The extremal number for (a, b)-paths and other hypergraph trees

Zoltan Füredi*

May 27, 2017

1 Introduction

We review classical and some latest results concerning Turán type hypergraph problems in the range where Razborov's method is not applicable. A sample of our new results (achieved with Jiang, Kostochka, Mubayi, and Verstaëte [8]) is as follows.

An (a, b)-path P of length $2t-1$ consists of $2t-1$ sets of size $r = a+b$ as follows (here a, b, t, and n are positive integers). Take t pairwise disjoint a-element sets $U_1, U_2, \ldots, U_t$ and another t pairwise disjoint b-element sets $V_1, \ldots, V_t$ and define the (hyper)edges of $P_{2t-1}(a, b)$ as the sets of the form $U_i \cup V_i$ and $V_i \cup U_{i+1}$. If the $at + bt$ elements are ordered linearly, then the members of P can be represented as intervals of length r. By adding one more set U_{t+1} together with the hyperedge $V_t \cup U_{t+1}$ we obtain the (a, b)-path of even length, $P_{2t}(a, b)$.

Let $\Psi_t(n, r)$ be the r-graph on n vertices in which there is a fixed vertex subset C of size $t-1$ and all possible edges intersecting C are present. Since $\tau(\Psi) = t-1$, no r-graph F with $\tau(F) \geq t$ is contained in Ψ. Esp., Ψ is P_{2t-1}-free, it yields the lower bound in the following asymptotic (r, t are fixed $n \to \infty$)

$$\mathrm{ex}_r(n, P_{2t-1}(a, b)) = (t-1)\binom{n}{r-1} + o(n^{r-1}).$$

This generalizes the Erdős–Gallai theorem for graphs, i.e., the case $a = b = 1$. We have asympotics for (a, b)-path of even length only if $a > b$, and exact results when $r - 2 > a > b \geq 2$. The other cases are still open.

These are instances of a more general result concerning Turán numbers of (a, b)-blow ups of trees.

2 Paths

2.1 Definitions concerning r-uniform hypergraphs

An r-uniform hypergraph, or simply r-*graph*, is a family of r-element subsets of a finite set. We associate an r-graph F with its edge set and call its vertex set $V(F)$. Usually we take $V(F) = [n]$, where $[n] := \{1, 2, 3, \ldots, n\}$. We also use the notation $F \subseteq \binom{[n]}{r}$. For a hypergraph H, a vertex

*Alfréd Rényi Institute of Mathematics, Budapest, Hungary. E-mail: `z-furedi@illinois.edu`. Research partially supported by grant (no. K116769) from the National Research, Development and Innovation Office NKFIH, by the Simons Foundation Collaboration Grant #317487, and by the European Research Council Advanced Investigators Grant 267195.

subset C of H that intersects all edges of H is called a *vertex cover* of H. With other notations $H \subseteq \left(\binom{[n]}{r} \setminus \binom{[n]\setminus C}{r}\right)$. Let $\tau(H)$ be the minimum size of a vertex cover of H. Let $\Psi_t(n,r)$ be the largest r-graph Ψ on n vertices with $\tau(\Psi) < t$. As r, t are fixed and $n \to \infty$

$$|\Psi_t(n,r)| = \binom{n}{r} - \binom{n-t+1}{r} = (t-1)\binom{n}{r-1} + o(n^{r-1}).$$

Given an r-graph F, let $\mathrm{ex}_r(n,F)$ denote the maximum number of edges in an r-graph on n vertices that does not contain a copy of F.

2.2 Paths in graphs

A fundamental result in extremal graph theory is the Erdős-Gallai Theorem [2], which states

$$\mathrm{ex}_2(n, P_\ell) \leq \frac{1}{2}(\ell-1)n, \tag{1}$$

where P_ℓ is the ℓ*-edge path*. (Warning! This is a non-standard notation.) Equality holds if and only if ℓ divides n and all connected components of G are complete graphs on ℓ vertices. The Turán function $\mathrm{ex}(n,P_\ell)$ was determined for every n and ℓ by Faudree and Schelp [3] and independently by Kopylov [11]. Let $n \equiv r \pmod{\ell}$, $0 \leq r < \ell$. Then $\mathrm{ex}(n,P_\ell) = \frac{1}{2}(\ell-1)n - \frac{1}{2}r(\ell-r)$. They described the extremal graphs which are either

— vertex disjoint unions of $\lfloor n/\ell \rfloor$ complete graphs K_ℓ and a K_r, or

— ℓ is odd, $\ell = 2t-1$, and $r = t$ or $t-1$. Then another extremal graphs with completely different structures can be obtained by taking a vertex disjoint union of m copies of K_ℓ ($0 \leq m < \lfloor n/\ell \rfloor$) and a copy of $\Psi_t(n - m\ell, 2)$.

The variety of extremal graphs makes the solution difficult.

