A Shelah group in ZFC

Márk Poór

Cornell University

VOrST Workshop
Vienna

July 2025

joint work with Assaf Rinot

The problem

Definition (Jónsson group)

We say that the group G is a ***Jónsson group****, if for each proper subgroup*
$H \lneq G$

The problem

Definition (Jónsson group)

We say that the group G is a ***Jónsson group****, if for each proper subgroup $H \lneq G$ necessarily $|H| < |G|$,*

The problem

Definition (Jónsson group)

We say that the group G is a ***Jónsson group****, if for each proper subgroup $H \lneq G$ necessarily $|H| < |G|$, i.e.*

$$\forall S \in [G]^{|G|} : \quad \langle S \rangle = G.$$

The problem

Definition (Jónsson group)

We say that the group G is a Jónsson group, if for each proper subgroup $H \lneq G$ necessarily $|H| < |G|$, i.e.

$$\forall S \in [G]^{|G|}: \quad \langle S \rangle = G.$$

Examples.

- for each prime p, letting $G_n = \{z \in \mathbb{C} \mid z^{p^n} = 1\}$, then $\langle G_n \mid n \in \omega \rangle$ is an increasing chain of cyclic groups, its union is Jónsson:

The problem

Definition (Jónsson group)

We say that the group G is a *Jónsson group*, *if for each proper subgroup $H \lneq G$ necessarily $|H| < |G|$, i.e.*

$$\forall S \in [G]^{|G|} : \quad \langle S \rangle = G.$$

Examples.

- for each prime p, letting $G_n = \{z \in \mathbb{C} \mid z^{p^n} = 1\}$, then $\langle G_n \mid n \in \omega \rangle$ is an increasing chain of cyclic groups, its union is Jónsson:
  it satisfies that if $g \in G_{n+1} \setminus G_n$, then g generates G_{n+1}.

The problem

Definition (Jónsson group)

We say that the group G is a *Jónsson group*, *if for each proper subgroup $H \lneq G$ necessarily $|H| < |G|$, i.e.*

$$\forall S \in [G]^{|G|} : \quad \langle S \rangle = G.$$

Examples.

- for each prime p, letting $G_n = \{z \in \mathbb{C} \mid z^{p^n} = 1\}$, then $\langle G_n \mid n \in \omega \rangle$ is an increasing chain of cyclic groups, its union is Jónsson:
  it satisfies that if $g \in G_{n+1} \setminus G_n$, then g generates G_{n+1}.
- Ol'šanskiĭ's Tarski monsters

The problem

Definition (Jónsson group)

We say that the group G is a Jónsson group, if for each proper subgroup $H \lneq G$ necessarily $|H| < |G|$, i.e.

$$\forall S \in [G]^{|G|} : \ \langle S \rangle = G.$$

Examples.

- for each prime p, letting $G_n = \{z \in \mathbb{C} \mid \ z^{p^n} = 1\}$, then $\langle G_n \mid \ n \in \omega \rangle$ is an increasing chain of cyclic groups, its union is Jónsson:
  it satisfies that if $g \in G_{n+1} \setminus G_n$, then g generates G_{n+1}.
- Ol'šanskiĭ's Tarski monsters , i.e. (for some fixed prime p) there exists a countable group G for which

$$\forall H \lneq G : \ |H| = p \ \text{(and so } H \simeq \mathbb{Z}/p\mathbb{Z}\text{)},$$

The problem

Definition (Jónsson group)

We say that the group G is a Jónsson group, if for each proper subgroup $H \lneq G$ necessarily $|H| < |G|$, i.e.

$$\forall S \in [G]^{|G|} : \ \langle S \rangle = G.$$

Examples.

- for each prime p, letting $G_n = \{z \in \mathbb{C} \mid \ z^{p^n} = 1\}$, then $\langle G_n \mid \ n \in \omega \rangle$ is an increasing chain of cyclic groups, its union is Jónsson:
  it satisfies that if $g \in G_{n+1} \setminus G_n$, then g generates G_{n+1}.
- Ol'šanskiĭ's Tarski monsters , i.e. (for some fixed prime p) there exists a countable group G for which

  $$\forall H \lneq G : \ |H| = p \ \ (\text{and so } H \simeq \mathbb{Z}/p\mathbb{Z}),$$

  in particular, G is generated by 2 elements, and every $p+1$ elements generate G.

