \in [0, 2\pi] \,|\, |\langle \mathbf{x}, \mathbf{y}_1(\varphi) \rangle| \le \varepsilon\}\right)$$

$$= \mu\left(\{\varphi \in, 2\pi] \,|\, |\cos \varphi| \le \varepsilon\}\right) \le 2\pi\varepsilon$$

for all normalized vectors $\mathbf{x}$. Hence we have for arbitrary $\mathbf{x} \neq 0$

$$\mu\left(\{\varphi \in [0, 2\pi] \,|\, |\langle \mathbf{x}, \mathbf{y}_n(\varphi)\rangle| \le \varepsilon\}\right) \le \frac{2\pi\varepsilon}{\|\mathbf{x}\|_2} .$$

We set

$$M_{N,G,n} = \{\varphi \in [0, 2\pi] \,|\, |\langle \mathbf{x}_G, \mathbf{y}_{p(n)}(\varphi)\rangle| \le N^{-3s^2-3}\} ,$$

where $\mathbf{x}_G = (\alpha_i(G, X), \beta_i(G, X))$. Now the lemma of Borel—Cantelli implies that it is sufficient that the series

$$\sum_{N=1}^{\infty} \mu \left(\bigcup_{0<\|G\|\le 2\sqrt{N}\log N \log\log N} \; \bigcup_{n=1}^{N} M_{N,G,n} \right)$$

is convergent. By assumption we have $\|\mathbf{x}_G\|_2 \ge C\,\|G\|^{-5s^2}$, therefore

$$\mu(M_{N,G,n}) \le \frac{2\pi}{C}\,\|G\|^{5s^2}\,N^{-3s^2-3} \le \frac{2\pi}{C}\,N^{\frac{5}{2}s^2-3s^2-\frac{5}{2}}$$

for all G, n, and sufficiently large N. Thus

$$\mu \left(\bigcup_{0<\|G\|\le 2\sqrt{N}\log N \log\log N} \; \bigcup_{n=1}^{N} M_{N,G,n} \right) \le 2^{s^2}\,\frac{2\pi}{C}\,N^{-\frac{5}{4}}$$

for sufficiently large N, which completes the proof.

Let $\lambda > 1$ be the absolute value of the eigenvalue with largest modulus.Then w.l.o.g. either $\lambda = |\lambda_1|$ or $\lambda = \lambda_{\rho+1}$. According to Lemma 3.1. we consider $\tilde{u}_n^{(N)}(h, A)$ as a function of λ where all other parameters are fixed; the transformation matrix X is supposed to have the property guaranteed by Lemma 3.1.a) for almost all such matrices and in the case of a complex eigenvalue of largest modulus the argument $\varphi = \varphi_1$ is supposed to have the property of Lemma 3.1.b). Then for almost all $x := \log\lambda \in [\alpha, \beta] \subseteq (0, \infty)$ we have to estimate the following exponential sum:

$$\frac{1}{N}\sum_{n=1}^{N} e(\tilde{u}_n^{(N)}(h, A)) , \tag{3.9}$$

with

$$\tilde{u}_n^{(N)}(h, x) = u_n^{(N)}(h, x) + C ,$$

where C does not depend on x, i.e. on λ, but it may depend on all remaining parameters, and

$$u_n^{(N)}(h, x) = \sum_{l=1}^{k} \alpha(h_l, X) e^{xp(n+l-1)} = \sum_{i=1}^{\tilde{k}} \tilde{g}_i(n) e^{\tilde{b}_i(n)x}$$

in the case of a real eigenvalue of largest modulus,

$$\begin{aligned} u_n^{(N)}(h, x) &= \sum_{l=1}^{k} (\alpha(h_l, X)\cos(p(n+l-1)\varphi_{i-\rho}) \\ &\quad + \beta(h_l, X)\sin(p(n+l-1)\varphi_{i-\rho}))e^{xp(n+l-1)} \\ &= \sum_{i=1}^{\tilde{k}} \tilde{g}_i(n) e^{\tilde{b}_i(n)x} \end{aligned}$$

in the other case; $\alpha(h_l, X)$, $\beta(h_l, X)$ are traces as defined in (3.6) corresponding to λ and $\tilde{k} \leq k$. The integers $\tilde{b}_i(n)$ are the exponents of the type $p(n+l-1)$ corresponding to non-zero coefficients $\tilde{g}_i$ such that $\tilde{b}_1(n) > \tilde{b}_2(n) > \cdots > \tilde{b}_{\tilde{k}}(n) > 0$. We have

$$\sum_{i=1}^{\tilde{k}} |\tilde{g}_i| \leq k(N) \max_i |\tilde{g}_i| \leq C_1(X)\sqrt{N}\log N \ . \tag{3.10}$$

