The cofinality of the strong measure zero ideal

Diego A. Mejía
damejiag@people.kobe-u.ac.jp

Kobe University

The Vienna Oracle of Set Theory
TU Wien

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More about the cofinality and the covering of the ideal of strong measure zero sets

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1. Strong measure zero sets and Yorioka ideals

Strong measure zero sets

Definition 1.1

1. For $\sigma = \langle \sigma_i : i < \omega \rangle \in (2^{<\omega})^\omega$, define $\mathrm{ht}_\sigma : \omega \to \omega$ s.t. $\mathrm{ht}_\sigma(i) := |\sigma_i|$.
2. A set $Z \subseteq 2^\omega$ has strong measure zero (in 2^ω) if
$$\forall f \in \omega^\omega \, \exists \sigma \in (2^{<\omega})^\omega : f \leq \mathrm{ht}_\sigma \text{ and } Z \subseteq \bigcup_{i<\omega} [\sigma_i]$$
where $[s] := \{x \in 2^\omega : s \subseteq x\}$ for $s \in 2^{<\omega}$.
3. $\mathcal{SN}$: the collection of strong measure zero subsets of 2^ω.

Fact 1.2

A set $Z \subseteq 2^\omega$ has strong measure zero iff
$$\forall f \in \omega^\omega \, \exists \sigma \in (2^{<\omega})^\omega : f \leq^* \mathrm{ht}_\sigma \text{ and } Z \subseteq [\sigma]_\infty$$
where $[\sigma]_\infty := \{x \in 2^\omega : x \text{ extends infinitely many } \sigma_i\}$.

Yorioka ideals

Definition 1.3

1. For $x, y \in \omega^\omega$,

$$x \ll y \text{ iff } \forall\, k < \omega\ \exists\, m_k < \omega\ \forall\, i \geq m_k \colon x(i^k) \leq y(i).$$

2. $\omega^{\uparrow\omega} := \{f \in \omega^\omega : f \text{ is increasing}\}$.
3. For $f \in \omega^{\uparrow\omega}$ define the Yorioka ideal

$$\mathcal{I}_f := \{A \subseteq 2^\omega : \exists\, \sigma \in (2^{<\omega})^\omega \colon f \ll \mathrm{ht}_\sigma \text{ and } A \subseteq [\sigma]_\infty\}.$$

Theorem 1.4 (Yorioka 2002)

Each $\mathcal{I}_f$ is a σ-ideal and $\mathcal{SN} = \bigcap_{f \in \omega^{\uparrow\omega}} \mathcal{I}_f$.

Theorem 1.5 (Kamo & Osuga 2008)

We do not get an ideal when replacing $f \ll \mathrm{ht}_\sigma$ by $f \leq^* \mathrm{ht}_\sigma$.

Fact 1.6

1. *If* $f \leq^* g$ *then* $\mathcal{I}_g \subseteq \mathcal{I}_f$.
2. *If* $D \subseteq \omega^{\uparrow\omega}$ *is a dominating family, then* $\mathcal{SN} = \bigcap_{f \in D} \mathcal{I}_f$.

Definition 1.7

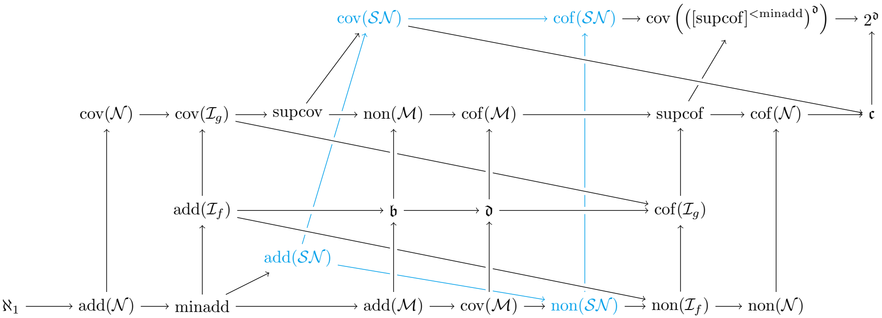
$$\mathsf{minadd} := \min_{f \in \omega^{\uparrow\omega}} \mathsf{add}(\mathcal{I}_f),$$

$$\mathsf{supcov} := \sup_{f \in \omega^{\uparrow\omega}} \mathsf{cov}(\mathcal{I}_f),$$

$$\mathsf{minnon} := \min_{f \in \omega^{\uparrow\omega}} \mathsf{non}(\mathcal{I}_f)$$

$$\mathsf{supcof} := \sup_{f \in \omega^{\uparrow\omega}} \mathsf{cof}(\mathcal{I}_f).$$

Expanded diagram



Yorioka's Characterization Theorem

Theorem 1.8 (Yorioka 2022)

If $\mathrm{minadd} = \mathrm{supcof} = \lambda$ *then* $\mathrm{add}(\mathcal{SN}) = \lambda$ *and* $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\lambda$.

