The Parity of the Zeckendorf Sum-of-Digits-Function

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Abstract

Let $s(n)$ denote the sum of digits of the Zeckendorf representation of n and $S_{q,i}(N) = \sum_{n<N, n\equiv i \bmod q}(-1)^{s(n)}$. The aim of this paper is to discuss the behaviour of $S_{q,i}(N)$. First it is shown that that the values of $S(N) = S_{1,0}(N)$ admit a Gaussian limit law with bounded mean and variance of order $\log N$. Conversely, for $q > 1$ $S_{q,i}(N)$ (mostly) has a periodic fractal structure. We also prove that $S_{3,0}(N) \geq 0$ which is an analogon to a well known result by Newman [11] for binary digit expansions.

Our methods can easily be extended to other digit expansions related to linear recurrences.

1 Introduction

The Fibonacci numbers F_n, $n \geq 0$, are defined by $F_0 = 0$, $F_1 = 1$, and by the second order linear recurrence

$$F_k = F_{k-1} + F_{k-2} \qquad (k \geq 2). \tag{1}$$

Any non-negative integer n can be uniquely represented as the sum of different Fibonacci numbers

$$n = \sum_{j=1}^{s(n)} F_{k_j}, \tag{2}$$

where $k_1 \geq 2$ and $k_{j+1} > k_j + 1$, $j \geq 1$. This representation is called *Zeckendorf representation* of n and can easily be determined by the following way. Let F_k be the greatest Fibonacci number $\leq n$. Then $n = F_k + n_1$, in which $n_1 < F_{k-1}$, and the Zeckendorf representation of n equals the Zeckendorff representation of n_1 added by F_k. The number $s(n)$ of summands in the Zeckendorf representation of n will be called *Zeckendorf sum-of-digits-function.*

Our first aim to discuss the parity of $s(n)$, $n < N$. For this purpose we will use the notation

$$S(N) = \sum_{n<N} (-1)^{s(n)}. \tag{3}$$

It will be easy to show that $S(N) = \Theta(\log N)$. Hence the number of $n < N$ with even $s(N)$ is $N/2 + S(N)/2 = N/2 + \mathcal{O}(\log N)$. However, although the bound $\log N$ is best possible we have $S(n) = \mathcal{O}((\log N)^{1/2})$ for most $n < N$, say 99 %. More precisely the values $S(n)$, $n < N$, attain a Gaussian limit law with zero mean and variance of order $\log N$.

Theorem 1 *For every* $\varepsilon > 0$

$$\frac{1}{N}|\{n < N \,:\, S(n) \leq x\sqrt{c \log N/\log\gamma}\}| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt + \mathcal{O}\left((\log N)^{-1/2+\varepsilon}\right), \tag{4}$$

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uniformly for all $x \in \mathbf{R}$ *(as* $N \to \infty$*) and*

$$|\{n < N \,:\, S(n) = m\}| = \frac{N}{\sqrt{2\pi c \log N/\log\gamma}} \left(\exp\left(\frac{-m^2}{2c\log N/\log\gamma} \right) + \mathcal{O}\left((\log N)^{-1/2+\varepsilon} \right) \right) \tag{5}$$

uniformly for all integers m *(as* $N \to \infty$*), where*

$$c = \frac{5+3\sqrt{5}}{30}$$

and $\gamma = (1+\sqrt{5})/2$ *denotes the golden number. Furthermore*

$$\frac{1}{N}\sum_{n<N} S(n) = G\left(\frac{\log N}{6\log\gamma}\right) + \mathcal{O}(1/N) \tag{6}$$

and

$$\frac{1}{N}\sum_{n<N} S(n)^2 = c\,\frac{\log N}{\log\gamma} + H\left(\frac{\log N}{6\log\gamma}\right) + \mathcal{O}(1/N), \tag{7}$$

where $G(x)$ *and* $H(x)$ *are real continuous periodic functions with period 1.*

Note that the corresponding sum $\sum\limits_{n<N} (-1)^{s_2(n)}$ in the binary case has a very simple behaviour. It has a discrete limit law with $\mathbf{Pr}(0) = 1/2$ and $\mathbf{Pr}(1) = \mathbf{Pr}(-1) = 1/4$. Nevertheless Gaussian limit laws have appeared in relation with the sum of digits function, e.g. in [13] it is shown that the values of $s_2(n)$, $n < N$, attain a Gaussian limit law with mean $\sim \frac{1}{2}\log_2 N$ and variance $\sim \frac{1}{4}\log_2 N$.

The situation changes drastically if we consider arithmetic progressions $n \equiv i \bmod q$, $q > 1$, and the corresponding sums

$$S_{q,i}(N) = \sum_{n<N, n\equiv i \bmod q} (-1)^{s(n)}. \tag{8}$$

Here we (mostly) have $S_{q,i}(N) = \Theta(N^\alpha)$, where $0 < \alpha < 1$. (In any case we have $S_{q,i}(N) = \mathcal{O}(N^\alpha)$ for some $\alpha < 1$. Hence, the number of $n < N$, $n \equiv i \bmod q$, with even $s(n)$ is $N/(2q) + \mathcal{O}(N^\alpha)$.) Moreover, we will identify many cases, where $S_{q,i}(N)$ has a periodic fractal structure. For example, we will prove the following theorem.

Theorem 2 *Let* $q = F_k$*, where* $k > 3$ *is odd and not divisible by* 3*. Then*

$$S_{q,i}(N) = N^{\alpha_q} G_{q,i}\left(\frac{\log N}{4k\log\gamma}\right) + \mathcal{O}(N^{\beta_q}) \tag{9}$$

as $N \to \infty$*, where* $0 < \alpha_q < 1$*,* $\beta_q < \alpha_q$*, and* $G_{q,i}(x)$ *is a real continuous periodic function with period 1.*

Remark. There is no doubt that Theorem 2 holds for $q = F_k$, too, if $k > 3$ is odd and divisble by 3. However, we did not find a rigorous proof that $\alpha_q > 0$ in this case, cf. the Remark at the end of section 3.

A very interesting situation which is not covered by Theorem 2 is the case $q = F_4 = 3$. Here we prove

Theorem 3 *For* $q = 3$ *we have*

$$S_{3,i}(N) = N^\alpha G_{3,i}\left(\frac{\log N}{8\log\gamma}\right) + \Theta(\log N) \tag{10}$$

as $N \to \infty$, where

$$\alpha = \frac{\log(4+\sqrt{15})}{8\log\gamma} \tag{11}$$

and $G_{3,i}(x)$ are real continuous periodic functions with period 1. Moreover, $G_{3,0}(x) > 0$ for all $x \in [0,1)$ and

$$S_{3,0}(N) \ge 0 \tag{12}$$

for all $N \ge 0$.

This theorem resembles in some sense a theorem by Newman [11], where it is shown that

$$\sum_{n<N, n\equiv 0 \bmod 3} (-1)^{s_2(n)} > 0$$

where $s_2(n)$ denotes the number of binary digits of n. The fractal representations (9,10) are also very similar to those known for binary digit expansions (see [8, 5]). However, phenomena of type (12) seem to be very rare for the Zeckendorf sum of digits function. In the binary case there are lots of examples (see [5]).

Remark. The results of Theorems 1 and 2 are stronlgy related to properties of substitution sequences discussed by Dumont and Thomas [2, 3, 4]. Consider the alphabet $A = \{1, \overline{2}, \overline{1}, 2\}$ and the substitution σ defined by $\sigma(1) = 1\overline{2}$, $\sigma(\overline{2}) = \overline{1}$, $\sigma(\overline{1}) = \overline{1}2$, $\sigma(2) = 1$ and let $(u_n)_{n\ge 0}$ be the fixed point of σ with $u_0 = 1$ (where σ is extended to $A^* = \bigcup_{k\ge 0} A^k$ by concatenation), i.e. $u_0u_1u_2\cdots = 1\overline{2}\overline{1}\overline{1}2\overline{1}2\overline{1}21\overline{1}2\cdots$. Furthermore, let $f : A \to \{-1, 1\}$ be defined by $f(1) = f(2) = 1$, $f(\overline{1}) = f(\overline{2}) = -1$. Then it is easy to show that

$$f(u_n) = (-1)^{s(n)}$$

and

$$f(\sigma^6(a)) = f(a)$$

for all $a \in A$ (where f is extended from A to A^* by $f(a_1a_2\cdots a_k) = f(a_1)+f(a_2)+\cdots+f(a_k)$). Hence we can apply [3, Theorem, p. 214 f.] for $k = 1$ and $k = 2$. For this purpose set

$$M_\sigma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

which has eigenvalues $(1\pm i\sqrt{3})/2$, $(1\pm\sqrt{5})/2$. Hence $M_\tau = (M_\sigma)^6$ has eigenvalues $((1\pm\sqrt{5})/2)^6$, 1 (double), and we directly obtain the leading term of the asymptotic expansions (6) and (7). We just have to evaluate the constants α, β of [3] to $\alpha = 0$, $\beta = (5+3\sqrt{5})/2$. Moreover, we obtain similiar expansions for

$$\frac{1}{N}\sum_{n<N} (S(n))^k$$

for any positive integer k and all involved periodic functions are continuous. Finally asymptotic normality of $S(n)$ follows from [4, Remark, p. 35] and it seems that expansions (8) and (9) (without positivity) may be deduced from [2, Theorem 2.6] (after some additional work).