The problem

Definition (Jónsson group)

We say that the group G is a *Jónsson group*, *if for each proper subgroup $H \lneq G$ necessarily $|H| < |G|$, i.e.*

$$\forall S \in [G]^{|G|} : \quad \langle S \rangle = G.$$

Examples.

- for each prime p, letting $G_n = \{z \in \mathbb{C} \mid \ z^{p^n} = 1\}$, then $\langle G_n \mid \ n \in \omega \rangle$ is an increasing chain of cyclic groups, its union is Jónsson:
  it satisfies that if $g \in G_{n+1} \setminus G_n$, then g generates G_{n+1}.
- Ol'šanskiĭ's Tarski monsters , i.e. (for some fixed prime p) there exists a countable group G for which

$$\forall H \lneq G : \ |H| = p \ \text{ (and so } H \simeq \mathbb{Z}/p\mathbb{Z}),$$

  in particular, G is generated by 2 elements, and every $p+1$ elements generate G.

Kurosh monster: The first instance for an uncountable Jónsson group (group theoretic motivations):

Theorem (Shelah, 1978)

There exists a Jónsson group G on $\aleph_1$.

Kurosh monster: The first instance for an uncountable Jónsson group (group theoretic motivations):

Theorem (Shelah, 1978)

There exists a Jónsson group G on $\aleph_1$.

Markov asked in the 1940's whether every infinite group admits a non-trivial Hausdorff topology.

Kurosh monster: The first instance for an uncountable Jónsson group (group theoretic motivations):

Theorem (Shelah, 1978)

There exists a Jónsson group G on $\aleph_1$.

Markov asked in the 1940's whether every infinite group admits a non-trivial Hausdorff topology.
In the above paper, the Jónsson group that Shelah constructed was also a counterexample to Markov's question assuming CH:

Kurosh monster: The first instance for an uncountable Jónsson group (group theoretic motivations):

Theorem (Shelah, 1978)

There exists a Jónsson group G on $\aleph_1$.

Markov asked in the 1940's whether every infinite group admits a non-trivial Hausdorff topology.
In the above paper, the Jónsson group that Shelah constructed was also a counterexample to Markov's question assuming CH: The points:

- There is an n such that whenever $X \subseteq G$ is of full size, every $g \in G$ can be written as a word of length n in X, i.e. $X^n = G$
- There exists a malnormal filtration $G = \bigcup_{\alpha \in \omega_1} G_\alpha$ (an increasing sequence $\langle G_\alpha : \alpha < \omega_1 \rangle$ of countable malnormal subgroups of G).

Kurosh monster: The first instance for an uncountable Jónsson group (group theoretic motivations):

Theorem (Shelah, 1978)

There exists a Jónsson group G on $\aleph_1$.

Markov asked in the 1940's whether every infinite group admits a non-trivial Hausdorff topology.
In the above paper, the Jónsson group that Shelah constructed was also a counterexample to Markov's question assuming CH: The points:

> There is an n such that whenever $X \subseteq G$ is of full size, every $g \in G$ can be written as a word of length n in X, i.e. $X^n = G$
> There exists a malnormal filtration $G = \bigcup_{\alpha \in \omega_1} G_\alpha$ (an increasing sequence $\langle G_\alpha : \alpha < \omega_1 \rangle$ of countable malnormal subgroups of G).

Definition (Boundedly Jónsson groups)

We say that the Jónsson group is ***boundedly Jónsson***, *if for some $n_G \in \omega$*

$$\forall S \in [G]^{|G|} : \quad (S \cup S^{-1})^{\leq n_G} = G.$$

Kurosh monster: The first instance for an uncountable Jónsson group (group theoretic motivations):

Theorem (Shelah, 1978)

There exists a Jónsson group G on $\aleph_1$.