We generalize these theorems for some hypergraph paths.

2.3 Paths in hypergraphs

Paths of length 2. Two r-sets with intersection size b can be considered to a hypergraph path $P_2(a,b)$ of length two, where $a+b=r$, and $1 \leq a, b \leq r-1$. If $H \subset \binom{[n]}{r}$ is $P_2(1,r-1)$-free then every $(r-1)$-set is covered by at most one member of H. The inequality $r|H| = |\partial(H)| \leq \binom{n}{r-1}$ yields the upper bound in the following result

$$\frac{1}{r}\binom{n}{r-1} - O(n^{r-2}) < \mathrm{ex}_r(n, P_2(1,r-1)) \leq \frac{1}{r}\binom{n}{r-1}. \tag{2}$$

Here for any given r equality holds if n is sufficiently large ($n > n_0(r)$) and certain divisibility conditions are satisfied (see, Keevash [10]).

The case $b=1$ was solved asymptotically by Frankl and the general case was handled in [5]. For

$1 \le b \le r-2$ and $n > n_1(r)$ one has

$$\mathrm{ex}_r(n, P_2(a,b)) = \Theta\left(\max\left\{\binom{n-b-1}{a-1}, n^b\right\}\right). \tag{3}$$

Note that the right hand side of (3) is $o(n^{r-1})$.

(a,b)-paths of length 3. In the case $\ell = 3$ an (a,b)-path has three r-sets, two of them are disjoint and they cover the third in a prescribed way. Füredi, and Özkahya [9] showed that given $1 \le a, b < r$, $r = a+b$ and for $n > n_2(r)$ one has

$$\mathrm{ex}_r(n, P_3(a,b)) = \binom{n-1}{r-1}. \tag{4}$$

Longer paths. Our goal is to prove an extension of the Erdős-Gallai Theorem for r-graphs. Here we collect what we can prove about an (a,b)-blow up of P_ℓ. Since the case $\ell = 2$ behaves differently, see (2), (3), and also the case $\ell = 3$ was presented above (4), we only discuss the case $\ell \ge 4$.

Theorem 1. *Suppose that $a+b = r$, $a, b \ge 1$, $r \ge 3$ and suppose that $\ell \in \{2t-1, 2t\}$, $\ell \ge 4$. Suppose that these values are fixed and $n \to \infty$ or $n > n_3(r,t)$. We have $|\Psi_t(n,r)| \le \mathrm{ex}_r(n, P_\ell(a,b))$.*

If ℓ is odd, $\ell = 2t-1 \ge 5$, $a \ne b$, $a, b \ge 2$ then for sufficiently large n we have

$$\mathrm{ex}_r(n, P_{2t-1}(a,b)) = \binom{n}{r} - \binom{n-t+1}{r}.$$

In addition, the only example achieving this bound is $\Psi_t(n,r)$.

If ℓ is odd, $\ell = 2t-1 \ge 5$, then as $n \to \infty$ we have

$$\mathrm{ex}_r(n, P_{2t-1}(1, r-1)) = \mathrm{ex}_r(n, P_{2t-1}(r-1, 1)) = (t-1)\binom{n}{r-1} + o(n^{r-1}).$$

If ℓ is odd, $\ell = 2t-1 \ge 5$, and $a = b = r/2$ then

$$\binom{n}{r} - \binom{n-t+1}{r} \le \mathrm{ex}_r\left(n, P_{2t-1}\left(\frac{r}{2}, \frac{r}{2}\right)\right) \le (t-1)\binom{n}{r-1}. \tag{5}$$

If ℓ is even, $\ell = 2t \ge 4$ and $a > b$ then as $n \to \infty$ we have

$$\mathrm{ex}_r(n, P_{2t}(a,b)) = (t-1)\binom{n}{r-1} + o(n^{r-1}).$$

If ℓ is even, and $a = b = r/2$ then we have

$$(t-1)\binom{n}{r-1} + o(n^{r-1}) \le \binom{n}{r} - \binom{n-t+1}{r} \le \mathrm{ex}_r\left(n, P_{2t}\left(\frac{r}{2}, \frac{r}{2}\right)\right) \le \left(t - \frac{1}{2}\right)\binom{n}{r-1}. \tag{6}$$

If ℓ is even, $\ell = 2t \geq 4$ and $a < b$ then as $n \to \infty$ we have

$$(t-1)\binom{n}{r-1} + o(n^{r-1}) \leq \mathrm{ex}_r(n, P_{2t}(a,b)) \leq t\binom{n}{r-1} + o(n^{r-1}).$$

We *conjecture* that $\Psi_t(n,r)$ gives the correct order of magnitude of the Turán number in all the above cases, although there are larger constructions for some special values. There is only one further case we can prove an asymptotic; in a forthcoming work we show that

$$\binom{n-1}{2} + \lfloor\frac{n-2}{2}\rfloor \leq \mathrm{ex}_3(P_4(1,2)) \leq \binom{n}{2} + Cn$$

holds for some absolute constant C. The triple system giving the lower bound can be defined as

$$\left\{F : 1 \in F \in \binom{[n]}{3}\right\} \cup \{\{2, 2i-1, 2i\} : 1 < i \leq n/2\}.$$

The cases $a \neq b$ are immediate consequences of our main Theorem 3, the proofs of the upper bounds of the cases (5) and (6) concerning $a = b = r/2$ are proved by the method of [4].

