Can categorical classes in $L_{\omega_1,\omega}$ be bounded in size?
Shelah's 80th!

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Historical Background

What does categoricity in power mean?

1. [Mor65] A theory, T, (formalized in the first order predicate calculus) is categorical in power κ if it has exactly one isomorphism type of models of powerκ.
2. [Kei71] A class of models $\boldsymbol{K}$ for a language L is categorical in κ if any two models $\mathcal{A}, \mathcal{B}$ in $\boldsymbol{K}$ are isomorphic.

When do $\aleph_1$-categorical theories (AEC) have a bounded size of models?

In the mid-70's Shelah answered my question as to whether a sentence of $L_{\omega_1,\omega}(Q)$ could be *categorical in the philosophers sense*, have only one model. In different papers he proved in different ways that $\aleph_1$-categorical such sentence has a model in $\aleph_2$.

Two questions: Under what conditions does a sentence of $L_{\omega_1,\omega}$ (with LN $\aleph_0$) that is $\aleph_1$-categorical have models in $\aleph_2$, $2^{\aleph_0}$, or even larger?

More generally, *Grossberg's question* Must an aec categorical in λ with $I(\boldsymbol{K}, \lambda^+) < 2^{\lambda^+}$ have a model in λ^{++}?

We already know the second is independent of ZFC.

One Completely General Result

WGCH(λ): $2^\lambda < 2^{\lambda^+}$

Let $\boldsymbol{K}$ be an abstract elementary class (AEC).

Theorem

[WGCH (λ)] Suppose $\lambda \geq \mathrm{LS}(\boldsymbol{K})$ and $\boldsymbol{K}$ is λ-categorical. If amalgamation fails in λ there are 2^{λ^+} models in $\boldsymbol{K}$ of cardinality $\kappa = \lambda^+$.

Uses $[\hat{\Theta}_{\lambda^+}(S)]$ (weak diamond) for many S.

λ-categoricity plays a fundamental role.

No really specific model theoretic hypothesis but a set-theoretic one!

Definitely not provable in ZFC for AEC (even for $L_{\omega_1,\omega}(Q_1)$ maybe for $L_{\omega_1,\omega}$).

THE counterexample: Φ

$\boldsymbol{K}$ is the models in a vocabulary with two unary relations P, Q and two binary relations E, R which satisfy:
For any model $M \in \boldsymbol{K}$,

1. P and Q partition M.
2. E is an equivalence relation on Q.
3. P and each equivalence class of E is denumerably infinite.
4. R is a relation on $P \times Q$ so that each element of Q codes a subset of P.
5. R induces the independence property on $P \cup Q$.

This class is axiomatized by a sentence Φ in $L_{\omega_1,\omega}(Q_1)$.

Properties of models of Φ

amalgamation, ω-stablilty, and arbitrarily large models FAIL

Under MA Φ is $\aleph_1$-categorical but is not ω-stable, fails amalgamation in $\aleph_0$, and has no models beyond the continuum.

Shelah suggested a variant, axiomatized in $L_{\omega_1,\omega}$ with the same properties in $\aleph_0$. Laskowski showed that sentence had at least $2^{\aleph_0}$ models in $\aleph_1$.

[She87, She83, She],[Bal09, §17]

The $L_{\omega_1,\omega}$ case
Syntax and Semantics

The class of models

$\boldsymbol{K}_T$ is the class of atomic models of the countable first order theory T.

Definition

The atomic class $\boldsymbol{K}_T$ is extendible if there is a pair $M \preceq N$ of countable, atomic models, with $N \neq M$.

Equivalently, $\boldsymbol{K}_T$ is extendible if and only if there is an uncountable, atomic model of T.

We assume throughout that $\boldsymbol{K}_T$ is extendible. We work in the monster model of T, which is usually not atomic.

A complete sentence of $L_{\omega_1,\omega}$ has a representation as an atomic class by Chang's trick: Expanding the language by introducing predicates for countable conjunctions (theory T^*) and making them correct by omitting types.

The $L_{\omega_1,\omega}$-class is the reducts of atomic models of T^*.