We find an upper bound of the exponential sum (3.9) by setting $w(N) = [k(N)\log\log N]$ and estimating the integral

$$I_h(N) := \int_\alpha^\beta \left| \frac{1}{N}\sum_{n=1}^N e(\tilde{u}_n^{(N)}(h,A)) \right|^{2w} dx \leq \frac{1}{N^{2w}} \sum_\nu \left| \int_\alpha^\beta e(f_\nu(x))\, dx \right| \tag{3.11}$$

with $f_\nu(x) = u_{n_1}^{(N)}(x,h) - u_{n_2}^{(N)}(x,h) + \cdots + u_{n_{2w-1}}^{(N)}(x,h) - u_{n_{2w}}^{(N)}(x,h)$, $\nu = (n_1, \cdots, n_{2w})$. (W.l.o.g. $\beta - \alpha \leq 1$.)

Let $\overline{b}_i(\nu)$ $(i = 1, \ldots, \overline{r})$ be the sequence consisting of $\tilde{b}_j(n_i)$ $(j = 1, \ldots, \tilde{k},\ i = 1, \ldots, 2w)$ in strictly increasing ordering $(\overline{r} \leq 2w\tilde{k})$ and define the set

$$X_N = \Big\{ \nu \in \{1, \ldots, N\}^{2w} \mid \text{if} \quad \overline{b}_i(\nu) \quad \text{occurs among the elements} \\ \tilde{b}_j(n_i) \quad \text{only once then it is} \quad \leq \sqrt{N} \Big\} \ .$$

As in [9] one can prove

$$\operatorname{card} X_N \leq 2(8w)^{2w} N^w \ ,$$

hence

$$I_h(N) \leq \frac{1}{N^{2w}} \sum_{\nu \notin X_N} \left| \int_\alpha^\beta e(f_\nu(x))\, dx \right| + 2(8w)^{2w} N^{-w} \ . \tag{3.12}$$

In the sequel let us consider $\nu \notin X_N$. For $L = 0, 1, 2$ we set

$$f_\nu^{(L)} = \frac{d^L}{dx} f_\nu(x) = \sum_{i=1}^r b_i^L(\nu) g_i(\nu) e^{x b_i(\nu)} \ ,$$

where $b_i(\nu)$ $(i = 1, \ldots, r \leq \overline{r} \leq 2kw)$ is a stricly increasing subsequence of $\tilde{b}_i$ and $g_i(\nu)$ are non-vanishing linear combinations of the terms $\tilde{g}_i(n_j)$. The definition of $g_i(\nu)$ combined with an application of Lemma 3.1. yields to the estimate

$$c_1 N^{-d} \leq |g_i(\nu)| \leq c_2\sqrt{N} w(N) \tag{3.13}$$

for all $i = 1, \ldots, s$ and sufficiently large N, where $d = d(s)$ is a suitable exponent and c_1, c_2 are some positive constants. Defining

$$\sigma(j) := \frac{1}{4} - \frac{1}{j+4} \tag{3.14}$$

we distinguish three cases:

(i) $r = 1$

(ii) $r \geq 2$ and $b_1(\nu) - b_2(\nu) > N^{\sigma(2)}$

(iii) $r \geq 2$ and there exists a maximal s_0 with $2 \leq s_0 \leq r$ and $b_1(\nu) - b_{s_0}(\nu) \leq N^{\sigma(s_0)}$.

Case (i):

$$\left| \frac{f_\nu^{(L)}}{b_1^L(\nu)} \right| = |g_1(\nu) e^{x b_1(\nu)}| \geq c_1 N^{-d} e^{\alpha\sqrt{N}} \geq e^{\frac{\alpha}{2}\sqrt{N}}$$

for sufficiently large N.

