Topological games related to the Δ-sets of reals

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We start with the definitions of several special subsets of the real line $\mathbb{R}$.

(a) A Q-set X is a subset of $\mathbb{R}$ such that each subset of X is F_σ, or, equivalently, each subset of X is G_δ in X.

(b) A λ-set X is a subset of $\mathbb{R}$ such that each countable $A \subset X$ is G_δ in X.

(c) A Δ-set X is a subset of $\mathbb{R}$ such that for every decreasing sequence $\{D_n : n \in \omega\}$ of subsets of X with $\bigcap_{n\in\omega} D_n = \emptyset$, there is a decreasing sequence $\{V_n : n \in \omega\}$ consisting of open subsets of X such that $D_n \subset V_n$ for every n, and again with $\bigcap_{n\in\omega} V_n = \emptyset$.

Status of the existence of Q-sets

- Each Q-set must be a Δ-set.
- Each Δ-set must be a λ-set.

K. Kuratowski constructed uncountable λ-sets in ZFC.
The existence of an uncountable Q-set is one of the fundamental set-theoretical problems. F. Hausdorff (1933) observed that

(1) the cardinality of an uncountable Q-set X has to be strictly smaller than the continuum $\mathfrak{c} = 2^{\aleph_0}$, so assuming the Continuum Hypothesis (CH) there are no uncountable Q-sets.

This result was extended independently by W. Sierpiński (1938) and F. Rothberger (1948) (the existence of a Q-set implies $2^{\aleph_0} = 2^{\aleph_1}$). On the other hand,

(2) Martin's Axiom plus the negation of the Continuum Hypothesis (MA $+\neg$CH) implies that every subset $X \subset \mathbb{R}$ of cardinality less than $\mathfrak{c}$ is a Q-set (Martin-Solovay (1970), M.E. Rudin (1977)).

Status of the existence of Δ-sets

The definition of a Δ-set of reals was given by G.M. Reed and then improved by E. van Douwen in 1977.

(3) No Δ-set X can have cardinality $\mathfrak{c}$ (T. Przymusiński (1977)). Hence, under MA, every subset of $\mathbb{R}$ that is a Δ-set is also a Q-set.

Problem 1

Is it consistent that there exists a Δ-set $X \subset \mathbb{R}$ that is not a Q-set?

R. Knight published a paper (1993) aiming to show that the answer is "Yes". However, up to now no expert can confirm that the proof is correct. Let us state as an open and challenging problem to find an alternative and real proof of this claim.

The following problem also is open.

Problem 2 (Reed, 1980)

Does the existence of a Δ-set imply $2^{\aleph_0} = 2^{\aleph_1}$?

On products of Δ-sets

J. Brendle (2018) proved the consistency of the existence of a Q-set whose square is not a Q-set. R. Carvalho and V. Rodrigues (2024) modified Brendle's argument to prove the consistency of the existence of a Q-set whose square is not a Δ-set. They also proved that the existence of a Δ-set implies the existence of a strong Δ-set, that is, of a Δ-set whose all finite powers are Δ-sets.

2. Examples of using of Q-sets and Δ-sets in general topology

A Moore space is a developable regular topological space. For a long time, topologists were trying to prove the so-called normal Moore space conjecture: every normal Moore space is metrizable. This was inspired by the fact that all known Moore spaces that were not metrizable were also not normal.

A Moore space is a regular topological space X having a sequence of open covers $\{\mathcal{U}_n : n \in \omega\}$ such that for each point $p \in X$ and each open V containing p, there is an $n \in \omega$ such that $\bigcup\{U \in \mathcal{U}_n : p \in U\} \subset V$.

One of the most basic and amazing constructions in general topology is the Nemytskii plane L. It is an example of a Tychonoff non-normal Moore space.

Let L be the closed upper half-plane, L_1 be the x-axis and $L_2 = L \setminus L_1$. The topology on the Nemytskii plane L is generated by the following open base.
Basic open neighborhoods of $p \in L_2$ are the usual Euclidean balls; and if $p = (x, 0) \in L_1$, then we take a tangent ball $U(p, \epsilon) = B((x, \epsilon), \epsilon) \cup \{p\}$.

Subspace $M(X)$ of the Nemytskii plane L

In 1935, F.B. Jones was the first to consider the subspace $M(X)$ of Nemytskii plane L, which is obtained by using only a subset $X \subset \mathbb{R}$ of the x-axis. Observe that $M(X)$ always is a Tychonoff separable first-countable Moore space. Also,

(a) $M(X)$ is normal if and only if X is a Q-set (R.H. Bing, R.W. Heath).