However, the methods presented in this paper are completely different from those used in [2, 3, 4], in fact they are much more explicit. We obtain an error bound in the weak limit law (4) and additionally a local limit law (6), again with a uniform error term. Furthermore, the periodic functions $G(x)$ and $H(x)$ in (6) and (7) (see Proposition 2) are explicit and the error term $\mathcal{O}(1/N)$ is better than that of [3].

The paper is organized in the following way. In section 2 we will prove and discuss Theorem 1, Theorem 2 and related results are provided in section 3. Theorem 3 is proved in section 4 and in a final section 5 we give an outline which kinds of digit expansions can be treated by using the methods which are developed in this paper.

2 A Gaussian Limit Law

The plan of the proof of Theorem 1 is the following one. First we will show (4) and (5) for Fibonacci numbers $N = F_k$ and then extend it for all N. (6) and (7) will be proved afterwards.

Lemma 1 *Suppose that $N = F_k + n$, where $n \le F_{k-1}$. Then*

$$S(N) = S(F_k) - S(n). \tag{13}$$

Furthermore,

$$S(F_k) = \frac{2}{\sqrt{3}} \sin\left(\frac{k\pi}{3}\right) = \begin{cases} 0 & \text{if } k \equiv 0 \bmod 3, \\ 1 & \text{if } k \equiv 1 \bmod 6 \text{ or } k \equiv 2 \bmod 6, \\ -1 & \text{if } k \equiv 4 \bmod 6 \text{ or } k \equiv 5 \bmod 6. \end{cases} \tag{14}$$

Proof. By definition

$$\begin{aligned} S(F_k + n) &= \sum_{j<F_k} (-1)^{s(j)} + \sum_{F_k \le j < F_k+n} (-1)^{s(j)} \\ &= S(F_k) + \sum_{j'<n} (-1)^{s(F_k+j')} = S(F_k) - S(n). \end{aligned}$$

This proves (13). Moreover, we obtain the recurrence $S(F_k) = S(F_{k-1}) - S(F_{k-2})$. Since $S(F_2) = 1$ and $S(F_3) = 0$ we directly get (14). □

Corollary 1 *$S(N) = \Theta(\log N)$ as $N \to \infty$.*

Proof. If N has the Zeckendorf representation $N = \sum_{j=1}^{s(N)} F_{k_j}$ then (13) implies

$$S(N) = \sum_{j=1}^{s(N)} (-1)^{s(N)-j} S(F_{k_j}) = \mathcal{O}(\log N).$$

On the other hand set

$$N = \sum_{j=1}^{L} F_{3j-1}$$

then (14) gives $|S(N)| = L \ge c \log N$ for some constant $c > 0$. □

Now set

$$a_{N,m} = |\{n < N \,:\, S(n) = m\}|$$

and (for $y \in \mathbf{C}$)

$$a_N(y) = \sum_{n<N} y^{s(n)} = \sum_{m \in \mathbf{Z}} a_{N,m} y^m.$$

Then these functions satisfy the following recurrence.

Lemma 2 *Suppose that $N = F_k + n$, where $n \le F_{k-1}$. Then*

$$a_N(y) = a_{F_k}(y) + y^{S(F_k)} a_n(y^{-1}). \tag{15}$$

Especially we have

$$a_{F_k}(y) = a_{F_{k-1}}(y) + y^{S(F_{k-1})} a_{F_{k-2}}(y^{-1}). \tag{16}$$

Proof. Equation (13) implies that

$$\begin{aligned} a_N(y) &= \sum_{j<F_k} y^{S(j)} + \sum_{F_k \le j < F_k+n} y^{S(j)} \\ &= a_{F_k}(y) + y^{S(F_k)} a_n(y^{-1}). \end{aligned}$$

□

The apparent difficulty of recurrence (16) is that y and y^{-1} appear at once. However, it is possible to separate them.

Proposition 1 *Suppose that $0 \le i < 6$. Then the sequence $a_{F_{6k+i}}(y)$, $k \ge 0$, satisfies the recurrence*

$$\begin{aligned} a_{F_{6k+i}}(y) &= 4(y^2+y+y^{-1}+y^{-2})a_{F_{6(k-1)+i}}(y) \\ &\quad -(4y^2+8y+14+8y^{-1}+4y^{-2})a_{F_{6(k-2)+i}}(y) \\ &\quad +4(y+3+y^{-1})a_{F_{6(k-3)+i}}(y) - a_{F_{6(k-4)+i}}(y) \end{aligned} \tag{17}$$

Proof. Set

$$\mathbf{a}_k(y) = (a_{F_{6k}}(y), a_{F_{6k+1}}(y), \ldots, a_{F_{6k+5}}(y), a_{F_{6k}}(y^{-1}), a_{F_{6k+1}}(y^{-1}), \ldots, a_{F_{6k+5}}(y^{-1}))^t,$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & y^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ y^{-1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & y^{-1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

and

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & y^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then by (14) and (16) we have

$$\mathbf{a}_k(y) = \mathbf{A}\mathbf{a}_k(y) + \mathbf{B}\mathbf{a}_{k-1}(y)$$

or

$$\mathbf{a}_k(y) = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{a}_{k-1}(y).$$

Since the characteristic polynomial of $(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$ is given by

$$\det\left(\lambda\mathbf{I}-(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\right)=\lambda^8(\lambda-1)(\lambda^3-(4y^2+4y+3+4y^{-1}+4y^{-2})\lambda^2+(4y+11+4y^{-1})\lambda-1)$$

the components of $\mathbf{a}_k(y)$ satisfy the proposed recurrence (17). □

Note that in the case $y=1$ the characteristic polynomial of (17) factors as

$$(\lambda-1)^2(\lambda-9-4\sqrt{5})(\lambda-9+4\sqrt{5}),$$

i.e. the largest eigenvalue is given by $\lambda_m=9+4\sqrt{5}$. Furthermore, if y is in the vicinity of 1 then let $\lambda_m(y)$ denote the dominating root (i.e. of maximal modulus) of the polynomial

$$\lambda^3-(4y^2+4y+3+4y^{-1}+4y^{-2})\lambda^2+(4y+11+4y^{-1})\lambda-1.$$

By the implicit function theorem $\lambda_m(y)$ is an analytic function and is approximately given by

$$\lambda_m(y)=9+4\sqrt{5}+\frac{105+47\sqrt{5}}{10}(y+y^{-1}-2)+\mathcal{O}\left((y+y^{-1}-2)^2\right). \tag{18}$$

Lemma 3 *There exist $\varepsilon>0$ and $\delta>0$ such that*

$$a_{F_k}(y)=c_k(y)\lambda_m(y)^{k/6}+\mathcal{O}\left(\lambda_m^{(1-\delta)k/6}\right) \tag{19}$$

uniformly for $|y-1|<\varepsilon$ (as $k\to\infty$), where $c_k(y)$ are analytic functions (for $|y-1|<\varepsilon$) with $c_k(1)=1/\sqrt{5}$ and $c_{k+6}(y)=c_k(y)$. Furthermore, we have

$$a_{F_k}(y)=\mathcal{O}\left(\lambda_m^{(1-\delta)k/6}\right) \tag{20}$$

uniformly for $|y|=1$ and $|y-1|\ge\varepsilon$ (as $k\to\infty$).

Proof. (19) and (20) follow immediately from Proposition 1 by observing that the largest root $\lambda_m(y)$ is bounded away from λ_m if $|y|=1$ and $|y-1|\ge\varepsilon$. □

Corollary 2 *Let $c=(5+3\sqrt{5})/30$. There exists $\eta>0$ such that*

$$a_{F_k}(e^{it})=F_k e^{-ckt^2/2}e^{\mathcal{O}(|t|+kt^4)}+\mathcal{O}\left(\lambda_m^{(1-\delta)k/6}\right) \tag{21}$$

uniformly for (real) $|t|\le\eta$ (as $k\to\infty$).

Proof. By (18)

$$\lambda_m(e^{it})=\lambda_m e^{-ct^2/2+\mathcal{O}(t^4)}. \tag{22}$$

Hence, (21) follows from Lemma 3. □

Corollary 3 *Let $c=(5+3\sqrt{5})/30$. Then we have*

$$a_{F_km}=\frac{F_k}{\sqrt{2\pi ck}}\exp\left(-\frac{m^2}{2ck}\right)+\mathcal{O}(F_k/k) \tag{23}$$

uniformly for all $m\ge 0$ (as $k\to\infty$).

