Abstract Elementary Classes and their axiomatizations: a review

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- Shelah's Presentation Theorem
- Relational Presentation Theorem
- Leung's Axiomatization
- The semantic-syntactic correspondence
- Shelah-Villaveces Axiomatization
- Wath we have

AEC definition

Let τ a language and $\mathcal{K}$ a class of τ-structures. We say that $(\mathcal{K}, \prec_{\mathcal{K}})$ is an Abstract Elementary Class (AEC) if and only if:

1. $\prec_{\mathcal{K}}$ is a partial order over $\mathcal{K}$ and refines the $\subseteq$ relation.
2. $(\mathcal{K}, \prec_{\mathcal{K}})$ is closed under isomorphisms.
3. There is a Cardinal κ, called the Löwenheim-Skolem number of $\mathcal{K}$, such that for all $M \in \mathcal{K}$ and all $A \subseteq |M|$ there is $N \in \mathcal{K}$ such that $A \subseteq |N|$ and $||N|| \geq \kappa$.
4. For all $M_1,\ M_2,\ N \in \mathcal{K}$ such that $M_1, M_2 \prec_{\mathcal{K}} N$ and $M_1 \subseteq M_2$, then $M_1 \prec_{\mathcal{K}} M_2$.
5. For every increasing and continuous $\prec_{\mathcal{K}}$-chain $\langle M_i \rangle_{i<\alpha}$, we have:
    - $M_\alpha := \bigcup_{i<\alpha} M_i \in \mathcal{K}$.
    - For all $i < \alpha$, $M_i \prec_{\mathcal{K}} \bigcup M_\alpha$.
    - If $N \in \mathcal{K}$ is such that $M_i \prec_{\mathcal{K}} N$, then $M_\alpha M_1 \prec_{\mathcal{K}} N$.

Example

- If T is a first order theory, then $(Mod(T), \prec_{\mathcal{K}})$ is an AEC with $LS(\mathcal{K}) = \aleph_0$.
- If $\psi \in \mathbb{L}_{\omega_1,\omega}$ and $\Delta \subseteq \mathbb{L}_{\omega_1,\omega}$ a countable fragment that contains ψ, then $(Mod(\psi), \prec_{\Delta})$ is an AEC with $LS(\mathcal{K}) = \aleph_0$.
- (Mazari-Armida 23)Classes of abelian groups and modules using the pure subgroup or module relation.

Definition

Let $f : \mathcal{M} \longrightarrow \mathcal{N}$ be an embedding, f is a $\mathcal{K}$-embedding if $f[\mathcal{M}] \prec_{\mathcal{K}} \mathcal{N}$

Topics

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Shelah's Presentation Theorem and the EM-functor

Theorem (Sh:87)

Let $K = (\mathcal{K}, \prec_{\mathcal{K}})$ *be an AEC with Löwenheim-Skolem number* κ *in a vocabulary* τ *such that* $|\tau| \leq \kappa$. *Then, there exists a vocabulary* $\tau' \supset \tau$ *with* $|\tau'| = \kappa$, *a first order* τ'*-theory* T' *and a set* Γ' *of quantifier free* T'*-types such that*

$$\mathcal{K} = \{M' \restriction_\tau : M' \vDash T' \text{ and omits all the types in } \Gamma'\}.$$

Furthermore,

1. *If* $M', N' \vDash T'$ *are such that that omit all the types in* Γ' *and* $M' \subseteq N'$, *then* $M' \restriction_\tau \prec_{\mathcal{K}} N' \restriction_\tau$ *and,*
2. *If* $M, N \in K$ *are such that* $M \prec_{\mathcal{K}} N$, *then there are expansions* M' *of* M *and* N' *of* N *to* τ' *such that* $M', N' \vDash T'$, *omit all the types in* Γ' *and* $M' \subseteq N'$.

$$(\mathcal{K}, \prec_{\mathcal{K}}) = (PC_\tau(T', \Gamma'), \subseteq)$$

Extracting indiscernibles

Definition 11.5. Let Φ be an EM blueprint. Let I, J be a linear orders, let δ be a limit ordinal and let $\langle \bar{a}_j : j \in J \rangle$ be a sequence. We say that $\langle \bar{a}_j : j \in J \rangle$ is (Φ, I)-*strictly indiscernible* if:

(1) J is infinite.
(2) For some α, for all $j \in J$, $\bar{a}_j \in {}^{\alpha}\,\mathrm{EM}_\tau(I, \Phi)$.
(3) There exists a sequence $\langle \bar{a}'_j : j \in J \rangle$ and a sequence of terms $\bar{\rho}$ such that $\bar{a}_j = \bar{\rho}(\bar{a}'_j)$ for all $j \in J$ and $\langle \bar{a}'_j : j \in J \rangle$ is quantifier-free indiscernible in the vocabulary of linear orders inside I.

We call $\langle \bar{a}_j : j \in J \rangle$ (Φ, I)-*strictly indiscernible over* A if $\langle \bar{a}_j \bar{a} : j \in J \rangle$ is (Φ, I)-strictly indiscernible for some (any) enumeration $\bar{a}$ of A.

Theorem 11.7 (Strict indiscernible extraction)**.** Let $\mathbf{K}$ be an AEC with arbitrarily large models and let $\mathrm{LS}(\mathbf{K}) < \theta \leq \lambda$ be cardinals with θ regular. Let $\kappa < \theta$ be a (possibly finite) cardinal. Let $\Phi \in \Upsilon_{\mathrm{LS}(\mathbf{K})}[\mathbf{K}]$ be an EM blueprint for $\mathbf{K}$.

Let $N := \mathrm{EM}_{\tau(\mathbf{K})}(\lambda, \Phi)$. Let $M \in \mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$ be such that $M \leq_{\mathbf{K}} N$. Let $\langle \bar{a}_i : i < \theta \rangle$ be a sequence of distinct elements such that for all $i < \theta$, $\bar{a}_i \in {}^{\kappa}|N|$.

If $\theta_0^{\kappa} < \theta$ for all $\theta_0 < \theta$, then there exists $w \subseteq \theta$ with $|w| = \theta$ such that $\langle \bar{a}_i : i \in w \rangle$ is (Φ, λ)-strictly indiscernible over M.

Topics

Relational Presentation Theorem

Theorem (BaBo17)

Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC with Löwenheim-Skolem number κ in a vocabulary τ. Then there exists an expansion of τ by predicates of arity κ and a T' τ'-theory in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$ such that

$$\mathcal{K} = \{M' \restriction_\tau : M' \models T'\}.$$

Furthermore

1. *If $M',\ N' \models T'$ are such that $M' \subseteq N'$, then $M' \restriction_\tau \prec_{\mathcal{K}} N' \restriction_\tau$.*
2. *If $M, N \in K$ are such that $M \prec_{\mathcal{K}} N$, then there are expansions M' of M and N' of N to τ' such that $M', N' \models T'$ and $M' \subseteq N'$.*

$$(\mathcal{K}, \prec_{\mathcal{K}}) = (PC_\tau(T'), \subseteq)$$

Theorem (BaBo17)

Let κ be a strongly compact cardinal and let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC with $LS(\mathcal{K}) < \kappa$. If $\mathcal{K}_{[\mu,\kappa)}$ has AP, JEP..., then $\mathcal{K}_{>\mu}$ has AP, JEP....

Topics

Leung's Axiomatization

Theorem (Le23)

Let $\lambda = \kappa + I_2(\kappa, \mathcal{K})$ *where* $I_2(\kappa, \mathcal{K})$ *is the number of non-isomorphic pairs* (M, N) *such that* $M \prec_{\mathcal{K}} N$ *and bought have cardinality* $LS(\mathcal{K})$*. There is* $\sigma_{\mathcal{K}} \in \mathbb{L}_{\lambda^+, \kappa^+}(\omega \cdot \omega)(\tau)$ *such that*
$(\mathcal{K}, \prec_{\mathcal{K}}) = (\{M \in \tau - \text{structures} | M \vDash \sigma_{\mathcal{K}}\}, \prec_{\Delta})$.