ω-stability in Atomic Classes

Definitions

$p \in S_{at}(A)$ if $a \models p$ implies Aa is atomic.
$\boldsymbol{K}$ is ω-stable if for every countable model M, $S_{at}(M)$ is countable.

But, there may be $A \subseteq M$, $p \in S_{at}(A)$ that has no extension to $S_{at}(M)$.
Note also ϕ may be κ-stable in this sense while the associated AEC is not κ-stable (for Galois types) [BK09].

First order absoluteness

Theorem (Morley-Baldwin-Lachlan)

A first order theory T in a countable language is $\aleph_1$ categorical iff

1. T has no 2-cardinal models and
2. T is ω-stable.

1) is arithmetic and 2) is Π^1_1.

Fact

A first order theory T in a countable language whose class of atomic models satisfies 1) and 2) is $\aleph_1$-categorical.

I emphasize Morley because it is his direction:
'$\aleph_1$-categorical implies ω-stable'; that is problematic for $L_{\omega_1,\omega}$.

Getting ω-stability: I

Theorem: Keisler/Shelah

$\boldsymbol{K} = \mathrm{mod}(\psi)$, $\psi \in L_{\omega_1,\omega}$

1. *(Keisler) ZFC* If some uncountable model in $\boldsymbol{K}$ realizes uncountably many types (in a countable fragment) over $\emptyset$ then $\boldsymbol{K}$ has $2^{\aleph_1}$ models in $\aleph_1$.
2. *(Shelah)* $(2^{\aleph_0} < 2^{\aleph_1})$ If $\boldsymbol{K}$ has $< 2^{\aleph_1}$ models of cardinality $\aleph_1$, then $\boldsymbol{K}$ is ω-stable.

Two uses of WCH to prove 2) from 1)

A. WCH implies AP in $\aleph_0$. Thus, if $\boldsymbol{K}$ is not ω-stable there is a countable model M and an uncountable $N \in \boldsymbol{K}$ which realizes uncountably many types over M.
B. By Keisler, $\mathrm{Th}_M(M)$ has $2^{\aleph_1}$ models. From WCH we conclude $\mathrm{Th}(M)$ has $2^{\aleph_1}$ models in $\aleph_1$.

Getting ω-stability: II

Morley's original *first order* proof using Hanf number for omitting types, EM-models, and the Skolem hull gives:

Theorem

If a complete first order theory has arbitrarily large models and is $\aleph_1$-categorical then it is ω-stable.

More generally,

Theorem

An $\aleph_1$-categorical atomic class $\boldsymbol{K}$ that has arbitrarily large models and amalgamation in $\aleph_0$ is ω-stable.

Tradeoff: $\beth_{\omega_1}$ (Morley) for weak CH (Shelah/Keisler).

A new notion of closure

Definition

An atomic tuple **c** is in the pseudo-algebraic closure of the finite, atomic set B ($\mathbf{c} \in \mathrm{pcl}(B)$) if
for every atomic model M such that $B \subseteq M$, and $M\mathbf{c}$ is atomic, $\mathbf{c} \subseteq M$.

When this occurs, and **b** is any enumeration of B and $p(\mathbf{x}, \mathbf{y})$ is the complete type of **cb**, we say that $p(\mathbf{x}, \mathbf{b})$ *is pseudo-algebraic.*

Example

Our notion, pcl of *algebraic* differs from the classical first-order notion of algebraic as the following examples show:

Example

Suppose that an atomic model M consists of two sorts. The U-part is countable, but non-extendible (e.g., U infinite, and has a linear order of type $(Z, <)$ on it. On the other sort, V is an infinite set with no structure (hence arbitrarily large atomic models). Then, an element $x_0 \in U$ is not algebraic over $\emptyset$ in the normal sense but is in $\mathrm{pcl}(\emptyset)$.

Definability of pseudo-algebraic closure

Strong ω-homogeneity of the monster model of T yields:

Fact

The truth of $c \in \mathrm{pcl}(\mathbf{b})$ does not depend on an ambient atomic model.

Further, since a model which is atomic over the empty set is also atomic over any finite subset, moving M to N we have:

Fact

If $\mathbf{c} \notin \mathrm{pcl}(B)$, witnessed by M then for every countable, atomic $N \supset B$, there is a realization $\mathbf{c}'$ of $p(\mathbf{x}, B)$ such that $\mathbf{c}' \not\subseteq N$.