Case (ii):

$$\begin{aligned}
\left| \frac{f_\nu^{(L)}}{b_1^L(\nu)} \right| &\geq e^{x b_1(\nu)} \left(|g_1(\nu)| - \sum_{i=2}^{r} \frac{b_i^L(\nu)}{b_1^L(\nu)} g_i(\nu) e^{x(b_i(\nu) - b_1(\nu))} \right) \\
&\geq e^{\alpha b_1(\nu)} \left(|g_1(\nu)| - \sum_{i=1}^{r} |g_i(\nu)| e^{-x N^{\sigma(2)}} \right) \\
&\geq e^{\alpha\sqrt{N}} \left(c_1 N^{-d} - e^{-\alpha N^{\sigma(2)}} 2 c_2 w k N^d \right) \\
&\geq e^{\frac{\alpha}{2}\sqrt{N}}
\end{aligned}$$

for sufficiently large N.

For $L = 1$ we derive from Proposition 2 in cases (i),(ii)

$$\left| \int_\alpha^\beta e(f_\nu(x))\, dx \right| \leq e^{-\frac{\alpha}{2}\sqrt{N}} \tag{3.15}$$

for sufficiently large N.

Case (iii): We set

$$\frac{f_\nu^{(L)}(x)}{b_1^L(\nu) e^{x b_1(\nu)}} = U(x) + V(x) \tag{3.16}$$

with

$$\begin{aligned}
U(x) &= \sum_{i=1}^{s_0} \frac{b_i^L(\nu)}{b_1^L(\nu)} g_i(\nu) e^{x(b_i(\nu) - b_1(\nu))} \\
V(x) &= \sum_{i=s_0+1}^{r} \frac{b_i^L(\nu)}{b_1^L(\nu)} g_i(\nu) e^{x(b_i(\nu) - b_1(\nu))}\,.
\end{aligned} \tag{3.17}$$

Using the notation $\delta = b_1(\nu) - b_{s_0}(\nu)$ and $b_1(\nu) - b_{s_0+1}(\nu) > N^{\sigma(s_0+1)}$ we obtain

$$|V(x)| \leq e^{-x N^{\sigma(s_0+1)}} \sum_{i=s_0+1}^{r} |g_i(\nu)| \leq e^{-\alpha N^{\sigma(s_0+1)}} 2 c_2 k w^2 \sqrt{N} \tag{3.18}$$

and

$$|U(x)| = \left| \frac{b_{s_0}^L(\nu) e^{-\delta x}}{b_1^L(\nu)} F_\nu(x) \right| \geq \frac{e^{-\delta x}}{4} |F_\nu(x)| \tag{3.19}$$

for sufficiently large N, where

$$F_\nu(x) = \sum_{i=1}^{s_0} \frac{b_i^L(\nu)}{b_{s_0}^L(\nu)} g_i(\nu) e^{x(b_i(\nu)-b_{s_0}(\nu))}\,.$$

Setting $\gamma_i = b_i(\nu) - b_{s_0}(\nu)$, $\delta_i = g_i(\nu)\, b_i^L(\nu)/b_{s_0}^L(\nu)$, $H = 4kw^2 c_2\sqrt{N}$, $\psi(N) = \log N$, and $M = [\gamma_1 \log H\, \psi(N)]$ we obtain by Proposition 3

$$|F_\nu^{(M)}| \geq \frac{|\delta_1|}{2} e^{\alpha} (4kw^2 c_2 \sqrt{N})^{\log \psi(N)} \tag{3.20}$$

for sufficiently large N. By Proposition 4 for every $\varepsilon > 0$ there exist $M(M+1)/2+1$ intervals $I_j \subseteq [\alpha, \beta]$, such that for all $x \in W := \bigcup_j I_j$

$$|F_\nu(x)| \geq \frac{|\delta_1|}{2} e^{\alpha} (4kw^2 c_2 \sqrt{N})^{\log\log N} \varepsilon^M \tag{3.21}$$