Proof. By Cauchy's formula

$$a_{F_km}=\frac{1}{2\pi}\int_{-\pi}^{\pi}a_{F_k}(e^{it})e^{-imt}\,dt.$$

Since

$$\int_{\eta\le|t|\le\pi} |a_{F_k}(e^{it})|\,dt = \mathcal{O}\left(\lambda_m^{(1-\delta)k/6}\right) = \mathcal{O}(F_k/k)$$

we just have to evaluate

$$I = \frac{1}{2\pi}\int_{|t|\le k^{-\varepsilon}} a_{F_k}(e^{it})e^{-imt}\,dt + \frac{1}{2\pi}\int_{k^{-\varepsilon}\le|t|\le\eta} a_{F_k}(e^{it})e^{-imt}\,dt = I_1 + I_2,$$

where $\varepsilon > 0$ is sufficiently small. From (22) it follows that $|\lambda_m(e^{it})| \le e^{-ct^2}$ for $|t| \le \eta$. Hence,

$$I_2 \le \frac{1}{\pi}\int_{k^{-\varepsilon}}^{\infty} e^{-cjt^2}\,dt + \mathcal{O}\left(\lambda_m^{(1-\delta)/6}\right) = \mathcal{O}\left(e^{-ck^{1-2\varepsilon}}\right) + \mathcal{O}\left(\lambda_m^{(1-\delta)k/6}\right) = \mathcal{O}(F_k/k)$$

Finally,

$$\begin{aligned}
I_1 &= \frac{1}{2\pi}\int_{|t|\le k^{-\varepsilon}} F_k e^{-imt-kct^2/2}\left(1+\mathcal{O}(kt^4+|t|)\right)\,dt + \mathcal{O}\left(\lambda^{(1-\delta)k/6}\right) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_k e^{-imt-kct^2/2}\,dt + \mathcal{O}\left(\int_{|t|>k^{-\varepsilon}} F_k e^{-kct^2/2}\,dt\right) \\
&\quad +\mathcal{O}\left(\int_{|t|\le k^{-\varepsilon}} F_k e^{-kct^2/2}(kt^4+|t|)\,dt\right) + \mathcal{O}\left(\lambda_m^{(1-\delta)k/6}\right) \\
&= \frac{F_k}{\sqrt{2\pi ck}}\exp\left(-\frac{m^2}{2kc}\right) + \mathcal{O}(F_k/k)
\end{aligned}$$

This completes the proof of Corollary 3. □

Let X_N be a discrete random variable with probability distribution $\mathbf{Pr}\{X_N = m\} = a_{N,m}/N$. Then the characteristic function of X_{F_k} is given by

$$\varphi_{X_{F_k}}(t) = \mathbf{E}e^{itX_{F_k}} = \frac{1}{F_k}a_{F_k}(e^{it}).$$

Hence, for the normalized random variable $Y_{F_k} = X_{F_k}/(ck)^{1/2}$ we obtain

$$\varphi_{Y_{F_k}}(t) = \frac{1}{F_k}a_{F_k}(e^{it/\sqrt{ck}}) \sim e^{-t^2/2} \qquad (k\to\infty) \tag{24}$$

for every fixed t which means that X_{F_k} has a Gaussian limiting distribution (as $k\to\infty$). The final step of the proof of (4) is to show that X_N, N not necessarily a Fibonacci number, has a Gaussian limiting distribution, too, as $N\to\infty$.

Proposition 2 *Let $c = (5+3\sqrt{5})/30$ and let $\lg N = \log N/\log\gamma$ denote the logarithm to basis $\gamma = (1+\sqrt{5})/2$. Then, for every $\varepsilon > 0$ the characteristic function of $Y_N = X_N/(c\lg N)^{1/2}$ satisfies*

$$\varphi_{Y_N}(t) = e^{-t^2/2} + \mathcal{O}\left((\lg N)^{-1/2+\varepsilon}\right) \tag{25}$$

uniformly for (real) t with $|t| \le (\lg N)^{1/2-\varepsilon}$ (as $N\to\infty$).

Proof. Suppose that N has the Zeckendorf representation $N = \sum_{j=1}^{J} F_{k_j}$. where $k_{j+1} > k_j + 1$, $j \ge 1$. Then (15) implies

$$a_N(y) = a_{F_{k_J}}(y) + y^{S(F_{k_J})}a_{F_{k_{J-1}}}(y^{-1}) + y^{S(F_{k_J}+F_{k_{J-1}})}a_{F_{k_{J-2}}}(y) + \cdots + y^{S(N-F_{k_1})}a_{F_{k_1}}(y^{(-1)^{J-1}}). \tag{26}$$

Let $\varepsilon > 0$ be fixed and let L be defined by $k_L \ge k_J - k_J^\varepsilon > k_{L-1}$. Since we have for $j \ge L$

$$\left(e^{it/(c\lg N)^{1/2}}\right)^{S(F_{k_J}+F_{k_{J-1}}+\cdots+F_{k_{j+1}})} = e^{\mathcal{O}(|t|(J-j)(\lg N)^{-1/2})} = e^{\mathcal{O}(|t|(\lg N)^{\varepsilon-1/2})}$$

and (for all j)

$$\left|a_{F_{k_j}}\left(e^{it/(c\lg N)^{1/2}}\right)\right| \le F_{k_j},$$

we obtain from (24) and (26)

$$\begin{aligned} a_N\left(e^{it/(c\lg N)^{1/2}}\right) &= Ne^{-t^2/2}\sum_{j=L}^{J}\frac{F_{k_j}}{N}e^{-((k_j-\lg N))t^2)/(2\lg N)}e^{\mathcal{O}(|t|(\lg N)^{\varepsilon-1/2}+|t^4|(\lg N)^{-1})} \\ &\quad +\mathcal{O}\left(F_{k_L}\right) \\ &= Ne^{-t^2/2}\sum_{j=L}^{J}\frac{F_{k_j}}{N}e^{\mathcal{O}(|t|(\lg N)^{\varepsilon-1/2}+|t|^2(\lg N)^{\varepsilon-1}+|t^4|(\lg N)^{-1})}+\mathcal{O}\left(F_{k_L}\right) \end{aligned}$$

Hence, if $|t| \le (\lg N)^\varepsilon$

$$a_N\left(e^{it/(c\lg N)^{1/2}}\right) = Ne^{-t^2/2} + \mathcal{O}\left(N(\lg N)^{2\varepsilon-1/2}\right).$$

(Note that $k_J = \lg N + \mathcal{O}(1)$.) Finally, if $(\lg N)^\varepsilon \le |t| \le (\lg N)^{1/2-\varepsilon}$ we can use (24) to obtain

$$|a_{F_{k_j}}\left(e^{it/(c\lg N)^{1/2}}\right)| \le e^{-c'(\lg N)^{2\varepsilon}}$$

for $j \ge L$ (and a sufficiently small constant $c' > 0$) which leads to

$$a_N\left(e^{it/(c\lg N)^{1/2}}\right) = \mathcal{O}\left(N(\lg N)^{2\varepsilon-1/2}\right).$$

This proves (25). $\square$

Now, (4) follows from Esseen's inequality [6, p. 32]. Set

$$\Delta_N(t) = e^{-t^2/2} - \frac{1}{N}a_N\left(e^{it/(c\lg N)^{1/2}}\right)$$

Then we have

$$\frac{1}{N}|\{n < N : S(n) \le x\sqrt{c\lg N}\}| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\,dt + \mathcal{O}\left(\frac{1}{T}+\int_{-T}^{T}\left|\frac{|Delta_N(t)}{t}\right|\,dt\right).$$

Chossing $T = (\lg N)^{1/2-\varepsilon}$ we directly obtain from Proposition 2 (and from a trivial estimate for $|t| \le (\log N)^{-2}$)

$$\int_{-T}^{T}\left|\frac{\Delta_N(t)}{t}\right|\,dt = \mathcal{O}\left((\log N)^{-1/2+\varepsilon}(\log\log N)\right).$$

This completes the proof of (4).

The proof of (5) requires a similar procedure as in Proposition 2.

Proposition 3 *For every $\varepsilon > 0$ we have*

$$a_{Nm} = \frac{N}{\sqrt{2\pi c\lg N}}\left(\exp\left(\frac{-m^2}{2c\lg N}\right) + \mathcal{O}\left((\log N)^{-1/2+\varepsilon}\right)\right) \tag{27}$$

uniformly for all integers m (as $N \to \infty$)

Proof. As in the proof of Proposition 2 we suppose that N has the Zeckendorf representation $N = \sum\limits_{j=1}^{J} F_{k_j}$. where $k_{j+1} > k_j + 1$, $j \geq 1$. Then (26) implies

$$a_{Nm} = \sum_{j=1}^{J} a_{F_{k_j},(-1)^{J-j}m - S(F_{k_J}+\cdots+F_{k_{j+1}})} \tag{28}$$

Furthermore, let $\varepsilon > 0$ be fixed and let L be defined by $k_L \geq k_J - k_J^{\varepsilon} > k_{L-1}$. Then by (28) and (23)

$$a_{Nm} = \sum_{j=L}^{J} \frac{F_{k_j}}{\sqrt{2\pi c k_j}} \exp\left(-\frac{(m - (-1)^{J-j} S(F_{k_J} + \cdots + F_{k_{j+1}}))^2}{2 j_l \sigma^2} \right) + \sum_{j=L}^{J} \mathcal{O}(F_{k_j}/k_j) + \mathcal{O}(F_{k_L}).$$