Remark

This is used to simplify some resoults and extend it to other contexts.

Topics

The correspondence

Definition (Galois Morlization, Vas16)

Let κ be an infinite cardinal and let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC. The $(< \kappa)$-Galois Morlization of $(\mathcal{K}, \prec_{\mathcal{K}})$ is $(\hat{\mathcal{K}}, \prec_{\hat{\mathcal{K}}})$, an AEC in a $< \kappa$-ary language $\hat{\tau}$ extending τ such that:

1. $\mathcal{K} = \hat{\mathcal{K}}$ and $\prec_{\mathcal{K}} = \prec_{\hat{\mathcal{K}}}$.
2. For all $p \in ga\text{-}S^{<\kappa}(\emptyset; \mathbb{C})$, there exists $R_p \in \hat{\tau}$ such that $\mathbb{C} \vDash R_p[\bar{b}]$ iff $p = ga\text{-}tp(\bar{a}/\emptyset; \mathbb{C})$.
3. $tp_{\Delta}(\bar{b}/A; \mathbb{C}) := \{\psi(\bar{x}; \bar{a}) \in qf\text{-}\mathbb{L}_{\kappa,\kappa}(\hat{\tau})$- formulas$|\bar{a} \in A$ y $\mathbb{C} \vDash \psi[\bar{b}; \bar{a}]\}$.
4. $qf\text{-}\mathbb{L}_{\kappa,\kappa}\text{-}S^{<\kappa}(A; \mathbb{C}) := \{tp_{\Delta}(\bar{b}/A; \mathbb{C}) | \bar{b} \in |\mathbb{C}|\}$.

Fact (The semantic-syntactic correspondence, Vas16)

$\mathcal{K}$ is $(< \kappa)$-tame iff $ga\text{-}tp(\bar{b}/A; \mathbb{C}) \mapsto tp_{\Delta}(\bar{b}/A; \mathbb{C})$ from $ga\text{-}S^{<\kappa}(A; \mathbb{C})$ to $qf\text{-}\mathbb{L}_{\kappa,\kappa}\text{-}S^{<\kappa}(A; \mathbb{C})$ is a bijection.

Order Property

Definition

We say that $\mathcal{K}$ has the $(\kappa_1, \kappa_2, \theta)$-order property of length μ if there are $A \subseteq |\mathbb{C}|$ with $|A| \leq \theta$, $\langle \overline{a}_i | i < \mu \rangle$ where $\overline{a}_i \in^{\kappa_1} |\mathbb{C}|$ and $\langle \overline{b}_i | i < \mu \rangle$ where $\overline{b}_i \in^{\kappa_2} |\mathbb{C}|$ such that if $i_0 < j_0 < \mu$, $i_1 < j_1 < \mu$, then $ga\text{-}tp(\overline{a}_{i_0} \overline{b}_{j_0} / A; \mathbb{C}) \neq ga\text{-}tp(\overline{a}_{j_1} \overline{b}_{i_1} / A; \mathbb{C})$.

Fact

If $\mathcal{K}$ has AP and is κ-tame, then $\mathcal{K}$ is λ-stable iff does not have the $(\kappa_1, \kappa_2, \lambda)$-order property.

Independence property

Definition

Let λ be a cardinal.

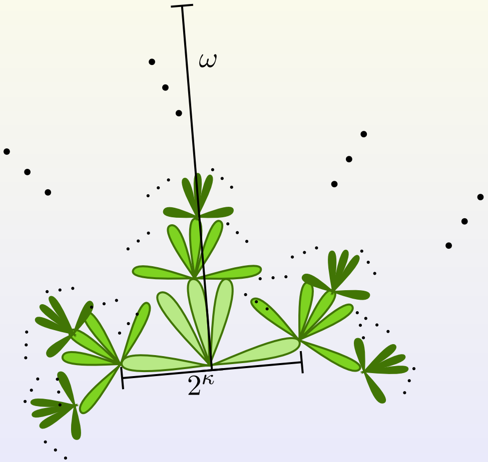
$$ded(\lambda) = \sup\{\kappa : \text{there is a linear order of size } \kappa \text{ which has a dense subset of size } \lambda\}.$$