and $\mu([\alpha,\beta] \setminus W) \leq 2\varepsilon M(M+1)$. Thus we have for $x \in W$ and $\varepsilon = N^{-2w}$ (N sufficiently large)

$$\begin{aligned} |U(x) - V(x)| &\geq \left|\frac{e^{-\delta\beta}}{8}\right| \delta_1 \left| e^{\alpha} (2kw^2 c_2 \sqrt{N})^{\log\log N} N^{-2wM} \right| \\ &\quad - \left| e^{-\alpha N^{\sigma(s_0+1)}} 2c_2 kw^2 \sqrt{N} \right| \\ &\geq \left| \frac{|\delta_1|}{8} e^{\alpha - N^{\sigma(s_0)}\beta + \log\log N \log(2kw^2 c_2 \sqrt{N}) - 2wM \log N} \right| \\ &\quad - \left| 2c_2 e^{-\alpha N^{\sigma(s_0+1)} + \log(kw^2\sqrt{N})} \right| \geq e^{-N^{\frac{1}{4}}}\,. \end{aligned} \tag{3.22}$$

Hence by (3.17) and $b_1(\nu) \geq \sqrt{N}$

$$|f_\nu^{(L)}(x)| \geq e^{\alpha\sqrt{N} - N^{\frac{1}{4}}} \geq e^{\frac{\alpha}{2}\sqrt{N}} \tag{3.23}$$

for sufficiently large N, $x \in W$, and $L = 0, 1, 2$. Thus f_ν is strictly monotonic on every interval I_j. Hence Proposition 2 yields

$$\begin{aligned} \left| \int_W e(f_\nu(x))\, dx \right| &\leq \sum_j \left| \int_{I_j} e(f_\nu(x))\, dx \right| \\ &\leq \left(\frac{M(M+1)}{2} + 1\right) e^{-\alpha/2\sqrt{N}} \leq e^{-\alpha/4\sqrt{N}}\,. \end{aligned} \tag{3.24}$$

Consequently in case (iii) we have shown

$$\begin{aligned} \left| \int_\alpha^\beta e(f_\nu(x))\, dx \right| &\leq \left| \int_{W^c} e(f_\nu(x))\, dx \right| + \left| \int_W e(f_\nu(x))\, dx \right| \\ &\leq \frac{2M(M+1)}{N^{2w}} + e^{-\alpha/4\sqrt{N}} \leq N^{-w}\,. \end{aligned} \tag{3.25}$$

From (3.12) and (3.25) we obain

$$I_h(N) \leq 3(8w)^{2w} N^{-w} \tag{3.26}$$

for sufficiently large N. To conclude the proof of our theorem we define for arbitrary $\eta > 0$ the sets

$$\mathcal{M}(N,h) = \left\{ x \in [\alpha,\beta] \mid \left| \sum_{n=1}^{N} e(\tilde{u}_n^{(N)}(h,x)) \right| \ge 3\,8w\, N^{\frac{1}{2}+\eta} \right\} , \tag{3.27}$$

and obtain for their Lebesgue measures by (3.11) and (3.26)

$$\mu(\mathcal{M}(N,h)) \le I_h(N) \left(24w\, N^{\frac{1}{2}+\eta} \right)^{-2w} \le N^{-\eta w} . \tag{3.28}$$

Hence we have

$$\mu(\mathcal{M}(N)) := \mu \left(\bigcup_{0<\|h\|\le\sqrt{N}} \mathcal{M}(N,h) \right) \le (2\sqrt{N}+1)^{ks^2} N^{-\eta w} \le N^{-2} \tag{3.29}$$

for sufficiently large N. From the Borel—Cantelli-lemma we conclude that for almost all $x \in [\alpha,\beta]$ there is a constant C such that

$$\left| \sum_{n=1}^{N} e(\tilde{u}_n^{(N)}(h,x)) \right| \le C\, N^{\frac{1}{2}+\eta} \tag{3.30}$$

for all N and h with $0 < \|h\| \le \sqrt{N}$. Setting $G = [\sqrt{N}]$, Proposition 1 yields

$$\begin{aligned} &D_n(k(N)) \le \\ &\le 2(s^2k(N))^2 3^{s^2k(N)+1} \left(\frac{2}{\sqrt{N}} + \sum_{0<\|h\|\le G} \frac{1}{R(h)} \left| \frac{1}{N} \sum_{n=1}^{N} e(\tilde{u}_n^{(N)}(h,x)) \right| \right) \\ &\le 6s^4 (\log N)^{2\theta} N^{s^2 \log 3 \log N^{\theta-1}} \left(\frac{2}{\sqrt{N}} + C\, N^{-\frac{1}{2}+\eta} \sum_{0<\|h\|\le G} \frac{1}{R(h)} \right) . \end{aligned} \tag{3.31}$$