(b) $M(X)$ is countably paracompact if and only if X is a Δ-set (T.C. Przymusiński).

(c) $M(X)$ is pseudonormal and nonmetrizable if X is an uncountable λ-set (F.B. Jones).

3. General topological spaces: Q-set spaces and Δ-spaces

Q-set space

A Hausdorff topological space X is called a *Q-space* if each subset of X is F_σ, or, equivalently, each subset of X is G_δ in X.
Z. Balogh defined a *Q-set space* which requires the Q-space to be not σ-discrete. He gave a beautiful ZFC construction (1998) of a topological space X which is a hereditarily paracompact, perfectly normal Q-set space with $|X| = \mathfrak{c}$.

Δ-space

A Tychonoff space X is called a *Δ-space* if for every decreasing sequence $\{D_n : n \in \omega\}$ of subsets of X with $\bigcap_{n\in\omega} D_n = \emptyset$, there is a decreasing sequence $\{V_n : n \in \omega\}$ consisting of open subsets of X such that $D_n \subset V_n$ for every n, and again with $\bigcap_{n\in\omega} V_n = \emptyset$.

Connections with the function space $C(X)$

Below topological spaces X are assumed to be Tychonoff, which means that real-valued continuous functions on X separate points and closed subsets of X.

By $C_p(X)$ we denote the space of all real-valued continuous functions on a topological space X in the topology of pointwise convergence, i.e. $C_p(X)$ is considered as a subspace of the product of the real lines $\mathbb{R}^X$ equipped with the product topology.

1. A collection of sets $\{U_\gamma : \gamma \in \Gamma\}$ is called an *expansion* of a collection of sets $\{X_\gamma : \gamma \in \Gamma\}$ in X if $X_\gamma \subseteq U_\gamma \subseteq X$ for every index $\gamma \in \Gamma$.
2. A collection of sets $\{U_\gamma : \gamma \in \Gamma\}$ is called *point-finite* if no point belongs to infinitely many U_γ -s.

Characterization theorem for $C_p(X)$

Theorem 3.1 (A.L, J. Kąkol, 2021)

For a Tychonoff space X, the following conditions are equivalent:

(1) $C_p(X)$ is a large subspace of the product $\mathbb{R}^X$, i.e. for every $f \in \mathbb{R}^X$ there is a bounded set $B \subset C_p(X)$ such that $f \in cl_{\mathbb{R}^X}(B)$.

(2) Any countable partition of X admits a point-finite open expansion in X.

(3) Any countable disjoint collection of subsets of X admits a point-finite open expansion in X.

(4) X is a Δ-space.

Theorem 3.2

Let X be a normal topological space. The following conditions are equivalent:

(1) For every $f \in \mathbb{R}^X$ there exists a sequence $S = \{f_n : n \in \omega\} \subset C_p(X)$ such that $f_n \to f$ in $\mathbb{R}^X$.

(2) X is a Q-space.

Some examples of Δ-spaces

- The one-point compactification of an uncountable discrete space provides the simplest example of a Δ-space which is not a Q-space.
- Every scattered Eberlein compact space is a Δ-space.
- If $\mathcal{A}$ is an almost disjoint family of subsets of $\mathbb{N}$, let $\Psi(\mathcal{A})$ denote the corresponding Isbell–Mrówka topological space. The underlying set of $\Psi(\mathcal{A})$ is $\mathbb{N} \bigcup \mathcal{A}$, the points of $\mathbb{N}$ are isolated and a base of neighborhoods of $A \in \mathcal{A}$ is the collection of all sets of the form $\{A\} \cup B$, where $A \setminus B$ is finite. Take X be the one-point compactification of $\Psi(\mathcal{A})$. Then X is a compact Δ-space.

Some topological properties of compact Δ-spaces

- Every compact Δ-space X is scattered. However, the compact ordinal space $[0, \omega_1]$ is not a Δ-space.
- Every compact Δ-space X has countable tightness, i.e. if $x \in cl(A)$ in X then there is a countable $M \subset A$ such that $x \in cl(M)$ in X.
- If X is a compact Δ-space and Y is its continuous image, then Y also is a compact Δ-space.
- If $X = \bigcup_{n\in\omega} X_n$, and every X_n is a compact Δ-space, then X also is a Δ-space.