Theorem 1.9 (Yorioka 2002)

ZFC *does not prove any relation between* $\mathrm{cof}(\mathcal{SN})$ *and* $\mathfrak{c}$.

2. The cofinality of $\mathcal{SN}$

Powers of ideals

Definition 2.1 (Power of ideals)

For an ideal $\mathcal{I} \subseteq \mathcal{P}(X)$, $\mathcal{I}^\kappa$ generates an ideal on X^κ. Denote by $\mathrm{add}(\mathcal{I}^\kappa)$, $\mathrm{cov}(\mathcal{I}^\kappa)$, $\mathrm{non}(\mathcal{I}^\kappa)$ and $\mathrm{cof}(\mathcal{I}^\kappa)$ the cardinal characteristics associated with this ideal.

Fact 2.2

$\mathrm{add}(\mathcal{I}^\kappa) = \mathrm{add}(\mathcal{I})$ *and* $\mathrm{non}(\mathcal{I}^\kappa) = \mathrm{non}(\mathcal{I})$, *while* $\mathrm{cov}(\mathcal{I}) \leq \mathrm{cov}(\mathcal{I}^\kappa)$ *and* $\mathrm{cof}(\mathcal{I}) \leq \mathrm{cof}(\mathcal{I}^\kappa)$.

An upper bound of $\mathrm{cof}(\mathcal{SN})$

Theorem 2.3 (Cardona & M. 2025)

$\mathrm{cof}(\mathcal{SN}) \leq \mathrm{cov}\left(([\mathrm{supcof}]^{<\mathrm{minadd}})^{\mathfrak{d}}\right)$.

Fact 2.4

If $\mathrm{minadd} = \mathrm{supcof} = \lambda$ *then* λ *is regular,* $\mathfrak{d} = \lambda$, *and* $\mathrm{cov}\left(([\mathrm{supcof}]^{<\mathrm{minadd}})^{\mathfrak{d}}\right) = \mathfrak{d}_\lambda$.

Proof of Theorem 2.3

Pick $D \subseteq \omega^{\uparrow\omega}$ dominating of size $\mathfrak{d}$.
For each $f \in D$, pick some cofinal family $\mathcal{C}_f := \{A^f_\alpha \colon \alpha < \mathsf{supcof}\}$ in $\mathcal{I}_f$.

Let $\mathcal{F}$ be a witness of cov $\left(([\mathsf{supcof}]^{<\mathsf{minadd}})^{\mathfrak{d}}\right)$.
Each $H \in \mathcal{F}$ can be seen as a function $D \to [\mathsf{supcof}]^{<\mathsf{minadd}}$.
Since $|H(f)| < \mathsf{minadd}$ for all $f \in \omega^{\uparrow\omega}$, $C_H(f) := \bigcup_{\alpha \in H(f)} A^f_\alpha$ is in $\mathcal{I}_f$.
Set $C_H := \bigcap_{f \in D} C_H(f)$.

$\{C_H \colon H \in \mathcal{F}\}$ is cofinal in $\mathcal{SN}$:
For $A \in \mathcal{SN}$ and $f \in D$ choose $\alpha_f < \mathsf{supcof}$ s.t. $A \subseteq A^f_{\alpha_f}$.
There is some $H \in \mathcal{F}$ s.t. $\alpha_f \in H(f)$ for all $f \in D$, hence
$A \subseteq A^f_{\alpha_f} \subseteq C_H(f)$ $(\therefore)$ $A \subseteq C_H$.

Dominating systems

Definition 2.5

Let I be a set, δ an ordinal, and let $\bar{f} = \langle f_\alpha \colon \alpha < \delta \rangle$ be dominating in $\omega^{\uparrow\omega}$. We say that $\langle \bar{A}^\alpha \colon \alpha < \delta \rangle$ is an $\bar{f}$-dominating system in I if, for any $\alpha < \delta$:

1. $\bar{A}^\alpha = \langle A^\alpha_i : i \in I \rangle$ is cofinal in $\mathcal{I}_{f_\alpha}$, and
2. $\forall\, \alpha < \delta\; \forall\, z \in I^\alpha \colon \bigcap_{\xi<\alpha} A^\xi_{z(\xi)} \notin \mathcal{I}_{f_\alpha}$.

The existence of such a system implies $\mathfrak{d} \leq \delta$ and $\mathrm{supcof} \leq |I|$.