Since $\eta > 0$ is arbitrary and

$$\begin{aligned} \sum_{0<\|h\|\le\sqrt{N}} \frac{1}{R(h)} &= \sum_{0<\|h\|\le\sqrt{N}} \prod_{i=1}^{s^2k} \frac{1}{\max(1,|h_i|)} \\ &\le \prod_{i=1}^{s^2k} \sum_{0<\|h\|\le\sqrt{N}} \frac{1}{|h_i|} \le \prod_{i=1}^{s^2k} 2\log\sqrt{N} \\ &\le (\log N)^{s^2 \log N^{\theta}} = N^{s^2 \log\log N \log N^{\theta-1}} , \end{aligned}$$

the proof of our Theorem is complete.

4. A Constructive Example

We will consider a matrix A of the form

$$A = X^{-1}\, Y\, X = (a_{i,j})_{1\le i,j\le s}\ , \tag{4.1}$$

where

$$X = (x_{i,j})_{1\le i,j\le s} = \left(\xi^{3^{s(i-1)+j}}\right)_{1\le i,j\le s}$$

with $\xi = 2^{1/3^{s^2+1}}$ and Y is the diagonal matrix $\operatorname{diag}(\lambda, 0, \dots, 0)$ for some special chosen $\lambda > 0$. Since the determinant $\det(X)$ is a non-zero polynomial $Q(\xi)$ with integer coefficients and degree $< 3^{s^2+1}$ and the irreducible polynomial of ξ $x^{3^{s^2+1}} - 2 = 0$ has degree 3^{s^2+1} it follows that $\det(X) \ne 0$. Thus X^{-1} exists and $A^n = \lambda^n \overline{A} = \lambda^n (a'_{i,j})_{1\le i,j\le s}$, where the numbers $a'_{i,j}$ are algebraic and $1, a'_{i,j}$ $(1 \le i,j \le s)$ are linear independent over the rationals. This can be seen by following observation.

We have

$$X^{-1} = (\overline{x}_{ij})_{1\le i,j\le s} = (P_{ij}(\xi)/Q(\xi))_{1\le i,j\le s}\ ,$$

where $P_{ij}(\xi)$ are polynomials with integer coefficients and

$$a'_{i,j} = P_{i1}(\xi)\xi^{3^j}/Q(\xi)\ .$$

Since the triadic expansion is unique, for every power ξ^n there is only one polynomial $P_{i1}(\xi)\xi^{3^j}$ $(1 \le i,j \le s)$ or $Q(\xi)$ such that the corresponding coefficient of ξ^n is different from zero. Thus a linear combination

$$\begin{aligned} h_0 1 + \sum_{i,j=1}^{s} h_{ij} a'_{i,j} &= \frac{1}{Q(\xi)}\left(h_0 Q(\xi) + \sum_{i,j=1}^{s} h_{ij} P_{i1}(\xi)\xi^{3^j}\right) \\ &= \frac{P(\xi)}{Q(\xi)} \end{aligned} \tag{4.2}$$

cannot be zero unless all coefficients h_0, h_{ij} are zero, since the degree of the polynomial $P(\xi)$ is less than 3^{s^2+1}.

Therefore by W. Schmidt's theorem [8] there exists a constant $C \le 1$ such that

$$\left|\sum_{i,j=1}^{s} h_{ij} a'_{i,j}\right| \ge \frac{C}{R(h)^2} \tag{4.3}$$

for all lattice points $h = (h_{ij})_{1\le i,j\le s}$ with $\|h\| > 0$.