If $j \geq L$ and $m = \mathcal{O}(k_J^{1/2} \lg k_J)$ then

$$\begin{aligned}
\frac{m^2}{2c\lg N} &- \frac{\left(m - (-1)^{J-j} S(F_{k_J} + \cdots + F_{k_{j+1}})\right)^2}{2ck_j} \\
&= \frac{m^2 - \left(m - (-1)^{J-j} S(F_{k_J} + \cdots + F_{k_{j+1}})\right)^2}{2c\lg N} \\
&\quad + \left(m - (-1)^{J-j} S(F_{k_J} + \cdots + F_{k_{j+1}})\right)^2 \left(\frac{1}{2c\lg N} - \frac{1}{2ck_j} \right) \\
&= \mathcal{O}\left(k_J^{\varepsilon - 1/2} \lg k_J\right) + \mathcal{O}\left(k_J^{\varepsilon-1} (\lg k_J)^2\right),
\end{aligned}$$

where we have used $\lg N = k_J + \mathcal{O}(1)$. Hence, from

$$\sqrt{\frac{k_J + \mathcal{O}(1)}{k_j}} \exp\left(\frac{m^2}{2c\lg N} - \frac{\left(m - (-1)^{J-j} S(F_{k_J} + \cdots + F_{k_{j+1}})\right)^2}{2ck_j} \right) = 1 + \mathcal{O}\left(k_J^{\varepsilon-1/2} \lg k_J\right)$$

we obtain

$$\begin{aligned}
a_{Nm} &= \frac{N}{\sqrt{2\pi c \lg N}} \exp\left(-\frac{m^2}{2c\lg N}\right) \sum_{j=L}^{J} \frac{F_{k_l}}{N} \left(1 + \mathcal{O}\left(k_J^{\varepsilon-1/2}\lg k_J\right)\right) + \mathcal{O}(F_{k_J}/k_J) \\
&= \frac{N}{\sqrt{2\pi c \lg N}} \left(\exp\left(-\frac{m^2}{2c\lg N}\right) + \mathcal{O}\left(k_J^{\varepsilon-1/2}\lg k_J\right) \right).
\end{aligned}$$

If $|m| \geq k_J^{1/2} \lg k_J$ then we have for $j \geq L$

$$\begin{aligned}
a_{F_{k_j},(-1)^{J-j}m - S(F_{k_J}+\cdots+F_{k_{j+1}})} &= \mathcal{O}\left(F_{k_j} k_J^{-1/2} \exp\left(-\frac{(\lg k_J)^2}{4c} \right) \right) \\
&= \mathcal{O}\left(F_{k_j} k_J^{-1} \right)
\end{aligned}$$

which finally gives

$$\begin{aligned}
a_{Nm} &= \mathcal{O}\left(\alpha^{k_J} k_J^{-1}\right) + \mathcal{O}\left(\sum_{j=0}^{L-1} \frac{F_{k_j}}{k_j^{1/2}} \right) \\
&= \mathcal{O}(F_{k_J}/k_J).
\end{aligned}$$

This completes the proof of Proposition 3. □

Now set $E_N = \mathbf{E}X_N = \sum\limits_{n<N} S(n)$ and $M_N = \mathbf{E}X_N^2 = \sum\limits_{n<N} S(n)^2$. In order to prove (6) and (7) we need recurrences for E_N and M_N and asymptotic formulae for E_{F_k} and M_{F_k}.

Lemma 4 *Suppose that $N = F_k + n$, where $n \le F_{k-1}$. Then*

$$\begin{aligned} E_N &= E_{F_k} - E_n + nS(F_k), \quad \textit{and} && (29)\\ M_N &= M_{F_k} + M_n - 2S(F_k)E_n + nS(F_k)^2. && (30)\end{aligned}$$

Furthermore,

$$E_{F_k} = \frac{1}{2}\left(F_{k+1}S(F_{k-1}) - F_{k-1}S(F_k) - S(F_{k-1})\right) \tag{31}$$

and

$$\begin{aligned} M_{F_k} &= \frac{k}{15}L_{k+2} - \frac{1}{4}(S(F_k)S(F_{k+1})+1)L_k \\ &\quad -\frac{1}{12}S(F_{2k+1})F_k - \frac{4}{15}F_k + \frac{1}{2}(1 - S(F_k)S(F_{k-1})), \end{aligned} \tag{32}$$

where L_k, $k \ge 0$ denote the Lucas numbers defined by $L_0 = 2$, $L_1 = 1$, and by $L_k = L_{k-1} + L_{k-2}$ for $k \ge 2$.

Proof. Equation (29) follows from

$$\begin{aligned} E_N &= \sum_{j<F_k} S(j) + \sum_{j'<n} S(j'+F_k) \\ &= E_{F_k} - E_n + nS(F_k). \end{aligned}$$

Similarly one can prove (30).

In order to prove (31) and (32) we just have to check the initial conditions $E_{F_1} = E_{F_2} = 0$, $M_{F_1} = M_{F_2} = 0$ and that the recurrences (29) and (30) are satisfied. With help of the representations $F_k = (\gamma^k - (-\gamma)^{-k})/\sqrt{5}$, $L_k = \gamma^k + (-\gamma)^{-k}$, $S(F_k) = 2\sin(k\pi/3)/\sqrt{3}$ this is an easy exercise. $\Box$

Note that $M_{F_k} \sim k/15 L_{k+2} \sim (5+3\sqrt{5})/30 F_k$ fits to the normalization in (24).

In order to complete the proof of Theorem 1 we need the following lemma (for a proof see [7]) which is useful to define proper functions $G(x)$ and $H(x)$.

Lemma 5 *Any positive real number y can be uniquely represented by*

$$y = \sum_{j=0}^{J} \gamma^{k_j}, \qquad 0 \le J \le \infty, \tag{33}$$

where $k_{j+1} < k_j - 1$, $j \ge 0$, and if $J = \infty$ then there infinitely many j with $k_{j+1} < k_j - 2$.

Proposition 4 *For $x \in \mathbf{R}$ set*

$$\begin{aligned} G(x) &= \gamma^{-6x}\sum_{j=0}^{J}(-1)^j\left(\frac{1}{\sqrt{15}}\gamma^{k_j+1}\sin\frac{(k_j-1)\pi}{3} - \frac{1}{\sqrt{15}}\gamma^{k_j-1}\sin\frac{k_j\pi}{3} - \frac{2}{\sqrt{3}}\sin\frac{(k_j-1)\pi}{3}\right) \\ &\quad +\gamma^{-6x}\frac{1}{\sqrt{15}}\sum_{j=0}^{J}(-1)^j\sin\frac{k_j\pi}{3}\sum_{i>j}\gamma^{k_j} \end{aligned}$$

and

$$\begin{aligned} H(x) &= \gamma^{-6x}\sum_{j=0}^{J}\left(\frac{\cos\left(\frac{(2k_j+1)\pi}{3}\right)}{6} - \frac{\sin\left(\frac{(2k_j+1)\pi}{3}\right)}{6\sqrt{15}} - \frac{4}{15\sqrt{5}} - \frac{1}{3}\right)\gamma^{k_j} \\ &\quad -\gamma^{-6x}\frac{4}{\sqrt{15}}\sum_{j=0}^{J}\sin\left(\frac{k_j\pi}{3}\right)\sum_{i>j}\gamma^{k_i}\,G\left(\frac{1}{6\log\gamma}\log\left(\frac{1}{\sqrt{5}}\sum_{i>j}\gamma^{k_i}\right)\right) \\ &\quad +\gamma^{-6x}\frac{4}{\sqrt{45}}\sum_{j=0}^{J}\sin^2\left(\frac{k_j\pi}{3}\right)\sum_{i>j}\gamma^{k_i}, \end{aligned}$$

where $y=\sqrt{5}\gamma^{6x}$ *has the representation (33). Then (6) and (7) are satisfied.*

Proof. Let $y>0$ be represented by (33) and set

$$\begin{aligned} G_0(y) &= \sum_{j=0}^{J}(-1)^j\left(\frac{1}{\sqrt{15}}\gamma^{k_j+1}\sin\frac{(k_j-1)\pi}{3}-\frac{1}{\sqrt{15}}\gamma^{k_j-1}\sin\frac{k_j\pi}{3}-\frac{2}{\sqrt{3}}\sin\frac{(k_j-1)\pi}{3}\right) \\ &\quad +\frac{1}{\sqrt{15}}\sum_{j=0}^{J}(-1)^j\sin\frac{k_j\pi}{3}\sum_{i>j}\gamma^{k_i}. \end{aligned}$$

Obviously $G_0(\gamma^6 y)=\gamma^6 G_0(y)$ and

$$G(x)=\gamma^{-6x}G_0\left(\gamma^{6x}\sqrt{5}\right).$$

Hence $G(x+1)=G(x)$. Furthermore, if N has the Zeckendorf expansion $N=\sum_{j=0}^{J}F_{k_j}$, where $F_{k_{j+1}}<F_{k_j}-1$, $j\geq 0$. Then by (29), (31), and by the representations $F_k=(\gamma^k-(-\gamma)^{-k})/\sqrt{5}$ and $S(F_k)=2\sin(k\pi/3)/\sqrt{3}$

$$\begin{aligned} E_N &= \sum_{j=0}^{J}\frac{(-1)^j}{2}\left(F_{k_j+1}S(F_{k_j-1})-F_{k_j-1}S(F_{k_j})-S(F_{k_j-1})\right) \\ &\quad +\sum_{j=0}^{J}(-1)^j S(F_{k_j})\sum_{i>j}F_{k_i} \\ &= \sum_{j=0}^{J}(-1)^j\left(\frac{1}{\sqrt{15}}\gamma^{k_j+1}\sin\frac{(k_j-1)\pi}{3}-\frac{1}{\sqrt{15}}\gamma^{k_j-1}\sin\frac{k_j\pi}{3}-\frac{2}{\sqrt{3}}\sin\frac{(k_j-1)\pi}{3}\right) \\ &\quad +\frac{1}{\sqrt{15}}\sum_{j=0}^{J}(-1)^j\sin\frac{k_j\pi}{3}\sum_{i>j}\gamma^{k_i}. \\ &- \sum_{j=0}^{J}(-1)^j\left(\frac{1}{\sqrt{15}}(-\gamma)^{-k_j-1}\sin\frac{(k_j-1)\pi}{3}-\frac{1}{\sqrt{15}}(-\gamma)^{-k_j+1}\sin\frac{k_j\pi}{3}-\frac{2}{\sqrt{3}}\sin\frac{(k_j-1)\pi}{3}\right) \\ &\quad -\frac{1}{\sqrt{15}}\sum_{j=0}^{J}(-1)^j\sin\frac{k_j\pi}{3}\sum_{i>j}(-\gamma)^{-k_i} \\ &= G_0(\sqrt{5}N)+\mathcal{O}(1)=N\,G\left(\frac{\log N}{6\log\gamma}\right)+\mathcal{O}(1). \end{aligned}$$

The continuity of $G_0(y)$ (and consequently that of $G(x)$) can be proved by exactly the same methods as in [1] (compare also with [3]). This completes the proof of (6).