Markov asked in the 1940's whether every infinite group admits a non-trivial Hausdorff topology.
In the above paper, the Jónsson group that Shelah constructed was also a counterexample to Markov's question assuming CH: The points:

- There is an n such that whenever $X \subseteq G$ is of full size, every $g \in G$ can be written as a word of length n in X, i.e. $X^n = G$
- There exists a malnormal filtration $G = \bigcup_{\alpha \in \omega_1} G_\alpha$ (an increasing sequence $\langle G_\alpha : \alpha < \omega_1 \rangle$ of countable malnormal subgroups of G).

Definition (Boundedly Jónsson groups)

We say that the Jónsson group is ***boundedly Jónsson***, *if for some $n_G \in \omega$*

$$\forall S \in [G]^{|G|} : \quad (S \cup S^{-1})^{\leq n_G} = G.$$

(note that being Jónsson is equivalent to $G = (S \cup S^{-1})^{<\omega}$)

Kurosh monster: The first instance for an uncountable Jónsson group (group theoretic motivations):

Theorem (Shelah, 1978)

There exists a Jónsson group G on $\aleph_1$.

Markov asked in the 1940's whether every infinite group admits a non-trivial Hausdorff topology.
In the above paper, the Jónsson group that Shelah constructed was also a counterexample to Markov's question assuming CH: The points:

- There is an n such that whenever $X \subseteq G$ is of full size, every $g \in G$ can be written as a word of length n in X, i.e. $X^n = G$
- There exists a malnormal filtration $G = \bigcup_{\alpha \in \omega_1} G_\alpha$ (an increasing sequence $\langle G_\alpha : \ \alpha < \omega_1 \rangle$ of countable malnormal subgroups of G).

Definition (Boundedly Jónsson groups)

*We say that the Jónsson group is **boundedly Jónsson**, if for some $n_G \in \omega$*

$$\forall S \in [G]^{|G|} : \quad (S \cup S^{-1})^{\leq n_G} = G.$$

(note that being Jónsson is equivalent to $G = (S \cup S^{-1})^{<\omega}$)

Definition

*The subgroup $H \leq G$ is **malnormal** (in symbols, $H \leq_m G$), if*

$$(\forall g \in G \setminus H)\ (\forall h \in H \setminus \{1\}) : \ g^{-1}hg \notin H.$$

Nontopologizable groups

Theorem (Shelah, 1978)

($2^\lambda = \lambda^+$) There exists a boundedly Jónsson group G of size λ^+ with $n_G = 6640$, and even $S \in [G]^{|G|} \implies S^{6640} = G$, moreover, G admits a malnormal filtration.

Nontopologizable groups

Theorem (Shelah, 1978)

($2^\lambda = \lambda^+$) There exists a boundedly Jónsson group G of size λ^+ with $n_G = 6640$, and even $S \in [G]^{|G|} \implies S^{6640} = G$, moreover, G admits a malnormal filtration.

The conjunction of these properties in turn imply that G admits no nondiscrete compatible T_1 topology:

Corollary (Shelah)

($2^\lambda = \lambda^+$ for some λ) There exists a group that does not admit any T_2 (in fact any T_1) group topology other than the discrete topology.

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson,

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson, (for some n, $S \in [G]^{|G|}$ implies $(S \cup S^{-1})^{\leq n} = G$),

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson, (for some n, $S \in [G]^{|G|}$ implies $(S \cup S^{-1})^{\leq n} = G$), and,

ii) admits a malnormal filtration $\langle G_\alpha : \alpha < \kappa \rangle$,

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson, (for some n, $S \in [G]^{|G|}$ implies $(S \cup S^{-1})^{\leq n} = G$), and,

ii) admits a malnormal filtration $\langle G_\alpha : \alpha < \kappa \rangle$,

then G admits no non-discrete T_1 group topology.