Fact

Let $\mu := \beth_{(2^{LS(\mathcal{K})})^+}$. *If* $\mathcal{K}$ *is* $< \aleph_0$*-tame,* $C \subseteq |\mathbb{C}|$ *with* $|C| = \lambda > \beth_3(LS(\mathcal{K}))$ *and* $|ga\text{-}S^1(C;\mathbb{C})| > Ded(\lambda)$, *y* $Ded(\lambda)^{2^\kappa} = Ded(\lambda)$, *then there are* $\psi(\overline{x}, \overline{y}) \in qf - \mathbb{L}_{\kappa,\kappa}(\hat{\tau})$, $\langle \overline{a_i} \in |\mathbb{C}| : i < \mu \rangle$ *y* $\langle \overline{b}_w \in |\mathbb{C}| : w \subseteq \mu \rangle$ *such that* $\mathbb{C} \vDash \psi[\overline{a}_i; \overline{b}_w]$ *iff* $i \in w$.

Topics

Canonical tree



The formulas

We define by induction on $\gamma < \lambda^+$ formulas $\varphi_{N,\gamma,n}(\overline{x}_n)$ in $\mathbb{L}_{\lambda^+,\kappa^+}$ for all $n < \omega$ and $N \in \mathcal{S}_n$.

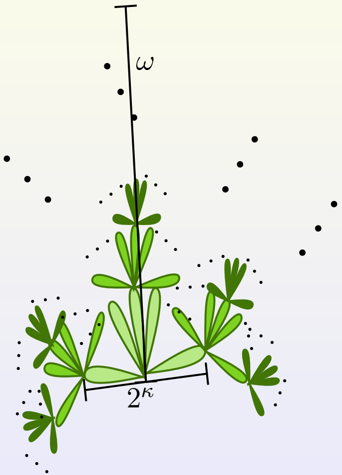
1. If $\gamma = 0$:
   1. If $n = 0$, then $\varphi_{\emptyset,0,0}(\overline{x}_n) := \top$.
   2. For $n > 0$, let $\varphi_{N,0,n}(\overline{x}_n) := \bigwedge Diag^\kappa_n(N)$ where $Diag^\kappa_n(N) := \{\varphi(x_{\alpha_0}, ..., x_{\alpha_{k-1}}) : \alpha_0, ..., \alpha_{k-1} < \kappa \cdot n,\ \varphi(x_{\alpha_0}, ..., x_{\alpha_{k-1}})$ is an atomic or negation of an atomic formula and $N \vDash \varphi(a^*_{\alpha_0}, ..., a^*_{\alpha_{k-1}})\}$.
2. If $\gamma = \beta + 1$, then

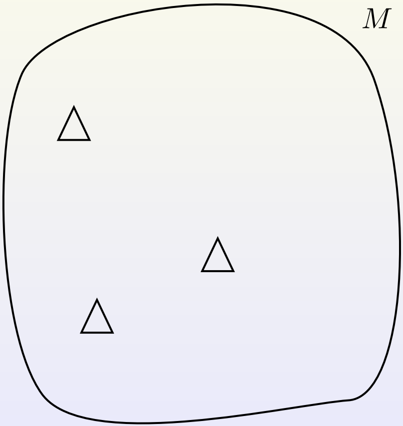
$$\varphi_{N,\gamma,n}(\overline{x}_n) := \forall z_{[\kappa]} \bigvee_{N \prec_{\mathcal{K}} N',\ N' \in \mathcal{S}_{n+1}} \exists \overline{x}_{=n}$$

$$\left[\varphi_{N',\beta,n+1}(\overline{x}_{n+1}) \wedge \bigwedge_{\alpha<\kappa} \bigvee_{\delta < \kappa \cdot (n+1)} z_\alpha = x_\delta \right].$$