By using (30) it easily follows that

$$M_N=cN\frac{\log N}{\log\gamma}+NH\left(\frac{\log N}{6\log\gamma}\right)+\mathcal{O}(1).$$

Again it is possible to prove that $H(x)$ is continuous (see [1]). Hence (7) follows. □

3 Representations of $S_{q,i}(N)$.

Let $q>1$ be an integer. Then it is clear that the Fibonacci numbers F_n mod q are periodic with minimal period s. We will first identify this period.

Lemma 6 *Let $l = \min\{n > 0 \,:\, q|F_n\}$. Then*

$$s = \begin{cases} l & \text{if } F_{l-1} \equiv 1 \bmod q, \\ 2l & \text{if } F_{l-1} \not\equiv 1 \bmod q \text{ and } l \text{ is even}, \\ 4l & \text{if } F_{l-1} \not\equiv 1 \bmod q \text{ and } l \text{ is odd}. \end{cases}$$

Especially, if $q = F_k$, $k > 2$, then $l = k$.

Proof. First let us show that the minimal period s of $F_k \bmod q$ is multiple of $l = \min\{n > 0 \,:\, q|F_n\}$. Since $F_0 = 0$ we surely have $F_s \equiv 0 \bmod q$ which implies $\gcd(F_s, F_l) = F_{\gcd(s,l)} \equiv 0 \bmod q$. Thus $\gcd(s,l) \ge l$ and consequently s is a multiple of l. Note that $F_{kl+j} \equiv F_{l+1}^k F_j \equiv F_{l-1}^k F_j \bmod q$.

If $F_{l+1} \equiv F_{l-1} \equiv 1 \bmod q$ then we surely have $s = l$. Next, if $F_{l-1} \not\equiv 1 \bmod q$ and l is even then $F_{l-1}^2 = (-1)^l + F_{l-2}F_l \equiv 1 \bmod q$. Hence $F_{l+1} \not\equiv 1 \bmod q$ and $F_{2l+1} \equiv 1 \bmod q$. This means that $s = 2l$. Similarly, if $F_{l-1} \not\equiv 1 \bmod q$ and l is odd then $F_{l+1} \not\equiv 1 \bmod q$, $F_{2l+1} \not\equiv 1 \bmod q$, $F_{3l+1} \not\equiv 1 \bmod q$, and $F_{4l+1} \equiv 1 \bmod q$. Thus $s = 4l$. $\square$

Next set

$$G_N(x,y) = \sum_{n<N} y^{s(n)} x^n \tag{34}$$

For $x, y \in \mathbf{C}$. Especially we will need $G_N(x,y)$ for $y = -1$ and for $x = \varepsilon$, where ε is a q-th root of unity.

Lemma 7 *Suppose that $N = F_k + n$, where $n \le F_{k-1}$. Then*

$$G_N(x,y) = G_{F_k}(x,y) + y x^{F_k} G_n(x,y). \tag{35}$$

Furthermore, if $q > 1$ and ε is a primitive q-th root of unity then

$$S_{q,i}(N) = \frac{1}{q}\sum_{j=0}^{q-1} \varepsilon^{-ij} G_N(\varepsilon^j, -1). \tag{36}$$

Proof. Obvious. $\square$

Note that Lemma 7 also implies that

$$S_{q,i}(F_k + n) = S_{q,i}(F_k) - S_{q,i-F_k}(n) \tag{37}$$

if $n \le F_{k-1}$.

Thus, in order to obtain an expression for $S_{q,i}(N)$ it suffices to evaluate $G_{F_k}(\varepsilon, -1)$, where ε is a q-th root of unity. By (35) they satisfy the recurrence

$$G_{F_k}(\varepsilon, -1) = G_{F_{k-1}}(\varepsilon, -1) - \varepsilon^{F_{k-1}} G_{F_{k-2}}(\varepsilon, -1). \tag{38}$$

Note that the factor $\varepsilon^{F_{k-1}}$ is periodic with period s. Hence the recurrence of $G_{F_k}(\varepsilon, -1)$ can be explicitly solved.

Proposition 5 *Suppose that $q > 1$ and that s is the minimal period of $F_n \bmod q$. Let ε be a q-th root of unity and set*

$$A_0 = 1,\ A_1 = 1 - \varepsilon^{F_1},\ A_k = A_{k-1} - \varepsilon^{F_k} A_{k-2} \quad (k \ge 2) \tag{39}$$

$$B_0 = 1,\ B_1 = 1,\ B_k = B_{k-1} - \varepsilon^{F_k} B_{k-2} \quad (k \ge 2) \tag{40}$$

$$\tag{41}$$

and

$$C = A_{s-1} - B_{s-2}. \tag{42}$$

Then for any i, $0 \le i < s$, the sequence $G_{F_{ls+i}}(\varepsilon, -1)$, $l \ge 0$, satisfies the recurrence

$$G_{F_{ls+i}}(\varepsilon, -1) = C\, G_{F_{(l-1)s+i}}(\varepsilon, -1) - G_{F_{(l-2)s+i}}(\varepsilon, -1). \tag{43}$$

Proof. Let $\mathbf{G}_k = \left(G_{F_{ks+1}}(\varepsilon,-1), G_{F_{ks+2}}(\varepsilon,-1), \ldots, G_{F_{ks+s}}(\varepsilon,-1)\right)^t$. Then (38) implies that $\mathbf{G_k}$ satisfies the recurrence

$$\mathbf{G}_k = \mathbf{A}\mathbf{G}_k + \mathbf{B}\mathbf{G}_{k-1} \qquad \text{resp.} \qquad \mathbf{G}_k = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{G}_{k-1},$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 1 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ -\varepsilon^{F_2} & 1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & -\varepsilon^{F_3} & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & -\varepsilon^{F_{s-1}} & 1 & 0 \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & \cdot & \cdot & 0 & -\varepsilon^{F_0} & 1 \\ 0 & 0 & \cdot & \cdot & 0 & 0 & -\varepsilon^{F_1} \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

Since $\mathbf{B}$ has rank 2 the eigenvalue $\lambda_0 = 0$ of $(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$ has multiplicity $s-2$. In order to calculate the remaining two eigenvalues we just have to solve the equation

$$\det\left(\mathbf{I} - \mathbf{A} - \lambda^{-1}\mathbf{B}\right) = 0.$$

Using the abbreviation $p = \lambda^{-1}$ this may be written down as

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & p & -p \\ -1 & 1 & 0 & 0 & \cdot & \cdot & 0 & p\varepsilon^{F_1} \\ \varepsilon^{F_2} & -1 & 1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & \varepsilon^{F_3} & -1 & 1 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \varepsilon^{F_{s-2}} & -1 & 1 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & \varepsilon^{F_{s-1}} & -1 & 1 \end{vmatrix} = 0. \tag{44}$$

By elementary operations on the last two colums it easily follows by induction that (44) is equivalent to the following equation:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ \varepsilon^{F_2} & -1 & 1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & \varepsilon^{F_3} & -1 & 1 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \varepsilon^{F_{s-3}} & -1 & 1 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \varepsilon^{F_{s-2}} & -1 & a & b \\ 0 & 0 & \cdot & \cdot & 0 & \varepsilon^{F_{s-1}} & c & d \end{vmatrix} = 0, \tag{45}$$

where

$$\begin{aligned} a &= 1 + pB_{s-2}, \\ b &= -pA_{s-2}, \\ c &= -1 - p\varepsilon^{F_{s-1}}B_{s-3}, \\ d &= 1 + p\varepsilon^{F_{s-1}}A_{s-3}. \end{aligned}$$

The first step is to add $-p$ times the first column to the $(s-1)$-st column and p times the first column to the s-th column:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdot & \cdot & pB_1 & -pA_1 \\ \varepsilon^{F_2} & -1 & 1 & 0 & \cdot & \cdot & -p\varepsilon^{F_2}B_0 & p\varepsilon^{F_2}A_0 \\ 0 & \varepsilon^{F_3} & -1 & 1 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \varepsilon^{F_{s-2}} & -1 & 1 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & \varepsilon^{F_{s-1}} & -1 & 1 \end{vmatrix} = 0.$$

Now add $-pB_1$ times the second column to the $(s-1)$-st column and pA_1 times the second column to the s-th column and so on. After j steps you obtain

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \varepsilon^{F_2} & -1 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \varepsilon^{F_3} & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & pB_j & -pA_j \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -p\varepsilon^{F_{j+1}}B_{j-1} & p\varepsilon^{F_{j+1}}A_{j-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \varepsilon^{F_{j+2}} & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \varepsilon^{F_{s-2}} & -1 & 1 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & 0 & \varepsilon^{F_{s-1}} & -1 & 1 \end{vmatrix} = 0.$$

which finally gives (45).