Ladder system spaces

In my recently published joint paper
A.L, P. Szeptycki, *On Δ-spaces*, Israel J. Math. 2025,
we investigated several set-theoretical constructions which produce locally compact first-countable topological spaces. In particular, we considered ladder system spaces.

Let L be a *ladder system* over a stationary subset of limit ordinals $D \subset \omega_1$. I.e. $L = \{s_\alpha : \alpha \in D\}$, where each s_α is an ω-sequence in α cofinal in α. Traditionally, we denote by X_L the topological space $\omega_1 \times \{0\} \cup D \times \{1\}$, where every point $(\alpha, 0)$ is isolated and for each $\alpha \in D$, a basic neighborhood of $(\alpha, 1)$ consists of $\{(\alpha, 1)\}$ along with a cofinite subset of $s_\alpha \times \{0\}$.
We ask the following natural question: under which conditions on the ladder system L, X_L is a Δ-space?

Remark 3.3

$MA(\omega_1)$ implies that every ladder system space X_L is a normal σ-closed discrete space, hence it is consistent that all X_L are Δ-spaces.

Remark 3.4

In ZFC there is a ladder system L on ω_1 such that the corresponding space X_L is a Δ-space.

However, consistently there are ladder system spaces X_L not Δ-.

Theorem 3.5

In forcing extension of ZFC obtained by adding one Cohen real, there is a ladder system L on ω_1 whose corresponding space X_L is not a Δ-space.

In our paper we posed the following problem.

Question

Does there exist in ZFC a ladder system L on some cardinal κ whose corresponding space X_L is not a Δ-space?

Very recently, this problem was solved, for large κ and μ-bounded ladder systems.

Theorem 3.6 (R. Carvalho, T. Inamdar, A. Rinot, 2025)

For $\kappa = cf(\beth_{\omega+1})$ there are co-boundedly many regular cardinals $\mu < \beth_\omega$ such that $E^\kappa_\mu = \{\delta < \kappa : cf(\delta) = \mu\}$ carries a μ-bounded ladder system L such that X_L is not a Δ-space.

The construction uses Shelah's middle diamonds.

Assuming $\beth_\omega = \aleph_\omega$ the authors produced an example at a much lower cardinality, namely $\kappa = 2^{2^{2^{\aleph_0}}}$, and a ladder system L is moreover ω-bounded.

Whether it is possible to get such an ω-bounded ladder system L for some κ in ZFC, is still unknown.

The scale of certain classes of topological spaces naturally extending the class Δ of Δ-spaces has been introduced and investigated in our recent paper
J. Kąkol, O. Kurka, A. L., *Some classes of topological spaces extending the class of* Δ*-spaces*, Proc. Amer. Math. Soc., 152 (2024), 883–898.

Definition 4.1

Let $\mathcal{P}$ be a family of subsets of a Hausdorff space X with $\emptyset \in \mathcal{P}$. We say that X has the Δ-property with respect to $\mathcal{P}$ if for every countable disjoint sequence $\langle X_n \mid n \in \omega \rangle$ of sets $X_n \in \mathcal{P}$, there is a point-finite expansion $\langle U_n \mid n \in \omega \rangle$ consisting of open sets in X.

We consider the classes of topological spaces which possess the Δ-property with respect to several natural families $\mathcal{P}$.

Definition 4.2

For a Hausdorff space X we say that

(0) $X \in \Delta_0$ if $\mathcal{P} = \mathcal{P}_0$ is the family consisting of $\emptyset$ and all singletons $\{x\}$ for $x \in X$.

(1) $X \in \Delta_1$ if $\mathcal{P} = \mathcal{P}_1$ is the family consisting of all $A \subseteq X$ with $|A| \leq \aleph_0$.

(2) $X \in \Delta_2$ if $\mathcal{P} = \mathcal{P}_2$ is the family consisting of all compact subsets of X.

(3) $X \in \Delta$ if $\mathcal{P} = \mathcal{P}(X)$ is the family consisting of all subsets of X.

The following inclusions hold:

$$\Delta \subsetneq \Delta_2 \subsetneq \Delta_1 = \Delta_0$$

(namely, Δ is a proper subclass of Δ_2).

It was established that the Δ_1-property and Δ_2-property coincide in the class of Čech-complete spaces as well as in the class of spaces of countable character. In particular,

For a set $X \subseteq \mathbb{R}$, $X \in \Delta_1$ iff $X \in \Delta_2$ iff X is a λ-set.