3 Trees, our main result

Ajtai, Komlós, Simonovits and Szemerédi [1] claimed a proof of the Erdős-Sós Conjecture, showing that if T is any tree with ℓ edges, where ℓ is large enough, then for all n,

$$\mathrm{ex}_2(n,T) \leq \frac{1}{2}(\ell-1)n.$$

To define hypergraph trees we make the following more general definition:

Definition 2. *Let $r, s, t, a, b > 0$ be integers with $b \leq a < r$, $a + b = r$, and let $G = G(U,V)$ denote a bipartite graph with parts $U = \{u_1, u_2, \ldots, u_s\}$ and $V = \{v_1, v_2, \ldots, v_t\}$. Then the (a,b)-blowup of G, denoted $G(a,b)$ is the r-uniform hypergraph with edge set $\{U_i \cup V_j : u_iv_j \in E(G)\}$ where*

1) $|U_i| = a$ and $|V_j| = b$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$, and

2) for all $i \neq j$, $U_i \cap U_j = \emptyset = V_i \cap V_j$ and for all i, j, $U_i \cap V_j = \emptyset$.

We investigate the problem of determining when the construction $\Psi_t(n,r)$ is asymptotically extremal for (a,b)-blowups of trees. For other instances of hypergraph trees for which the crosscut constructions are asymptotically extremal, see [6, 7, 12]. Let $\mathcal{T}_{s,t}(a,b)$ denote the family of (a,b)-blowups of trees T with parts U and V where $a > b$ and $|U| = s$ and $|V| = t$. A *crosscut leaf* in $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$ is a b-set V_j in the part of size t whose degree in $\mathcal{T}$ is one. Our main result is the following theorem.

Theorem 3. *If $\mathcal{T} \in \mathcal{T}_{s,t}(a,b)$ where $a+b=r$ and $b<a<r$, then*

$$\mathrm{ex}(n,\mathcal{T}) \le (t-1)\binom{n}{r-1} + o(n^{r-1}).$$

This is asymptotically sharp whenever $t \le s$. If $t \le s$, $a < r-1$, and $\mathcal{T}$ has a crosscut leaf, then for large enough n,

$$\mathrm{ex}(n,\mathcal{T}) = \binom{n}{r} - \binom{n-t+1}{r}.$$

In addition, the only example achieving the bound is $\Psi_t(n,r)$.

References

[1] Ajtai, M., J. Komlós, M. Simonovits, and E. Szemerédi, The solution of the Erdős-Sós conjecture for large trees. (Manuscript, in preparation).

[2] Erdős, P., and T. Gallai, On maximal paths and circuits of graphs. Acta Math. Acad. Sci. Hungar. 10 (1959), 337–356.

[3] Faudree, R. J., and R. H. Schelp, Path Ramsey numbers in multicolorings, J. Combin. Theory Ser. B 19 (1975), 150–160.

[4] Frankl, P., Asymptotic solution of a Turán-type problem. Graphs and Combin. 6 (1990), 223–227.

[5] Frankl, P., and Z. Füredi, Forbidding just one intersection, Journal of Combinatorial Theory, Ser. A 39 (1985), 160–176.

[6] Füredi, Z., Linear trees in uniform hypergraphs. European J. Combin. 35 (2014), 264–272.

[7] Füredi, Z., and T. Jiang, Turán numbers of hypergraph trees. To appear in J. Combin. Theory Ser. A, see also arxiv1505.03210v1

[8] Füredi, Z., T. Jiang, A. Kostochka, D. Mubayi, and J. Verstraëte, The extremal number for (a,b)-paths and other hypergraph trees. Manuscript.

[9] Füredi, Z., and L. Özkahya, Unavoidable subhypergraphs: **a**-clusters, J. Combinatorial Theory, Ser. A 118 (2011), 2246–2256.

[10] Keevash, P., The existence of design. arXiv:1401.3665.

[11] Kopylov, G. N., Maximal paths and cycles in a graph, Dokl. Akad. Nauk SSSR 234 (1977), no. 1, 19–21. (English translation: Soviet Math. Dokl. 18 (1977), no. 3, 593–596.)

[12] Kostochka, A., D. Mubayi, and J. Versraëte, Turán problems and shadows II: Trees. J. Combin. Theory Ser. B. 122 (2017), 457–478.