Hence

$$\begin{aligned} \det(\mathbf{I}-\mathbf{A}-p\mathbf{B}) &= (1+pB_{s-2})(1+p\varepsilon^{F_{s-1}}A_{s-3}) - pA_{s-2}(1+p\varepsilon^{F_{s-1}}B_{s-3}) \\ &= \varepsilon^{F_{s-1}}(B_{s-2}A_{s-3}-A_{s-2}B_{s-3})p^2 + (B_{s-2}+\varepsilon^{F_{s-1}}A_{s-3}-A_{s-2})p + 1. \end{aligned}$$

The coefficient of p is given by

$$B_{s-2}+\varepsilon^{F_{s-1}}A_{s-3}-A_{s-2} = B_{s-2}-A_{s-1} = -C$$

and that of p^2 by

$$\begin{aligned} \varepsilon^{F_{s-1}}(B_{s-2}A_{s-3}-A_{s-2}B_{s-3}) &= \varepsilon^{F_{s-1}}\left((B_{s-3}-\varepsilon^{F_{s-2}}B_{s-4})A_{s-3} - B_{s-3}(A_{s-3}-\varepsilon^{F_{s-2}}A_{s-4})\right) \\ &= \varepsilon^{F_{s-1}}\varepsilon^{F_{s-2}}(B_{s-3}A_{s-4}-A_{s-3}B_{s-4}) \\ &= \cdots \\ &= \varepsilon^{F_{s-1}}\varepsilon^{F_{s-2}}\cdots\varepsilon^{F_2}(B_1A_0-A_1B_0) \\ &= \varepsilon^{\sum_{i=1}^{s-1}F_i} = \varepsilon^{F_{s+1}-1} = 1. \end{aligned}$$

Therefore the characteristic polynomial of $(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$ is given by

$$\det\left(\lambda\mathbf{I}-(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\right) = \lambda^{s-2}\left(\lambda^2-C\lambda+1\right)$$

which proves Proposition 5 □

Remark. Note that the coefficient C in the recurrence (43) has an interesting combinatorial intepretation:

$$C = \sum_{M\in\mathcal{M}} (-1)^{|M|}\,\varepsilon^{\sum_{k\in M}F_k}, \tag{46}$$

where $\mathcal{M}$ denotes the set of subsets $M = \{k_1, k_2, \ldots, k_L\} \subseteq \{0, 1, \ldots, s-1\}$, $L \geq 0$, such that $k_{j+1} > k_j + 1$, $j \geq 1$ and $\{0, s-1\} \not\subseteq M$.
Proof. It is clear that

$$A_{s-1} = \sum_{M \in \mathcal{M}_1} (-1)^{|M|} \varepsilon^{\sum_{k \in M} F_k},$$

where $\mathcal{M}_1$ denotes the set of subsets $M = \{k_1, k_2, \ldots, k_L\} \subseteq \{1, \ldots, s-1\}$, $L \geq 0$, such that $k_{j+1} > k_j + 1$, $j \geq 1$, and that

$$B_{s-2} = \sum_{M \in \mathcal{M}_2} (-1)^{|M|} \varepsilon^{\sum_{k \in M} F_k},$$

where $\mathcal{M}_2$ denotes the set of subsets $M = \{k_1, k_2, \ldots, k_L\} \subseteq \{2, \ldots, s-2\}$, $L \geq 0$, such that $k_{j+1} > k_j + 1$, $j \geq 1$. Since $C = A_{s-1} - \varepsilon^{F_0} B_{s-2}$ representation (46) directly follow. □

With help of Proposition 5 is it easy to calculate $G_{F_{ls+i}}(\varepsilon, -1)$ explicitly:

$$G_{F_{ls+i}}(\varepsilon, -1) = \frac{G_{F_{s+i}}(\varepsilon, -1) - \lambda_2 G_{F_i}(\varepsilon, -1)}{\lambda_1 - \lambda_2} \lambda_1^l + + \frac{G_{F_{s+i}}(\varepsilon, -1) - \lambda_1 G_{F_i}(\varepsilon, -1)}{\lambda_2 - \lambda_1} \lambda_2^l, \tag{47}$$

where

$$\lambda_{1,2} = \frac{C \pm \sqrt{C^2 - 4}}{2}.$$

It is interesting to see that there are lots of cases, where λ_1 and λ_2 are positive and real.

Lemma 8 *Suppose that $s = 4l$, where $l = min\{n > 0 : q|F_n\}$ then C is real. Furthermore, if $|C| > 2$ then λ_1 and λ_2 are positive and real.*

Proof. If $s = 4l$ then $F_{k+2l} \equiv -F_k \bmod q$. Hence (46) gives

$$\begin{aligned} C &= \sum_{M \in \mathcal{M}} (-1)^{|M|} \varepsilon^{\sum_{k \in M} F_k} \\ &= \sum_{M \in \mathcal{M}} (-1)^{|M|} \varepsilon^{-\sum_{k \in M} F_{k+2l}} \\ &= \sum_{(M+2l) \in \mathcal{M}} (-1)^{|M+2l|} \varepsilon^{-\sum_{k \in (M+2l)} F_k} = \overline{C} \end{aligned}$$

which proves that C is real.

Now assume that $|C| > 2$. Then $\lambda_1 \neq \lambda_2$ are non-zero and real. Let

$$\mathbf{G}'_k = \left(G_{F_{ks+1}}(\varepsilon, -1), G_{F_{ks+2}}(\varepsilon, -1), \ldots, G_{F_{ks+s/2}}(\varepsilon, -1)\right)^t$$

and

$$\mathbf{G}''_k = \left(G_{F_{ks+s/2+1}}(\varepsilon, -1), G_{F_{ks+s/2+2}}(\varepsilon, -1), \ldots, G_{F_{ks+s}}(\varepsilon, -1)\right)^t$$

Then (38) and the relation $F_{k+s/2} \equiv -F_k \bmod q$ implies that $\mathbf{G}'_k$ and $\mathbf{G}''_k$ satisfy the recurrences

$$\mathbf{G}'_k = \mathbf{A}\mathbf{G}'_k + \mathbf{B}\mathbf{G}''_{k-1} \qquad \text{and} \qquad \mathbf{G}''_k = \overline{\mathbf{B}}\mathbf{G}'_k + \overline{\mathbf{A}}\mathbf{G}''_k,$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 1 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ -\varepsilon^{F_2} & 1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & -\varepsilon^{F_3} & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & -\varepsilon^{F_{s/2-1}} & 1 & 0 \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & \cdot & \cdot & 0 & -\varepsilon^{F_0} & 1 \\ 0 & 0 & \cdot & \cdot & 0 & 0 & -\varepsilon^{F_1} \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

Set $\mathbf{D} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$. Then we have

$$\mathbf{G}'_k = \mathbf{D}\overline{\mathbf{D}}\mathbf{G}'_{k-1} \qquad \text{and} \qquad \mathbf{G}''_k = \overline{\mathbf{D}}\mathbf{D}\mathbf{G}''_{k-1}.$$

Since $\mathbf{D}\overline{\mathbf{D}}$ and $\overline{\mathbf{D}}\mathbf{D}$ have rank 2 and we already know the solution of $\mathbf{G}'_k$ and $\mathbf{G}''_k$ the non-zero eigenvalues of $\mathbf{D}\overline{\mathbf{D}}$ resp. of $\overline{\mathbf{D}}\mathbf{D}$ are λ_1 and λ_2. Their eigenspaces are of dimension 1.

Let $\mathbf{x}$ denote an eigenvector of $\mathbf{D}\overline{\mathbf{D}}$ corresponding to λ_1, i.e. $\mathbf{D}\overline{\mathbf{D}}\mathbf{x} = \lambda_1\mathbf{x}$. Obviously $\overline{\mathbf{D}}\mathbf{D}\overline{\mathbf{x}} = \overline{\lambda_1}\overline{\mathbf{x}} = \lambda_1\overline{\mathbf{x}}$, which means that $\overline{\mathbf{x}}$ is an eigenvector of $\overline{\mathbf{D}}\mathbf{D}$ corresponding to λ_1. Furthermore

$$\overline{\mathbf{D}}(\mathbf{D}\overline{\mathbf{D}}\mathbf{x}) = (\overline{\mathbf{D}}\mathbf{D})(\overline{\mathbf{D}}\mathbf{x}) = \overline{\mathbf{D}}(\lambda_1\mathbf{x}) = \lambda_1(\overline{\mathbf{D}}\mathbf{x}).$$

Hence there exists a (complex) number μ such that $\overline{D}\mathbf{x} = \mu\overline{\mathbf{x}}$. Since we also have $\mathbf{D}\overline{\mathbf{x}} = \overline{\mu}\mathbf{x}$ we obtain

$$\mathbf{D}\overline{\mathbf{D}}\mathbf{x} = \mathbf{D}(\overline{\mathbf{D}}\mathbf{x}) = \mathbf{D}(\mu\overline{\mathbf{x}}) = \mu(\mathbf{D}\overline{\mathbf{x}}) = \mu(\overline{\mu}\mathbf{x}) = (\mu\overline{\mu})\mathbf{x}$$

providing $\lambda_1 = \mu\overline{\mu} > 0$. Finally $\lambda_2 = 1/\lambda_1 > 0$, too. □

Proposition 6 *Suppose that ε is a q-th root of units and that C is real and of modulus greater than 2. Then there exists a continuous periodic function $H(x)$ with period 1 such that*

$$G_N(\varepsilon, -1) = N^\alpha H\left(\frac{\log N}{s \log \gamma}\right) + \mathcal{O}(1), \tag{48}$$

where

$$\alpha = \frac{\log \lambda_1}{s \log \gamma}.$$

($\lambda_1 = (C + \sqrt{C^2 - 4})/2 > 1$.)