Pseudo-minimal sets

Definition

1. A *possibly incomplete* type q over $\mathbf{b}$ is *pseudominimal* if for any finite, $\mathbf{b}^* \supseteq \mathbf{b}$, $\boldsymbol{a} \models q$, and $\mathbf{c}$ such that $\mathbf{b}^*\mathbf{c}\boldsymbol{a}$ is atomic, if $\mathbf{c} \subset \mathrm{pcl}(\mathbf{b}^*\boldsymbol{a})$, and $\mathbf{c} \notin \mathrm{pcl}(\mathbf{b}^*)$, then $\boldsymbol{a} \in \mathrm{pcl}(\mathbf{b}^*\mathbf{c})$.
2. M is pseudominimal if $x = x$ is pseudominimal in M.

I.e, pcl satisfies exchange (and more); we have a geometry.

'Density'

Definition

$\boldsymbol{K}_T$ satisfies *'density' of pseudominimal types* if for every atomic $\mathbf{e}$ and atomic type $p(\mathbf{e}, \mathbf{x})$ there is a $\mathbf{b}$ with $\mathbf{eb}$ atomic and $q(\mathbf{e}, \mathbf{b}, \mathbf{x})$ extending p such that q is pseudominimal.

So density fails if there is a single type $p(\mathbf{e}, \mathbf{x})$ over which exchange fails.

Method: 'Consistency implies Truth':I

[BL16]
Let ϕ be a τ-sentence in $L_{\omega_1,\omega}(Q)$ such that it is consistent that ϕ has a model.
Let A be the countable ω-model of set theory, containing ϕ, that thinks ϕ has an uncountable model.

Construct B, an uncountable model of set theory, which is an elementary extension of A, such that B is correct about uncountability. Then the model of ϕ in B is actually an uncountable model of ϕ.

Main Theorem

Goal Theorem [BLS16]

If $\boldsymbol{K}_T$ fails 'density of pseudominimal types' then $\boldsymbol{K}_T$ has $2^{\aleph_1}$ models of cardinality $\aleph_1$.

We prove this in two steps

1. Force to construct a model (M, E) of set theory in which a model of T codes model theoretic and combinatorial information sufficient to guarantee the non-isomorphism of its image in the different ultralimits.
2. Apply Skolem ultralimits of the models of set theory from 1) to construct $2^{\aleph_1}$ atomic models of T with cardinality $\aleph_1$ in V.

Getting Models in the continuum

Getting models in $2^{\aleph_0}$: Method

In the novel *White Light* [Ruc80], Rudy Rucker proposes a metaphor for the continuum hypothesis. One can reach $\aleph_1$ by a laborious climb up the side of Mt. ON, pausing at ϵ_0.

Or one can take
Cantor's elevator An instantaneous trip up a shaft at the center of the mountain.

For atomic models we take the slightly slower
Shelah's elevator The elevator is a bit slower but has only countably many floors. After building finitely many rooms at each step we reach an object of cardinality $2^{\aleph_0}$.

Asymptotic similarity

Definition

Fix an L-structure M. A subset of M, indexed by $\{a_\eta : \eta \in 2^\omega\}$, is *asymptotically similar* if, for every k-ary L-formula θ, there is an integer N_θ such that for every $\ell \geq N_\theta$,

$$M \models \theta(a_{\eta_0}, \ldots, a_{\eta_{k-1}}) \leftrightarrow \theta(a_{\tau_0}, \ldots, a_{\tau_{k-1}})$$

whenever $(\eta_0, \ldots, \eta_{k-1})$ and $(\tau_0, \ldots, \tau_{k-1})$ satisfy: $\eta_i \restriction \ell = \tau_i \restriction \ell$ and the $\eta_i \restriction \ell$ are distinct.

Remark

Asymptotic similarity is a type of indiscernibility; the indiscernibility is only formula by formula. Let $M = (2^\omega, \{U_i : i < \omega\}$ where the U_i are independent unary predicates. The entire universe is asymptotically similar, although no two elements have the same 1-type.