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson, (for some n, $S \in [G]^{|G|}$ implies $(S \cup S^{-1})^{\leq n} = G$), and,

ii) admits a malnormal filtration $\langle G_\alpha : \alpha < \kappa \rangle$,

then G admits no non-discrete T_1 group topology.

Proof. Assume on the contrary that τ is a T_1 group topology on G, such that $\{1\} \notin \tau$.

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson, (for some n, $S \in [G]^{|G|}$ implies $(S \cup S^{-1})^{\leq n} = G$), and,

ii) admits a malnormal filtration $\langle G_\alpha : \alpha < \kappa \rangle$,

then G admits no non-discrete T_1 group topology.

Proof. Assume on the contrary that τ is a T_1 group topology on G, such that $\{1\} \notin \tau$.
Fix an open U so that $1 \in U \subsetneq G$,

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson, (for some n, $S \in [G]^{|G|}$ implies $(S \cup S^{-1})^{\leq n} = G$), and,

ii) admits a malnormal filtration $\langle G_\alpha : \alpha < \kappa \rangle$,

then G admits no non-discrete T_1 group topology.

Proof. Assume on the contrary that τ is a T_1 group topology on G, such that $\{1\} \notin \tau$.
Fix an open U so that $1 \in U \subsetneq G$, pick an open $1 \in V = V^{-1} \subseteq U$ with

$$V^{\leq n} = V \cup V^2 \cup \cdots \cup V^n \subseteq U,$$

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson, (for some n, $S \in [G]^{|G|}$ implies $(S \cup S^{-1})^{\leq n} = G$), and,

ii) admits a malnormal filtration $\langle G_\alpha : \alpha < \kappa \rangle$,

then G admits no non-discrete T_1 group topology.

Proof. Assume on the contrary that τ is a T_1 group topology on G, such that $\{1\} \notin \tau$.
Fix an open U so that $1 \in U \subsetneq G$, pick an open $1 \in V = V^{-1} \subseteq U$ with

$$V^{\leq n} = V \cup V^2 \cup \cdots \cup V^n \subseteq U,$$

in particular, $V^{\leq n} \subsetneq G$.

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson, (for some n, $S \in [G]^{|G|}$ implies $(S \cup S^{-1})^{\leq n} = G$), and,

ii) admits a malnormal filtration $\langle G_\alpha : \alpha < \kappa \rangle$,

then G admits no non-discrete T_1 group topology.

Proof. Assume on the contrary that τ is a T_1 group topology on G, such that $\{1\} \notin \tau$.
Fix an open U so that $1 \in U \subsetneq G$, pick an open $1 \in V = V^{-1} \subseteq U$ with

$$V^{\leq n} = V \cup V^2 \cup \cdots \cup V^n \subseteq U,$$

in particular, $V^{\leq n} \subsetneq G$. Now

(1) Case 1: $|V| = |G|$, in which case $V^{\leq n} = G$,

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson, (for some n, $S \in [G]^{|G|}$ implies $(S \cup S^{-1})^{\leq n} = G$), and,

ii) admits a malnormal filtration $\langle G_\alpha : \alpha < \kappa \rangle$,

then G admits no non-discrete T_1 group topology.

Proof. Assume on the contrary that τ is a T_1 group topology on G, such that $\{1\} \notin \tau$.
Fix an open U so that $1 \in U \subsetneq G$, pick an open $1 \in V = V^{-1} \subseteq U$ with

$$V^{\leq n} = V \cup V^2 \cup \cdots \cup V^n \subseteq U,$$

in particular, $V^{\leq n} \subsetneq G$. Now

(1) Case 1: $|V| = |G|$, in which case $V^{\leq n} = G$, a contradiction,

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson, (for some n, $S \in [G]^{|G|}$ implies $(S \cup S^{-1})^{\leq n} = G$), and,

ii) admits a malnormal filtration $\langle G_\alpha : \alpha < \kappa \rangle$,

then G admits no non-discrete T_1 group topology.