Proof. If N has the Zeckendorf expansion $N = \sum_{j=0}^J F_{k_j}$, where $F_{k_{j+1}} < F_{k_j} - 1$, $j \ge 0$, then by (35)

$$E_N = \sum_{j=0}^J (-1)^j \varepsilon^{\sum_{i<j} F_{k_i}} G_{F_{k_j}}(\varepsilon, -1).$$

Furthermore, $G_{F_k}(\varepsilon, -1)$ is explicitely given by (47). Hence, by using Lemma 5 it is easy to construct a contiuous function $H_0(y)$, $y > 0$, satisfying

$$G_N(\varepsilon, -1) = H_0(\sqrt{5}N) + \mathcal{O}(1)$$

and

$$H_0(\gamma^s y) = \lambda_1 H_0(y).$$

(Compare with the proof of Proposition 4.) Therefore

$$H(x) = \lambda_1^{-x} H_0\left(\sqrt{5}\gamma^{sx}\right)$$

is continuous and satisfies $H(x+1) = H(x)$ and relation (48). □

Now, Lemma 7 and Proposition 6 immediately imply Theorem 2 if there exists a F_k-th root of unity ε such that $|C| > 2$. In order to prove $|C| > 2$ we will use a Theorem of Kronecker (see [10, p. 46]) saying that a non-zero totally real algebraic integer a is either of the form $a = 2\cos(r\pi)$ for some rational number r or there exists a conjugate a' of a with $|a'| > 2$.

Lemma 9 *Let $q = F_k$, where $k > 3$ is odd and not divisible by 3 (i.e. F_k is odd). Then for $\varepsilon = e^{2\pi i/p} = \left(e^{2\pi i/q}\right)^{q/p}$, where p is a prime divisor of q, we have $C \neq 0$.*

Proof. By (39), (40), and (42) it follows that C can be represented as a polynomial in ε, i.e. if we define

$$A_0(x) = 1,\ A_1(x) = 1 - x^{F_1},\ A_m(x) = A_{m-1}(x) - x^{F_m} A_{m-2}(x) \quad (m \geq 2) \tag{49}$$

$$B_0(x) = 1,\ B_1(x) = 1,\ B_m(x) = B_{m-1}(x) - x^{F_m} B_{m-2}(x) \quad (m \geq 2) \tag{50}$$

and

$$C(x) = A_{4k-1}(x) - B_{4k-2}(x). \tag{51}$$

then $C(x)$ is a polynomial with integer coefficients and $C(\varepsilon) = C$. If $C = C(\varepsilon) = 0$ then $C(x)$ is divisble by $f_p(x) = x^{p-1} + x^{p-2} + \cdots + 1$. Hence, $C(1)$ is divisible by $p = f_p(1)$. However, it follows from (49), (50), and (51) by induction that $C(1) = -1$. Thus, $C \neq 0$. □

Proposition 7 *Let $q = F_k$, where $k > 3$ is odd and not divisible by 3. Then there exists a q-the root of unity ε for which $|C| > 2$.*

Proof. Since $\gcd(F_k, 3) = \gcd(F_k, F_4) = F_{\gcd(k,4)} = 1$ there exists a be a prime $p > 3$ dividing $q = F_k$. Let $C(x)$ be defined by (51). Then $a = C(e^{2\pi i/p})$ is a non-zero algebraic integer. Its conjugates are of the form $C(e^{2\pi i l/p})$ for some $l \in \{1, 2, \ldots, p-1\}$. Hence, by Lemma 8 a is totally real and by Kronecker's theorem there either exists a conjugate $a' = C(e^{2\pi i l'/p})$ of a with $|a'| > 2$ or $a = \pm(e^{2\pi i l''/p} + e^{-2\pi i l''/p})$ for some $l'' \in \{1, 2, \ldots, p-1\}$. In the second case the polynomial

$$C'(x) = C(x) \mp (x^{l''} + x^{p-l''})$$

satisfies $C'(\varepsilon) = 0$. Hence, as in the proof of Lemma 9 we have $p | C'(1)$. However $C'(1) = C(1) \mp 2 \in \{-3, 1\}$ which leads to a contradiction. □

Remark. Note that for $q = F_k$, where $k > 3$ is odd and divisble by 3, we have $C(1) = C(-1) = 2$. So the above method fails to prove that there exists a q-th root of unity ε for which $|C(\varepsilon)| > 2$. However, there is numerical evidence that ε with $|C(\varepsilon)| > 2$ exists, especially $C(\varepsilon)$ seems to very large if $\arg(\varepsilon) \sim 2\pi/\sqrt{5}$. There is also a heuristic reason for this phenomenon. If $\arg(\varepsilon) \sim 2\pi/\sqrt{5}$ then (for small m) $A_m(\varepsilon) \sim 4A_{m-4}(\varepsilon) - A_{m-8}(\varepsilon)$ and $B_m(\varepsilon) \sim 4B_{m-4}(\varepsilon) - B_{m-8}(\varepsilon)$. So it might be expected that $C(\varepsilon)$ is of order $(2+\sqrt{3})^k$.

4 A Zeckendorf Analogon to Newman's Theorem

We now restrict to the case $q = F_4 = 3$. By Lemma 7

$$S_{3,r}(N) = \frac{1}{3}\left(G_N(1,-1) + \omega^{2r} G_N(\omega,-1) + \omega^r G_N(\omega^2,-1)\right),$$

where $\omega = e^{2\pi/3}$. Since $G_N(1,-1) = S(N) = \Theta(\log N)$ it remains to discuss $G_N(\omega,-1)$. We have $s = 8$ and a simple computation shows that

$$C = 8 > 2.$$

Thus Proposition 6 directly implies (10).

In order to complete the proof of Theorem 3 we will show that there exists a constant $c > 0$ such that

$$\Re(G_N(\omega,-1)) \geq cN^{\alpha} \tag{52}$$

for sufficiently large N, where α is given in (11). In fact, (52) implies $S_{3,0}(N) > 0$ for sufficiently large $N \geq N_0$. Furthermore all appearing computations and constants are effective. Therefore it is easy to estimate N_0 explicitly and to observe that $S_{3,0}(N) \geq 0$ ever for $N < N_0$.

The roots of the characteristic equation of

$$G_{F_{8k+i}}(\omega,-1) - 8\,G_{F_{8(k-1)+i}}(\omega,-1) + G_{F_{8(k-2)+i}}(\omega,-1) = 0$$

are

$$\lambda_1 = 4+\sqrt{15} \quad \text{and} \quad \lambda_2 = 4-\sqrt{15}$$

and so

$$G_{F_{8k+i}}(\omega,-1) = A_i\lambda_1^k + B_i\lambda_2^k,$$

where

$$A_i = \frac{G_{F_{8+i}}(\omega,-1) - (4-\sqrt{15})G_{F_i}(\omega,-1)}{2\sqrt{15}}.$$

(We will not take care of B_i since $\lambda_2 < 1$ and so $B_i\lambda_2^k$ has no importance.) Simple computations give

$$\begin{aligned}
A_1 &= \frac{-1+\sqrt{15}}{2\sqrt{15}},\\
A_2 &= \frac{(3+\sqrt{15})-2\omega}{2\sqrt{15}},\\
A_3 &= \frac{(3+\sqrt{15})-(1+\sqrt{15})\omega}{2\sqrt{15}},\\
A_4 &= \frac{(8+2\sqrt{15})+2\omega}{2\sqrt{15}},\\
A_5 &= \frac{(5+\sqrt{15})+(3+\sqrt{15})\omega}{2\sqrt{15}},\\
A_6 &= \frac{(11+3\sqrt{15})+(11+3\sqrt{15})\omega}{2\sqrt{15}},\\
A_7 &= \frac{(13+3\sqrt{15})+(16+4\sqrt{15})\omega}{2\sqrt{15}},\\
A_8 &= \frac{(24+6\sqrt{15})+(16+4\sqrt{15})\omega}{2\sqrt{15}}.
\end{aligned}$$

Note that $A_0 = A_8/\lambda_1$. With help of these expressions it is easy to obtain proper approximations of $S_{3,i}(F_{8k+i})$. For conveniece we will use the abbreviation $\lambda = \lambda_1^{1/8}$ and the representation

$$S_{3,r}(F_{8k+i}) = \frac{c_{ri}}{6\sqrt{15}}\lambda^{8k+i} + \mathcal{O}(1),$$

where the corresponding (approximate) values c_{ri} are listed in Table 1. From this list we see that $c_{0,i} > 0$ and so $S_{3,0}(F_k) \ge cF_k^\alpha$ for sufficiently large k. It remains to consider those cases N where the Zeckendorf representation of N has more than one digit. For the sake of shortness we will not treat all possible cases but we have selected two "critical" ones.