Dominating system principle

Lemma 2.6

Let δ be an ordinal, $|I| \geq \text{supcof}$, and $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle \subseteq \omega^{\uparrow\omega}$ dominating. TFAE:

1. *There is an $\bar{f}$-dominating system in I.*
2. *There is some $\bar{A} = \langle A_\alpha : \alpha < \delta \rangle$ s.t. for all $\alpha < \delta$,*
   - S1 $A_\alpha \in \mathcal{I}_{f_\alpha}$ *and*
   - S2 $\bigcap_{\xi<\alpha} A_\xi \notin \mathcal{I}_{f_\alpha}$.

Definition 2.7

A $\mathbf{DS}$-pair of length δ is a pair $(\bar{f}, \bar{A})$ satisfying II.

$\mathbf{DS}(\delta)$: There is some $\mathbf{DS}$-pair of length δ.

Existence of dominating systems

Theorem 2.8 (Cardona & M. 2025)

$\mathrm{cov}(\mathcal{M}) = \mathfrak{d}$ *implies* $\mathbf{DS}(\mathfrak{d})$.

Some properties

Lemma 2.9

$\mathbf{DS}(\delta)$ *implies:*

- a. $\mathbf{DS}(\delta - \beta)$ *for any* $\beta < \delta$.
- b. δ *is a limit ordinal and* $\mathfrak{d} \leq \delta < \mathfrak{c}^+$.
- c. *There is some* $\delta' \leq \delta$ *s.t.* $|\delta'| = \mathfrak{d}$, $\mathrm{cf}(\delta') = \mathrm{cf}(\delta)$ *and* $\mathbf{DS}(\delta')$ *holds.*
- d. $\mathfrak{b} \leq \mathrm{cf}(\delta) \leq \mathfrak{d}$.

Corollary 2.10

If κ *is regular and* $\mathbf{DS}(\kappa)$ *holds, then* $\kappa = \mathfrak{d}$.

Corollary 2.11

If $\exists\, \delta\ \mathbf{DS}(\delta)$ *and* $\mathfrak{b} = \mathfrak{d}$ *then* $\mathbf{DS}(\mathfrak{d})$ *holds.*

Main Lemma

Main Lemma 2.12 (Cardona & M. 2025)

Under $\mathbf{DS}(\delta)$*: Let* $(\bar{f}, \bar{A})$ *be a* $\mathbf{DS}$*-pair of length* δ*.*
If $\langle \mathcal{C}_\alpha : \alpha < \delta \rangle$ *satisfies*

$$\mathcal{C}_\alpha \subseteq \mathcal{I}_{f_\alpha} \text{ and } \sum_{\xi<\alpha} |\mathcal{C}_\xi| < \mathrm{non}(\mathcal{SN}) \text{ for all } \alpha < \delta,$$

then there is some $K \in \mathcal{SN}$ *s.t.* $K \nsubseteq C$ *for all* $C \in \bigcup_{\alpha<\delta} \mathcal{C}_\alpha$ *and*

$$|K| = \sum_{\alpha<\delta} |\mathcal{C}_\alpha| \leq \mathrm{non}(\mathcal{SN}).$$

Applications

Theorem 2.13 (Cardona & M. 2025)

Assume $\mathbf{DS}(\delta)$.

- (a) *If* $\delta \leq \mathrm{non}(\mathcal{SN})$ *then* $\delta < \mathrm{cof}(\mathcal{SN})$.
- (b) *If* $\mathrm{cf}(\mathrm{non}(\mathcal{SN})) = \mathrm{cf}(\delta)$ *then* $\mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$ *and there is some* $K \in \mathcal{SN}$ *of size* $\mathrm{non}(\mathcal{SN})$.

Corollary 2.14

If $\mathfrak{d} \leq \mathrm{cof}(\mathcal{SN})$ *then* $\mathrm{cov}(\mathcal{M}) < \mathrm{cof}(\mathcal{SN})$.

Goldstern & Judah & Shelah 1993

There is a forcing extension satisfying $\mathfrak{c} = \aleph_2$ and $\mathcal{SN} = [2^\omega]^{<\mathfrak{c}}$.
Here, $\mathfrak{d} = \aleph_1$, $\mathrm{add}(\mathcal{SN}) = \mathrm{cof}(\mathcal{SN}) = \aleph_2$ and $\mathbf{DS}(\omega_1)$ holds.