Proof. Assume on the contrary that τ is a T_1 group topology on G, such that $\{1\} \notin \tau$.
Fix an open U so that $1 \in U \subsetneq G$, pick an open $1 \in V = V^{-1} \subseteq U$ with

$$V^{\leq n} = V \cup V^2 \cup \cdots \cup V^n \subseteq U,$$

in particular, $V^{\leq n} \subsetneq G$. Now

(1) Case 1: $|V| = |G|$, in which case $V^{\leq n} = G$, a contradiction,

(2) or Case 2: $|V| < |G|$, and then $1 \in V \subseteq G_\alpha \lneq_{\mathrm{m}} G$ for some $\alpha < \kappa$.

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson, (for some n, $S \in [G]^{|G|}$ implies $(S \cup S^{-1})^{\leq n} = G$), and,

ii) admits a malnormal filtration $\langle G_\alpha : \alpha < \kappa \rangle$,

then G admits no non-discrete T_1 group topology.

Proof. Assume on the contrary that τ is a T_1 group topology on G, such that $\{1\} \notin \tau$.
Fix an open U so that $1 \in U \subsetneq G$, pick an open $1 \in V = V^{-1} \subseteq U$ with

$$V^{\leq n} = V \cup V^2 \cup \cdots \cup V^n \subseteq U,$$

in particular, $V^{\leq n} \subsetneq G$. Now

(1) Case 1: $|V| = |G|$, in which case $V^{\leq n} = G$, a contradiction,

(2) or Case 2: $|V| < |G|$, and then $1 \in V \subseteq G_\alpha \lneq_{\mathrm{m}} G$ for some $\alpha < \kappa$. But then for any fixed $g \notin G_\alpha$: $V \cap g^{-1}Vg \subseteq G_\alpha \cap g^{-1}G_\alpha g = \{1\}$ by the malnormality of G_α,

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson, (for some n, $S \in [G]^{|G|}$ implies $(S \cup S^{-1})^{\leq n} = G$), and,

ii) admits a malnormal filtration $\langle G_\alpha : \alpha < \kappa \rangle$,

then G admits no non-discrete T_1 group topology.

Proof. Assume on the contrary that τ is a T_1 group topology on G, such that $\{1\} \notin \tau$.
Fix an open U so that $1 \in U \subsetneq G$, pick an open $1 \in V = V^{-1} \subseteq U$ with

$$V^{\leq n} = V \cup V^2 \cup \cdots \cup V^n \subseteq U,$$

in particular, $V^{\leq n} \subsetneq G$. Now

(1) Case 1: $|V| = |G|$, in which case $V^{\leq n} = G$, a contradiction,

(2) or Case 2: $|V| < |G|$, and then $1 \in V \subseteq G_\alpha \lneq_{\mathrm{m}} G$ for some $\alpha < \kappa$. But then for any fixed $g \notin G_\alpha$: $V \cap g^{-1}Vg \subseteq G_\alpha \cap g^{-1}G_\alpha g = \{1\}$ by the malnormality of G_α, hence

$$\{1\} = V \cap g^{-1}Vg \in \tau,$$

Proof of the corollary

Lemma

If G is a group on a regular κ,

i) boundedly Jónsson, (for some n, $S \in [G]^{|G|}$ implies $(S \cup S^{-1})^{\leq n} = G$), and,

ii) admits a malnormal filtration $\langle G_\alpha : \alpha < \kappa \rangle$,

then G admits no non-discrete T_1 group topology.

Proof. Assume on the contrary that τ is a T_1 group topology on G, such that $\{1\} \notin \tau$.
Fix an open U so that $1 \in U \subsetneq G$, pick an open $1 \in V = V^{-1} \subseteq U$ with

$$V^{\leq n} = V \cup V^2 \cup \cdots \cup V^n \subseteq U,$$

in particular, $V^{\leq n} \subsetneq G$. Now

(1) Case 1: $|V| = |G|$, in which case $V^{\leq n} = G$, a contradiction,

(2) or Case 2: $|V| < |G|$, and then $1 \in V \subseteq G_\alpha \lneq_{\mathrm{m}} G$ for some $\alpha < \kappa$. But then for any fixed $g \notin G_\alpha$: $V \cap g^{-1}Vg \subseteq G_\alpha \cap g^{-1}G_\alpha g = \{1\}$ by the malnormality of G_α, hence

$$\{1\} = V \cap g^{-1}Vg \in \tau$$

a contradiction.