5. Topological games on X from the classes $\Delta_{(i)}$

We introduce new topological games, inspired by the definitions of the classes $\Delta_{(i)}$. We consider topological games of two players, called player I and player II, where player I makes the first move. The result of a play cannot end in a draw. A strategy of a player is a function from a finite initial sequence of a play with the last move taken by the other player to legal moves of the player. A game G is said to be *determined* if either player I or player II has a winning strategy in G, where a strategy for a player is winning if every play of the game in which all moves of the player are determined by the strategy results in a win for the player.

Definition 5.1

For a Hausdorff space X let $\mathcal{P}$ be a family of subsets of X. Player I and Player II play on X a game $G_{\mathcal{P}}$ as follows. Player I starts and on each stage $n \in \omega$ chooses a subset $X_n \in \mathcal{P}$ such that $X_0 = \emptyset$ and every X_n is disjoint with all previously chosen sets $\langle X_i \mid i \leq n-1 \rangle$. Player II responds at stage $n \in \omega$ by choosing an open set $U_n \subseteq X$ such that $X_n \subseteq U_n$ for every $n \in \omega$. So, playing G, the players produce a sequence of pairs $\langle (X_n, U_n) \mid n \in \omega \rangle$. Player II wins if after ω moves the family of open sets $\langle U_n \mid n \in \omega \rangle$ is point-finite; otherwise Player I wins.

Definition 5.2

Denote by $G(\Delta)$ the game $G_{\mathcal{P}}$ where $\mathcal{P}$ is the family of all subsets of X; denote by $G(\Delta_0)$ the game $G_{\mathcal{P}}$ where $\mathcal{P}$ is the family $\{\emptyset\} \cup \{\{x\} \mid x \in X\}$; denote by $G(\Delta_1)$ the game $G_{\mathcal{P}}$ where $\mathcal{P}$ is the family consisting of all countable subsets of X; denote by $G(\Delta_2)$ the game $G_{\mathcal{P}}$ where $\mathcal{P}$ is the family consisting of all compact subsets of X.

The following assertion is straightforward.

Proposition 5.3

Let $X \notin \Delta$ ($X \notin \Delta_i$). Then player I has a winning strategy in the game $G(\Delta)$ ($G(\Delta_i)$, respectively).

Problem 3

- For which subsets $X \subseteq \mathbb{R}$ are any of the games $G(\Delta)$, $G(\Delta_2)$, $G(\Delta_1)$, and $G(\Delta_0)$ determined?
  Is the game $G(\Delta)$ determined, assuming that X is a Δ-set or a Q-set of reals?
- Find characterizations of the classes of topological spaces X for which the games $G(\Delta)$, $G(\Delta_2)$, $G(\Delta_1)$, and $G(\Delta_0)$ are determined.

Problem 4

- For which subsets $X \subseteq \mathbb{R}$ does player II have a winning strategy in any of the games $G(\Delta)$, $G(\Delta_2)$, $G(\Delta_1)$ or $G(\Delta_0)$?
- Find a characterization of the classes of topological spaces X for which player II has a winning strategy in any of the games $G(\Delta)$, $G(\Delta_2)$, $G(\Delta_1)$ or $G(\Delta_0)$.

Theorem 5.4 (jointly with W. Kubis)

Assume that X a first-countable space from the class Δ_0. Then player I does not have a winning strategy in the game $G(\Delta_0)$.
In particular, if X is a λ-set of reals then player I does not have a winning strategy in the game $G(\Delta_0)$.

Problem 5

- Let X be a Δ-set or a Q-set of reals. Is it possible to prove that player I does not have a winning strategy in the game $G(\Delta)$?
- Assume that $(\mathrm{MA} + \neg \mathrm{CH})$ holds. Let $X \subset \mathbb{R}$ be a set of cardinality less than $\mathfrak{c}$. Is it true that player I does not have a winning strategy in the game $G(\Delta)$?

For general topological spaces X we do have sufficient condition, in ZFC.

A compact space X is a scattered Eberlein compact if it can be embedded into σ-product of two-point spaces $\{0, 1\}$.

Theorem 5.5

Let X be a subspace of any topological space Y which can be represented as a countable union of scattered Eberlein compact spaces. Then player II has a winning strategy in the game $G(\Delta)$, hence also in all the games $G(\Delta_2)$, $G(\Delta_1)$, and $G(\Delta_0)$.

Evidently, Theorem 5.5 applies to any countable space X.

However, Theorem 5.5 cannot help if X is an uncountable set of reals.

Thank you!