Theorem 2.15 (Cardona & M. 2025)

Under $\mathbf{DS}(\delta)$*: if* $\mu := \mathrm{non}(\mathcal{SN}) = \mathrm{supcof}$ *and* $\kappa := \mathrm{cf}(\delta) = \mathrm{cf}(\mu)$ *then*

$$\mathrm{add}(\mathcal{SN}) \leq \kappa \text{ and } \mathfrak{d}_\kappa \leq \mathrm{cof}(\mathcal{SN}).$$

Moreover, $\mathfrak{d}_\kappa \neq \mu$ *and* $\mu < \mathrm{cof}(\mathcal{SN})$*.*

Corollary 2.16

If $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}$, $\mu := \mathrm{non}(\mathcal{SN}) = \mathrm{supcof}$ *and* $\kappa := \mathrm{cf}(\mathfrak{d}) = \mathrm{cf}(\mu)$ *then* $\mathrm{add}(\mathcal{SN}) \leq \kappa$ *and* $\mathfrak{d}_\kappa \leq \mathrm{cof}(\mathcal{SN})$*.*

As a consequence:

Theorem 1.8 (Yorioka's characterization, 2002)

If $\mathrm{minadd} = \mathrm{supcof} = \lambda$ then $\mathrm{add}(\mathcal{SN}) = \lambda$ and $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\lambda$.

3. Questions

Lower bounds of cofinality

Question 3.1

$\mathfrak{b} \leq \mathrm{cof}(\mathcal{SN})$? $\mathrm{cof}(\mathcal{N}) \leq \mathrm{cof}(\mathcal{SN})$?

Question 3.2 (Yorioka 2002)

$\aleph_1 < \mathrm{cof}(\mathcal{SN})$?

About $\mathbf{DS}(\delta)$

Question 3.3

Is there some δ s.t. $\mathbf{DS}(\delta)$ holds?

Question 3.4

Does $\exists\,\delta\ \mathbf{DS}(\delta)$ imply $\mathbf{DS}(\mathfrak{d})$?

Question 3.5

Assuming $\exists\,\delta\ \mathbf{DS}(\delta)$, let δ_0 be the smallest ordinal satisfying $\mathbf{DS}$.
Do we have $\delta_0 \leq \mathrm{non}(\mathcal{SN})$?

δ_0 is (additively) indecomposable and $|\delta_0| = \mathfrak{d}$.

4. おまけ

Forcing

Theorem 4.1

Assume $\lambda^{\aleph_0} = \lambda$, $0 < \gamma < \lambda^+$ *is indecomposable and* $\mathrm{cf}(\lambda\gamma) > \omega$ *(i.e.* $\gamma = 1$ *or* $\mathrm{cf}(\gamma) > \omega$*).*

Then, any ccc finite support iteration of length $\lambda\gamma$ *forces* $\mathfrak{c} = \lambda$ *and* $\mathbf{DS}(\lambda\beta)$ *for all* $\beta \leq \gamma$ *with* $\mathrm{cf}(\beta) = \mathrm{cf}(\gamma)$*.*

Corollary 4.2

If $\lambda^{\aleph_0} = \lambda$ *then* $\mathbb{C}_\lambda$ *forces* $\mathbf{DS}(\lambda\beta)$ *for all* $\beta < \lambda^+$ *s.t.* $\beta = 1$ *or* $\mathrm{cf}(\beta) > \omega$*.*

In particular $\mathfrak{d}_{\mathrm{cf}(\lambda)} \leq \mathrm{cof}(\mathcal{SN}) \leq \mathrm{cov}\left(([\lambda]^{<\aleph_1})^\lambda\right)$ *and* $\lambda < \mathrm{cof}(\mathcal{SN})$*.*

Fix $\kappa \leq \lambda = \lambda^{\aleph_0}$ with κ regular.

Corollary 4.3

The finite support iteration of amoeba forcing of length $\lambda\kappa$ forces $\text{add}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \kappa$ *and* $\mathfrak{c} = \lambda$.

In particular, $\mathbf{DS}(\kappa)$ *and* $\mathbf{DS}(\lambda\kappa)$ *hold, and* $\text{cof}(\mathcal{SN}) = \mathfrak{d}_\kappa$.

Corollary 4.4

The finite support iteration of Hechler forcing of length $\lambda\kappa$ forces $\text{cov}(\mathcal{SN}) = \aleph_1$, $\text{add}(\mathcal{M}) = \text{cof}(\mathcal{M}) = \kappa$ *and* $\text{non}(\mathcal{SN}) = \mathfrak{c} = \lambda$.

In particular, $\mathbf{DS}(\kappa)$ *and* $\mathbf{DS}(\lambda\kappa)$ *hold.*

If $\text{cf}(\lambda) = \kappa$ *then* $\mathfrak{d}_\kappa \leq \text{cof}(\mathcal{SN}) \leq \text{cov}\left(([\lambda]^{<\aleph_1})^\kappa\right)$ *and* $\lambda < \text{cof}(\mathcal{SN})$.