Boundedly Jónsson groups

Shortly after Shelah's non-topologizable group under CH:

Boundedly Jónsson groups

Shortly after Shelah's non-topologizable group under CH:

Theorem (Hesse)

There exists a non-topologizable group (in ZFC*).*

Boundedly Jónsson groups

Shortly after Shelah's non-topologizable group under CH:

Theorem (Hesse)

There exists a non-topologizable group (in ZFC*).*

Theorem (Ol'šanskiĭ)

There exists a countable non-topologizable group.

Boundedly Jónsson groups

Shortly after Shelah's non-topologizable group under CH:

Theorem (Hesse)

There exists a non-topologizable group (in ZFC*).*

Theorem (Ol'šanskiĭ)

There exists a countable non-topologizable group.

Variations.

Definition

The group G admits the Bergman property if

$$\forall S \subseteq G : \ (\langle S \rangle = G) \implies (S \cup S^{-1})^{\leq n} = G \text{ for some } n = n_S \in \omega.$$

Boundedly Jónsson groups

Shortly after Shelah's non-topologizable group under CH:

Theorem (Hesse)

There exists a non-topologizable group (in ZFC).

Theorem (Ol'šanskiĭ)

There exists a countable non-topologizable group.

Variations.

Definition

The group G admits the Bergman property if

$$\forall S \subseteq G : \ (\langle S \rangle = G) \implies (S \cup S^{-1})^{\leq n} = G \text{ for some } n = n_S \in \omega.$$

Remark

Being Jónsson is equivalent to $G = (S \cup S^{-1})^{<\omega}$ for each full-sized S, and boundedly Jónsson is equivalent to $G = (S \cup S^{-1})^{\leq n_G}$, and so

$$\text{Boundedly Jónsson} \implies (\text{Jónsson} + \text{Bergman})$$

Boundedly Jónsson groups

Shortly after Shelah's non-topologizable group under CH:

Theorem (Hesse)

There exists a non-topologizable group (in ZFC).

Theorem (Ol'šanskiĭ)

There exists a countable non-topologizable group.

Variations.

Definition

The group G admits the Bergman property if

$$\forall S \subseteq G : \ (\langle S \rangle = G) \implies (S \cup S^{-1})^{\leq n} = G \text{ for some } n = n_S \in \omega.$$

Remark

Being Jónsson is equivalent to $G = (S \cup S^{-1})^{<\omega}$ for each full-sized S, and boundedly Jónsson is equivalent to $G = (S \cup S^{-1})^{\leq n_G}$, and so

$$\text{Boundedly Jónsson} \implies (\text{Jónsson} + \text{Bergman})$$

Theorem (Bergman, 2006)

If Ω is infinite, then the permutation group S_Ω has the Bergman property.

Shelah groups

Definition

A group G is *n-Shelah* if $X^n = G$ for every $X \subseteq G$ of full size. A *Shelah group* is an n-Shelah group for some n.

Theorem (Shelah)

($2^\kappa = \kappa^+$) There exists a 6640*-Shelah group on* κ^+*.*

Shelah groups

Definition

A group G is *n-Shelah* if $X^n = G$ for every $X \subseteq G$ of full size. A *Shelah group* is an n-Shelah group for some n.

Theorem (Shelah)

($2^\kappa = \kappa^+$) There exists a 6640*-Shelah group on* κ^+.

Question

(*a*) *Can one use alternative polynomials to get* $n < 6640$*?*

(*b*) *Is it consistent to have a Shelah group of inaccessible size?*

(*c*) *What can be done on the ground of* ZFC *alone?*

Shelah groups

Definition

A group G is *n-Shelah* if $X^n = G$ for every $X \subseteq G$ of full size. A *Shelah group* is an n-Shelah group for some n.