Getting models in $2^{\aleph_0}$

Theorem [BL19]

If a countable first order theory T has an atomic pseudominimal model M of cardinality $\aleph_1$ then there is an atomic pseudominimal model N of T which a contains a set of *asymptotically similar* elements with cardinality $2^{\aleph_0}$.

A simple application of the method gives Borel models in the continuum of any theory with trivial definable closure.

$\boldsymbol{K}_{(2^{\aleph_0})^+} \neq \emptyset$ implies ω-stability

Goal Theorem [BLS24]

Theorem

If an atomic class At *is* $\aleph_1$*-categorical and has a model of size* $(2^{\aleph_0})^+$ *then* At *is* ω*-stable.*

Old and new definitions:

Definition

1. *A type* $p \in S_{at}(M)$ splits over $F \subseteq M$ *if there are tuples* $\mathbf{b}, \mathbf{b}' \subseteq M$ *and a formula* $\phi(\mathbf{x}, \mathbf{y})$ *such that* $\mathrm{tp}(\mathbf{b}/F) = \mathrm{tp}(\mathbf{b}'/F)$*, but* $\phi(\mathbf{x}, \mathbf{b}) \wedge \neg\phi(\mathbf{x}, \mathbf{b}') \in p$.
2. *We call* $p \in S_{at}(M)$ constrained *if* p *does not split over some finite* $F \subseteq M$ *and* unconstrained *if* p *splits over every finite subset of* M.
3. $C_M := \{p \in S_{at}(M) : p \text{ is constrained}\}$*, for an atomic model* M*. We say* At *has* **only constrained types** *if* $S_{at}(N) = C_N$ *for every atomic model* N.

Limit types

Definition

For $|N| = \aleph_1$, a type $p \in S_{at}(N)$ is a limit type if the restriction $p \restriction M$ is realized in N for every countable $M \preceq N$.

Trivially, for every N, every type in $S_{at}(N)$ realized in N is a limit type. Since we allow $M = N$ in the definition of a limit type, if M is countable, then the only limit types in $S_{at}(M)$ are those realized in M.

Proof Sketch

Studying constrained and limit types (over models) and investigate them under the assumption of $\aleph_1$-categoricity. From this, we prove the main theorem. These results depend on a major hypothesis, the existence of an uncountable model in which every limit type is constrained.
The construction of this model is another example of 'consistency implies truth'.

Basic Properties

Lemma

1. If M is a countable atomic model and $p \in S_{at}(M)$ then p is realized in an atomic extension of M.
2. For any atomic models $M \preceq N$ and finite $A \subseteq M$, then for any $q \in S_{at}(N)$ that does not split over A, the restriction $q\restriction M$ does not split over A; and any $p \in S_{at}(M)$ that does not split over A has a unique non-splitting extension $q \in S_{at}(N)$.
3. If some atomic N has an unconstrained $p \in S_{at}(N)$, then for every countable $A \subseteq N$, there is a countable $M \preceq N$ with $A \subseteq M$ for which the restriction $p\restriction M$ is unconstrained.
4. At has only constrained types if and only if $S_{at}(M) = C_M$ for every/some countable atomic model M.

'Consistency implies Truth': II

Using the technique of [BLS16]:

KEY Theorem:

If At admits an uncountable, atomic model, then there is some $N \in \mathrm{At}$ with $|N| = \aleph_1$ for which every limit type in $S_{at}(N)$ is constrained.

So if $\aleph_1$ -categorical: limit = constrained on the model in $\aleph_1$.

$(2^{\aleph_0})^+$ is enough

Theorem

If an atomic class At is $\aleph_1$-categorical and has a model of size $(2^{\aleph_0})^+$ then the model M in $\aleph_1$ satisfies $S_{At}(M)$ has only constrained types and so is ω-stable.

Pf. Use the existence of a model in $2^{\aleph_0})^+$ to construct an $\aleph_1$-saturated model in $\aleph_1$. If there is c realizing an unconstrained type,use relative $\aleph_1$-saturation to build an unconstrained limit type. This contra the KEY.

Similarly argue that if there is a unconstrained type over a countable model then there is a model in $\aleph_1$ with an unconstrained limit type. Apply KEY again [BLS24, Theorem 2.4.4]

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