First, suppose that N has the Zeckendorf representation

$$N = F_{8k+5} + F_{8k+3} + F_{8k+1} + \cdots.$$

Then by (37)

$$\begin{aligned}
S_{3,0}(N) &= S_{3,0}(F_{8k+5}) - S_{3,1}(F_{8k+3}) + S_{3,2}(F_{8k+1}) \mp \cdots\\
&\ge \frac{\lambda^{8k+5}}{6\sqrt{15}}\left(2.994 + \frac{7.665}{\lambda^2} - \frac{2.22}{\lambda^4} - 10.52\sum_{j\ge 3}\frac{1}{\lambda^{2j}}\right) - \mathcal{O}(\log N)\\
&\ge 0.052\lambda^{8k+5} - \mathcal{O}(\log N) \ge cN^\alpha.
\end{aligned}$$

i	$c_{0,i}$	$c_{1,i}$	$c_{2,i}$
0	8	2	-10
1	4.4396	-2.2198	-2.2198
2	9.4001	-6.4910	-2.9091
3	8.5882	-7.6656	-0.92252
4	10.5107	-4.1861	-6.3245
5	2.9940	1.3418	-4.3359
6	4.8124	4.8124	-9.6249
7	2.9172	6.3068	-9.2241

Table 1

Similarly, if

$$N = F_{8k+5} + F_{8k+3} + 0 \cdot F_{8k+1} + \cdots$$

then

$$\begin{aligned} S_{3,0}(N) &= S_{3,0}(F_{8k+5}) - S_{3,1}(F_{8k+3}) \pm \cdots \\ &\geq \frac{\lambda^{8k+5}}{6\sqrt{15}} \left(2.994 + \frac{7.665}{\lambda^2} - 10.52 \sum_{j \geq 2} \frac{1}{\lambda^{2j+1}} \right) - \mathcal{O}(\log N) \\ &\geq 0.016\lambda^{8k+5} - \mathcal{O}(\log N) \geq cN\alpha. \end{aligned}$$

As mentioned above all other case can be treated in the same fashion. This completes the proof of Theorem 3.

5 Linear Recurrences and Digit Expansions

Let $d \geq 2$ and $a_1 \geq a_2 \geq \cdots a_d > 0$ be integers and let Q_k, $k \geq 0$, be the solution of the linear recurrence

$$Q_k = a_1 Q_{k-1} + a_2 Q_{k-2} + \cdots + a_d Q_{k-d}, \qquad k \geq d, \tag{53}$$

with initial values $Q_0, Q_1, \ldots, Q_{d-1}$ satisfying

$$Q_0 = 1, \qquad Q_k = a_1 Q_{k-1} + a_2 Q_{k-2} + \cdots + a_k Q_0 + 1, \quad 1 \leq k \leq d-1. \tag{54}$$

The characteristic polynomial $P(x) = x^k - a_1 x^{k-1} - \cdots - a_d$ is irreducible and has a unique root $\gamma \in (a_1, a_1 + 1)$ which is a PV-number (see [12]). Hence $Q_k = C\gamma^k + \mathcal{O}(1)$ for some constant $C > 0$.

Every non-negative integer n has a unique Q-ary digit expansion

$$n = \sum_{k=0}^{L(n)} d_k Q_k,$$

where the digits $d_k = d_k(n) \geq 0$ satisfy

$$(d_k, d_{k-1}, \ldots, d_{k-l+1}) < (a_1, a_2, \ldots, a_l), \qquad l - 1 \leq k \leq L(n),\ 1 \leq l \leq d,$$

in lexicographic order (see [12, 9]). Let

$$s(n) = \sum_{k=0}^{L(n)} d_k(n)$$

denote the correspoding sum-of-digits-function and set

$$G_N(x,y) = \sum_{n<N} y^{s(n)} x^n \tag{55}$$

for $x, y \in \mathbf{C}$ which can be calculated by the following rules.

Lemma 10

- *Suppose that* $d_k \le a_1$. *Then*

$$\begin{aligned} G_{d_k Q_k}(x,y) &= \left(1 + yx^{Q_k} + (yx^{Q_k})^2 + \cdots + (yx^{Q_k})^{d_k-1}\right) G_{Q_k}(x,y) \\ &= \frac{1-(yx^{Q_k})^{d_k}}{1-yx^{Q_k}} G_{Q_k}(x,y). \end{aligned} \tag{56}$$

- *For* $k \ge d$ *we have*

$$\begin{aligned} G_{Q_k}(x,y) &= G_{a_1 Q_{k-1}}(x,y) + y^{a_1} x^{a_1 Q_{k-1}} G_{a_2 Q_{k-2}}(x,y) + \cdots \\ &\quad + y^{a_1+\cdots+a_{d-1}} x^{a_1 Q_{k-1}+\cdots+a_{d-1}Q_{k-d+1}} G_{a_d Q_{k-d}}(x,y). \end{aligned} \tag{57}$$

- *If* N *has the* Q*-ary digit expansion* $N = \sum_{j=1}^{J} d_{k_j} Q_{k_j}$, $k_{j+1} > k_j$, $d_{k_j} > 0$, $j \ge 1$, *then*

$$\begin{aligned} G_N(x,y) &= G_{d_{k_J} Q_{k_J}} + y^{d_{k_J}} x^{d_{k_J} Q_{k_J}} G_{d_{k_{J-1}} Q_{k_{J-1}}}(x,y) + \cdots \\ &\quad + y^{d_{k_J}+d_{k_{J-1}}+\cdots+d_{k_2}} x^{d_{k_J}Q_{k_J}+d_{k_{J-1}}Q_{k_{J-1}}+\cdots+d_{k_2}Q_{k_2}} G_{d_{k_1} Q_{k_1}}(x,y). \end{aligned} \tag{58}$$

Proof. If $t < a_1$ and $n < Q_k$ then $s(tQ_k + n) = t + s(n)$. Thus (56) follows directly:

$$\begin{aligned} G_{d_k Q_k}(x,y) &= \sum_{t=0}^{d_k-1} \sum_{n<Q_k} y^{s(tQ_k+n)} x^{tQ_k+n} \\ &= G_{Q_k}(x,y) \sum_{t=0}^{d_k-1} y^t x^{tQ_k}. \end{aligned}$$

Similarly we obtain (57):

$$\begin{aligned} G_{Q_k}(x,y) &= \sum_{n<a_1Q_{k-1}+\cdots+a_dQ_{k-d}} y^{s(n)} x^n \\ &= \sum_{n<a_1Q_{k-1}} y^{s(n)} x^n + \sum_{n<a_2Q_{k-2}} y^{s(a_1Q_{k-1}+n)} x^{a_1Q_{k-1}+n} + \cdots \\ &\quad + \sum_{n<a_dQ_{k-d}} y^{s(a_1Q_{k-1}+\cdots+a_{d-1}Q_{k-d+1}+n)} x^{a_1Q_{k-1}+\cdots+a_{d-1}Q_{k-d+1}+n} \\ &= G_{a_1 Q_{k-1}}(x,y) + y^{a_1} x^{a_1 Q_{k-1}} G_{a_2 Q_{k-2}}(x,y) + \cdots \\ &\quad + y^{a_1+\cdots+a_{d-1}} x^{a_1 Q_{k-1}+\cdots+a_{d-1}Q_{k-d+1}} G_{a_d Q_{k-d}}(x,y). \end{aligned}$$

The proof of (58) runs along the same lines. □

Let $q \ge 1$ be a fixed integer and let s be the minimal period of Q_k mod q. If $x = \varepsilon$ is a q-th root of unity then (56) and (57) induces system of linear recurrences with constant coefficients for $G_{F_{ks+i}}(\varepsilon, y)$. Therefore there always exists an explicit solution with exponential growth. With help of (58) it is also possible to obtain a proper representation of $G_N(\varepsilon, y)$. Let λ_1

denote a root of largest modulus of the characteristic polynomial of the above system of linear recurrences. Suppose that $|y| \le 1$. Then by continuity $|\lambda_1| < \alpha^s$ if $y \ne 1$ or $\varepsilon \ne 1$.

Let $r \ge 2$ be another fixed integer and let

$$A_{q,i;r,m}(N) = |\{n < N \,:\, n \equiv i \bmod q,\ s(n) \equiv m \bmod r\}|. \tag{59}$$

Obviously we have

$$A_{q,i;r,j}(N) = \frac{1}{qr}\sum_{j=0}^{q-i}\sum_{l=0}^{r}\varepsilon_q^{-ji}\varepsilon_r^{-lm}G_N(\varepsilon_q^j,\varepsilon_r^m), \tag{60}$$

where $\varepsilon_q = e^{2\pi i/q}$ and $\varepsilon_r = e^{2\pi i/r}$. Since $G_N(1,1) = N$ we immediately obtain

$$A_{q,i;r,j}(N) = \frac{N}{qr} + R_{q,i;r,j}(N),$$

where

$$R_{q,i;r,j}(N) = \mathcal{O}(N^{1-\eta})$$

for some $\eta > 0$.

As in the Fibonacci case there are many situations, where $R_{q,i;r,j}(N)$ has a proper fractal representation. If the dominating root $\lambda_{1,\max}$ of $G_{F_k}(\varepsilon_r^m,\varepsilon_q^i)$, $m \ne 0$ or $i \ne 0$, real and and greater than 1 then there exists a continuous periodic function $H_{q,i;r,j}(x)$ with period 1 such that

$$R_{q,i;r,j}(N) = N^{\alpha_{q,i;r,j}}H_{q,i;r,j}\left(\frac{\log N}{s\log\gamma}\right) + \mathcal{O}(N^{\alpha_{q,i;r,j}-\eta}),$$

where

$$\alpha_{q,i;r,j} = \frac{\log\lambda_{1,\max}}{s\log\gamma}.$$

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