Theorem (Shelah)

($2^\kappa = \kappa^+$) There exists a 6640*-Shelah group on* κ^+*.*

Question

(a) *Can one use alternative polynomials to get* $n < 6640$*?*

(b) *Is it consistent to have a Shelah group of inaccessible size?*

(c) *What can be done on the ground of* ZFC *alone?*

Exercise (Observation)

Whereas if there exists an n-Shelah group, then there exist colorings $c_0, c_1, \ldots, c_{n^n-1} : [\kappa]^n \to \kappa$*, such that for every* $\Gamma \in [\kappa]^\kappa$ *we have* $\bigcup_{i<n^n} c_i``[\Gamma]^n = \kappa$

Shelah groups

Definition

A group G is *n-Shelah* if $X^n = G$ for every $X \subseteq G$ of full size. A *Shelah group* is an n-Shelah group for some n.

Theorem (Shelah)

($2^\kappa = \kappa^+$) There exists a 6640-Shelah group on κ^+.

Question

(a) *Can one use alternative polynomials to get $n < 6640$?*

(b) *Is it consistent to have a Shelah group of inaccessible size?*

(c) *What can be done on the ground of ZFC alone?*

Exercise (Observation)

Whereas if there exists an n-Shelah group, then there exist colorings $c_0, c_1, \ldots, c_{n^n-1} : [\kappa]^n \to \kappa$, such that for every $\Gamma \in [\kappa]^\kappa$ we have $\bigcup_{i<n^n} c_i``[\Gamma]^n = \kappa$
(therefore, $\kappa \nrightarrow [\kappa]^n_\kappa$, which asserts that there is a coloring $c : [\kappa]^n \to \kappa$ such that, for every $\Gamma \in [\kappa]^\kappa$, $c``[\Gamma]^n = \kappa$).

Shelah groups

Definition

A group G is *n-Shelah* if $X^n = G$ for every $X \subseteq G$ of full size. A *Shelah group* is an n-Shelah group for some n.

Theorem (Shelah)

($2^\kappa = \kappa^+$) There exists a 6640-Shelah group on κ^+.

Question

(a) *Can one use alternative polynomials to get $n < 6640$?*

(b) *Is it consistent to have a Shelah group of inaccessible size?*

(c) *What can be done on the ground of ZFC alone?*

Exercise (Observation)

Whereas if there exists an n-Shelah group, then there exist colorings $c_0, c_1, \ldots, c_{n^n-1} : [\kappa]^n \to \kappa$, such that for every $\Gamma \in [\kappa]^\kappa$ we have $\bigcup_{i<n^n} c_i``[\Gamma]^n = \kappa$ (therefore, $\kappa \nrightarrow [\kappa]^n_\kappa$, which asserts that there is a coloring $c : [\kappa]^n \to \kappa$ such that, for every $\Gamma \in [\kappa]^\kappa$, $c``[\Gamma]^n = \kappa$).

If $\kappa = \aleph_0$ or if κ is weakly compact, then there is no n-Shelah group of size κ.

Shelah groups

Definition

A group G is *n-Shelah* if $X^n = G$ for every $X \subseteq G$ of full size. A *Shelah group* is an n-Shelah group for some n.

Theorem (Shelah)

($2^\kappa = \kappa^+$) There exists a 6640-Shelah group on κ^+.

Question

(a) *Can one use alternative polynomials to get $n < 6640$?*

(b) *Is it consistent to have a Shelah group of inaccessible size?*

(c) *What can be done on the ground of ZFC alone?*

Exercise (Observation)

Whereas if there exists an n-Shelah group, then there exist colorings $c_0, c_1, \ldots, c_{n^n-1} : [\kappa]^n \to \kappa$, such that for every $\Gamma \in [\kappa]^\kappa$ we have $\bigcup_{i<n^n} c_i``[\Gamma]^n = \kappa$
(therefore, $\kappa \nrightarrow [\kappa]^n_\kappa$, which asserts that there is a coloring $c : [\kappa]^n \to \kappa$ such that, for every $\Gamma \in [\kappa]^\kappa$, $c``[\Gamma]^n = \kappa$).