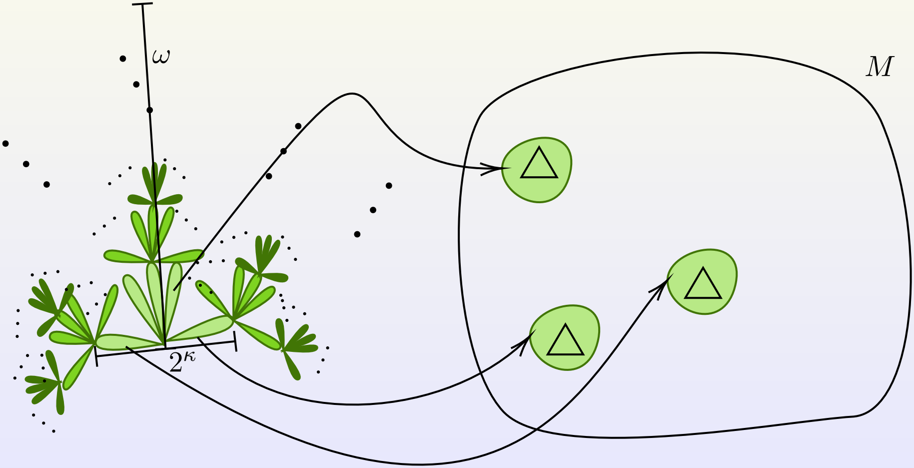
3. If γ is a limit ordinal, then $\varphi_{N,\gamma,n}(\overline{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{N,\beta,n}(\overline{x}_n)$

$M \vDash \varphi_{\emptyset,1,0}$

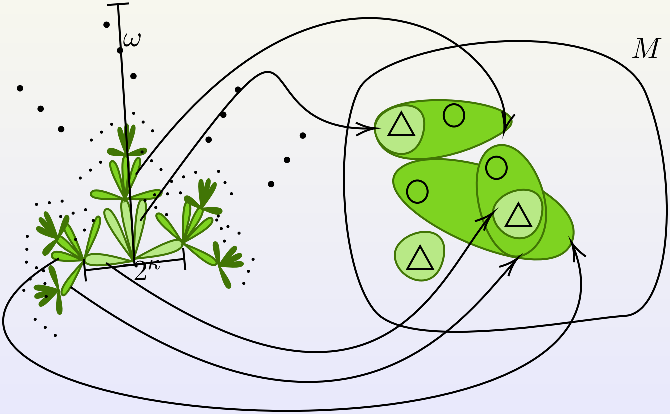




$M \models \varphi_{\emptyset,1,0}$



$M \models \varphi_{\emptyset,2,0}$



The sentence

Fact

Let $\lambda = \beth_2(\kappa)^{++}$. *We have* $\varphi_{\emptyset,\lambda+1,0} \in \mathbb{L}_{\lambda^+,\kappa^+}$.

Theorem (Shelah-Villaveces 2022)

$$M \in \mathcal{K} \text{ if and only if } M \vDash \varphi_{\emptyset,\lambda+1,0}$$

Proof sketch.

Notice that $M \vDash \varphi_{\emptyset,\lambda+1,0}$ if and only if for all $\gamma < \lambda$ and all $A \in [|M|]^\kappa$, $M \vDash \varphi_{N,\gamma,1}[A]$ for some $N \in \mathcal{S}_1$.
Left to right: induction on $\gamma < \lambda$. Use coherence and Löwenheim-Skolem.
Right to left: show that

$$\mathbb{S} := \{M^* \subseteq M| \text{ there are } N \in \mathcal{S}_1 \text{ with enumeration } \langle a^*_\alpha|\alpha < \kappa\rangle \text{ and}$$

$$f : N \cong M^* \text{ such that } M \vDash \varphi_{N,\lambda,1}[\langle f(a^*_\alpha)|\alpha < \kappa\rangle]\}$$

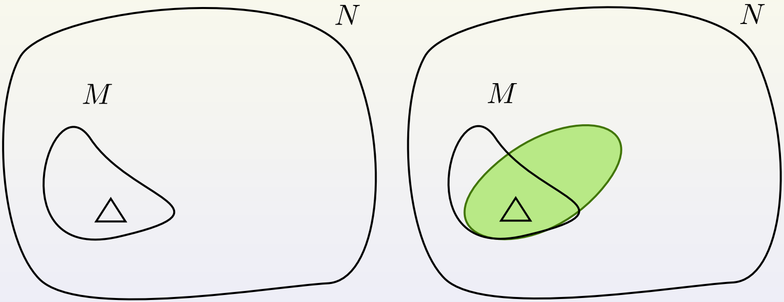
is a directed system. Use a complicate combinatorial principle. □