Let $\lambda_1 = 3\,, p_1 = 4$ and for $r \ge 1$

$$\begin{aligned} t_r &= (1 + [72/C])(2r+1)^{2s^2}((2r+1)^{2r}+1)^{(2r+1)s^2} \\ q_r &= \left[\frac{\lambda_r^{r^{2r}}}{t_r} + 1\right] t_r (r+1)^4 \\ p_{r+1} &= p_r q_r \\ \lambda_{r+1} &= \lambda_r + \frac{a_r}{q_r p_r} \end{aligned} \tag{3.4}$$

for some non-negative integers $a_r < q_r$ specified later.

Now we define

$$\lambda = \lambda_1 + \sum_{r=1}^{\infty} \frac{a_r}{p_r q_r} = \lim_{r\to\infty} \lambda_r \,. \tag{4.5}$$

To show that the sequence of the powers of the above chosen matrix A is c.d. we will proceed similarly to M.B. Levine [4].

Considering lattice points $h_n = (h_{ij,n})_{1\le i,j\le s}$ $(n = 1, \dots, 2r+1)$ we set

$$B_r(x) = \sum_{n=1}^{r} \left(\sum_{i,j=1}^{s} h_{ij,n} a'_{i,j} \right) x^{n-1} \tag{4.6}$$

and prove

Lemma 1. *There are at least $\frac{2}{3} q_r$ non-negative integers $a_r < q_r$ such that*

$$\min \left| B_{2r+1} \left(\lambda_r + \frac{a_r}{p_r q_r} \right) \right| \ge (p_r t_r)^{-2r} \,, \tag{4.7}$$

where the minimum is taken over all lattice points h_n with
$0 < \max_{1\le n\le 2r+1} \|h_n\| \le (2r+1)^{2r}$.

Proof. By Theorem 3.4.1 of [1] the set $S = \{x \in \mathbf{R} |\, |p(x)| \le K\}$ of a polynomial $p(x) = x^N + \cdots$ is the union of at most N intervals of measure $\mu(S) \le 12\, K^{1/N}$. Let n_0 denote the maximal number such that $\|h_{n_0}\| > 0$. Then

$$B_{2r+1} \left(\lambda_r + \frac{u}{p_r q_r} \right) = (p_r q_r)^{1-n_0} \left(\sum_{i,j=1}^{s} h_{ij,n_0} a'_{i,j} \right) P(u) \,,$$

where $P(u) = u^{n_0 - 1} + \cdots$ is a polynomial with integer coefficients. Choose

$$K = (p_r t_r)^{n_0 - 1 - 2r} \left(\frac{q_r}{t_r} \right)^{n_0 - 1} (2r+1)^{4rs^2} C^{-1} \,,$$

then the set of all u with

$$\left| B_{2r+1} \left(\lambda_r + \frac{u}{p_r q_r} \right) \right| < (p_r t_r)^{-2r} \,,$$

has measure

$$\begin{aligned} &\le 12\, K^{1/(n_0-1)} = 12 (p_r t_r)^{1-2r/(n_0-1)} \frac{q_r}{t_r} (2r+1)^{4rs^2/(n_0-1)} C^{-1/(n_0-1)} \\ &\le \frac{12}{C} (2r+1)^{2s^2} \frac{q_r}{t_r} \,, \end{aligned}$$

since

$$\left| \sum_{i,j=1}^{s} h_{ij,n_0} a'_{i,j} \right| \ge \frac{C}{(2r+1)^{4rs^2}} \,.$$

Therefore there are at most

$$((2r+1)^{2r}+1)^{(2r+1)s^2}\left(\frac{12}{C}(2r+1)^{2s^2}\frac{q_r}{t_r}+2r\right)\le\frac{q_r}{3}$$

non-negative integers $a_r < q_r$ not satisfying (4.7).

As in [4] we define

$$n_1=0\,, n_{r+1}=n_r+r^{2r}\;(r\ge 1) \tag{4.8}$$

and for $1\le k\le r$

$$S(c,h_1.\dots,h_k,h_{r+1})= \\ =\sum_{n=0}^{r^{2r}-1} e\left(\sum_{\nu=1}^{k}\left(\sum_{i,j=1}^{s} h_{ij,\nu}a'_{i,j}\right)\left(\lambda_r+\frac{c}{p_rq_r}\right)^{n_r+n+\nu-1}+h_{r+1}\frac{n}{r^{2r}}\right)\,, \tag{4.9}$$

where $h_\nu=(h_{ij,\nu})_{1\le i,j\le s}$ are integer points, $h_m\in\mathbf{Z}$,

$$D_{r,k}(c)=\sum_{0<\max_{1\le\nu\le k}(\|h_\nu\|,|h_m|)\le r^{r-1}}\frac{|S(c,h_1,\dots,h_k,h_{r+1})|}{R(h_1)\cdots R(h_k)R(h_{r+1})}\,,$$

and

$$M_{r,k}=\frac{2r}{q_r}\sum_{c=0}^{q_r-1}D_{r,k}(c)\,. \tag{4.10}$$

Next we specify the sequence a_r in (3.5). By a simple combinatorial argument we get

Lemma 2. *There are non-negative integers $a_r < q_r$ satisfying (3.7) such that for every integer $1\le k\le r$*

$$D_{r,k}(a_r)\le M_{r,k}\,. \tag{4.11}$$

Therefore it suffices to estimate $M_{r,k}$. For this purpose we set for $1\le k\le r$

$$E(h_1,\dots,h_k,h_{r+1})=\frac{1}{q_r}\sum_{c=0}^{q_r-1}|S(c,h_1,\dots,h_k,h_{r+1})|^2\,. \tag{4.12}$$

Using Cauchy—Schwarz's inequality we conclude

$$\begin{aligned} M_{r,k}=2r&\sum_{0<\max_{1\le\nu\le k}(\|h_\nu\|,|h_m|)\le r^{r-1}}\frac{1}{R(h_1)\cdots R(h_k)R(h_{r+1})}\times\\ &\times\frac{1}{q_r}\sum_{c=0}^{q_r-1}|S(c,h_1,\dots,h_k,h_{r+1})| \\ \le 2r&\sum_{0<\max_{1\le\nu\le k}(\|h_\nu\|,|h_m|)\le r^{r-1}}\frac{E(h_1,\dots,h_k,h_{r+1})^{1/2}}{R(h_1)\cdots R(h_k)R(h_{r+1})}\,. \end{aligned} \tag{4.13}$$

and derive as in [4]

Lemma 3. *For $r \ge r_0, 1 \le k \le r$ and $0 < \max_{1\le\nu\le k}(\|h_\nu\|, |h_m|) \le r^{r-1}$ we have*

$$E(h_1, \dots, h_k, h_{r+1}) < 16 r^{2r}\,. \tag{4.14}$$

Thus

$$D_{r,k}(a_r) \le M_{r,k} \le 8 r^{r+1}(3 + \log(r^{r-1}))^{s^2k+1}$$

or

$$D_{r,k}(a_r) = O(r^{r+1}(\log r^r)^{s^2k+1})\,, \tag{4.15}$$

where the O-constant does not depend on k. For fixed N let l_0 be defined by $n_{l_0} \le N < n_{l_0+1}$ and set $R = N - n_{l_0}$. For an interval $I \subseteq [0,1)^{s^2}$ let $A(I, N_1, N_2, A^n)$ denote the number of integers n satisfying $N_1 < n \le N_2$ and

$$(\{a'_{i,j}\lambda^{n+\nu-1}\})_{1\le i,j\le s} \in I \quad (\nu = 1, \dots, k)\,. \tag{4.16}$$

Then the discrepancy can be estimated by

$$\begin{aligned} D_N(k, A^n) \le \frac{1}{N}\sum_{r=1}^{l_0-1} \sup_I \left|A(I, n_r, n_{r+1}) - r^{2r}|I|\right| \\ + \frac{1}{N}\sup_I\left|A(I, n_{l_0}, N) - R|I|\right|\,. \end{aligned} \tag{4.17}$$