Syntactic substructure criteria

Theorem (Shelah-Villaveces 2022, N.S. 2023)

1. $M_1 \prec_{\mathcal{K}} M_2$,
2. *if* $\overline{a} \in |M_1|^{\leq \kappa}$ *then there are* $M_{\overline{a}} \prec_{\mathcal{K}} M_1$, $N_{\overline{a}} \in \mathcal{S}_1$ *with enumeration* $\langle a^*_\alpha | \alpha < \kappa \rangle$ *and* $f_{\overline{a}} : N_{\overline{a}} \cong M_{\overline{a}}$ *such that*
   1. $\overline{a} \in |M_{\overline{a}}|^{\leq \kappa}$ *and*
   2. $M_2 \vDash \varphi_{N_{\overline{a}},\lambda,1}[\langle f_{\overline{a}}(a^*_\alpha) | \alpha < \kappa \rangle]$.

Syntactic substructure criteria



$$(\mathcal{K}, \prec_{\mathcal{K}}) = (Mod(\psi_{\mathcal{K}}), \prec_{\Delta})$$

A game to know if $M \in \mathcal{K}$: $G_{AEC}(M)$

Let M be a $\tau(\mathcal{K})$-structure. Remember that $\lambda = \beth_2(\kappa)^{++}$ and $\kappa = LS(\mathcal{K})$. The states of the game are pairs (α, π) where $\alpha < \lambda$ and $\pi : N \longrightarrow M$ is a $\mathcal{K}$-embedding for $N \in \mathcal{S}_n$ and $n < \omega$.

Starting stage: is $(\lambda, \emptyset)$.

Further stages: At stage (α, π):

1. Player I: picks an ordinal $\alpha < \lambda$ and a tuple $\overline{a} \in |M|^\kappa$.
2. Player II: picks $N' \in \mathcal{S}_{n+1}$ and a $\mathcal{K}$-embedding $\pi' : N' \longrightarrow M$ such that $N \prec_{\mathcal{K}} N'$ and

$$\pi' = \pi \cup \{(n_i, m_i) : i \in [\kappa \cdot n, \kappa \cdot (n+1))\}$$

Fact

$M \in \mathcal{K}$ iff player II has a wining strategy in the game $G_{AEC}(M)$.

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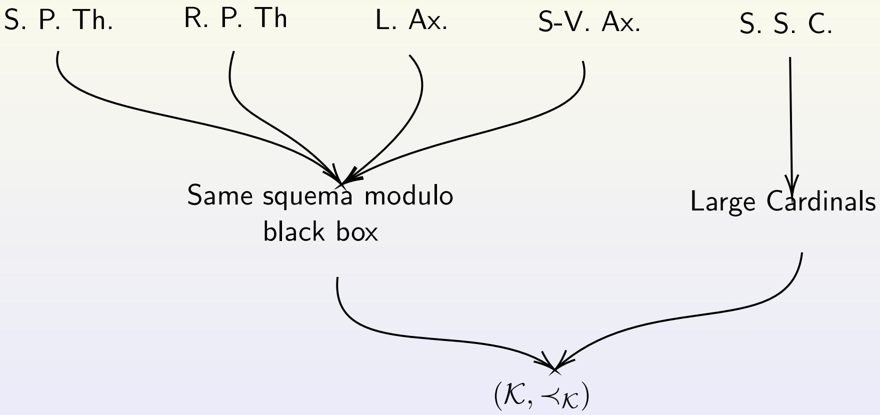
What we have


S. P. Th.
R. P. Th
L. Ax.
S-V. Ax.
S. S. C.
Same squema modulo black box
Large Cardinals
$(\mathcal{K}, \prec_{\mathcal{K}})$

Thank you! :)