By Proposition 1 we have for $k \le r$

$$\begin{aligned} &\left|A(I, n_r, n_r + R, A^n) - R r^{2r} r^{-2r}|I|\right| \le 2(ks^2+1)^2 3^{ks^2+2} \times \\ &\times \left(r^{r+1} + \sum_{0<\max_{1\le\nu\le k}(\|h_\nu\|,|h_m|)\le r^{r-1}} L(h_1, \dots, h_k, h_{r+1}) \right)\,, \end{aligned} \tag{4.18}$$

unifomly for all $R \le r^{2r}$, where

$$\begin{aligned} &L(h_1, \dots, h_k, h_{r+1}) = \\ &\quad = \frac{\left| \sum_{n=0}^{r^{2r}-1} e\left(\sum_{\nu=1}^{k} (\sum_{i,j=1}^{s} h_{ij,\nu} a'_{i,j}) \lambda^{n_r+n+\nu-1} + h_{r+1}\frac{n}{r^{2r}} \right) \right|}{R(h_1)\cdots R(h_k) R(h_{r+1})}\,. \end{aligned}$$

Since

$$\begin{aligned} \sum_{n=0}^{r^{2r}-1} e\left(\sum_{\nu=1}^{k} \left(\sum_{i,j=1}^{s} h_{ij,\nu} a'_{i,j} \right) \lambda^{n_r+n+\nu-1} + h_{r+1}\frac{n}{r^{2r}} \right) \\ - S(a_r, h_1, \dots, h_k, h_{r+1}) = O(s^2) \end{aligned} \tag{4.19}$$

for $r \to \infty$ uniformly in $k \le r$ (cf. [4]),we get for $k \le r$

$$\left|A(I, n_r, n_r + R, A^n) - R|I|\right| \le C k^2 3^{ks^2} r^{r+ks^2+2} (\log r)^{ks^2+1}\,. \tag{4.20}$$

Combining (4.20) with the trivial estimate for $k > r$ and with (4.17) we conclude

$$\begin{aligned} D_N(k, A^n) &\le \frac{1}{N}\left(2k^{2k} + Ck^2 3^{ks^2} \sum_{r=1}^{l_0} r^{r+ks^2+2} (\log r)^{ks^2+1}\right) \\ &\le \frac{2k^{2k}}{N} + C_1 k^2 6^{ks^2} \frac{(\log N)^{ks^2+3}}{\sqrt{N}}, \end{aligned} \tag{4.21}$$

where C_1 does not depend on N and k. Hence we immediately obtain for $k = k(N) \le (\log N)^\theta$ $(0 < \theta < 1)$ and every $\eta > 0$

$$D_N(k(N), A^n) = O(N^{-1/2+\eta}). \tag{4.22}$$

Thus we have found an explicitely constructed matrix A such that the discrepancy of the sequence of its powers is of the same magnitude as obtained in our metric theorem.

References

1. R.P. Boas, *Entire Functions*, Academic Press, New York, 1954.
2. E. Hlawka, *Theorie der Gleichverteilung*, Bibl. Inst., Mannheim—Wien—Zürich, 1974.
3. L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, John Wiley and Sons, New York, 1974.
4. M.B. Levine, *On the completely uniform distribution of fractional parts of the exponential function*, Tr. Semin. Im. I. G. Petrovskogo **7** (1981), 245–256 (Russian, English summary).
5. V. Losert, W.G. Nowak, and R.F. Tichy, *On the asymptotic distribution of the powers of $s \times s$-matrices*, Comp. Math. **45** (1982), 273–291.
6. H. Niederreiter and R.F. Tichy, *Solution of a problem of Knuth on complete uniform distribution of sequences*, Mathmatika **23** (1985), 26–32.
7. W.G. Nowak and R.F. Tichy, *An improved estimate on the distribution mod 1 of powers of real matrices*, Comp. Math. **49** (1983), 283–289.
8. W.M. Schmidt, *Simultanous approximation to algebraic numbers by rationals*, Acta Math. **125** (1970), 189–201.
9. R.F. Tichy, *Ein metrischer Satz ber vollständig gleichverteilge Folgen*, Acta Arith. **48** (1987), 197–207.

Michael Drmota
Department of Discrete Mathematics

Robert Franz Tichy
Department of Technical Mathematics

Reinhard Winkler
Department of Algebra

Technical University of Vienna
Wiedner Hauptstrasse 8–10
A-1040 Vienna, AUSTRIA