If $\kappa = \aleph_0$ or if κ is weakly compact, then there is no n-Shelah group of size κ.
We don't know if there exists a countable group with the Bergman property.

Recent developments

Theorem (Cornullier, 2018)

Every 3-Shelah group is finite.

Recent developments

Theorem (Cornullier, 2018)

Every 3-Shelah group is finite.

Theorem (Banakh, 2022)

If $2^{\lambda} = \lambda^{+}$, *then there exists a* 36*-Shelah group* $(G, \cdot)$ *of size* λ^{+}.

Theorem (Cornullier, 2018)

Every 3-Shelah group is finite.

Theorem (Banakh, 2022)

If $2^{\lambda} = \lambda^{+}$, then there exists a 36-Shelah group $(G, \cdot)$ of size λ^{+}.
Moreover, the group is not polybounded (for every system $p_0, p_1, \ldots, p_{k-1}$ of group polynomials there is some $g \in G$, $i < k$ for which $p_i(g) \neq 1$), and absolutely $T_1 - \mathcal{S}$-closed (G forms a closed set whenever it is embedded in a T_1 semigroup).

Theorem (Corson-Ol'šanskiĭ-Varghese, 2023)

There exists a group G on ω_1 that is Jónsson and has the Bergman property:

$$\forall Y \in [G]^{\aleph_1} \; \exists n_Y \in \omega : \; Y^{\leq n_Y} = G.$$

Theorem (Cornullier, 2018)

Every 3-Shelah group is finite.

Theorem (Banakh, 2022)

If $2^\lambda = \lambda^+$*, then there exists a* 36*-Shelah group* $(G, \cdot)$ *of size* λ^+*.*
Moreover, the group is not polybounded (for every system $p_0, p_1, \ldots, p_{k-1}$ *of group polynomials there is some* $g \in G$*,* $i < k$ *for which* $p_i(g) \neq 1$*), and absolutely* $T_1 - \mathcal{S}$*-closed (*G *forms a closed set whenever it is embedded in a* T_1 *semigroup).*

Theorem (Corson-Ol'šanskiĭ-Varghese, 2023)

There exists a group G *on* ω_1 *that is Jónsson and has the Bergman property:*

$$\forall Y \in [G]^{\aleph_1}\ \exists n_Y \in \omega :\ Y^{\leq n_Y} = G.$$

Main result

Theorem (P – Rinot, 2023)

(*a*) *For every regular* λ*, there is an n-Shelah group of size* λ^+*.*

Main result

Theorem (P – Rinot, 2023)

(*a*) *For every regular λ, there is an n-Shelah group of size λ^+.*

(*b*) *If there is a uniformly coherent κ-Souslin tree, then there is an n-Shelah group of size κ.*

Main result

Theorem (P – Rinot, 2023)

(*a*) *For every regular λ, there is an n-Shelah group of size λ^+.*

(*b*) *If there is a uniformly coherent κ-Souslin tree, then there is an n-Shelah group of size κ.*

(*c*) *If $\square(\kappa)$ holds, then there is an n-Shelah group of size κ.*
In fact, the same is true for weaker forms of $\square(\kappa)$, so an n-Shelah group may exist above a supercompact cardinal.

Main result

Theorem (P – Rinot, 2023)

(*a*) *For every regular λ, there is an n-Shelah group of size λ^+.*

(*b*) *If there is a uniformly coherent κ-Souslin tree, then there is an n-Shelah group of size κ.*

(*c*) *If $\square(\kappa)$ holds, then there is an n-Shelah group of size κ.*
In fact, the same is true for weaker forms of $\square(\kappa)$, so an n-Shelah group may exist above a supercompact cardinal.

In all the above cases, our n is 10120.

Main result

Theorem (P – Rinot, 2023)

(a) For every regular λ, there is an n-Shelah group of size λ^+.

(b) If there is a uniformly coherent κ-Souslin tree, then there is an n-Shelah group of size κ.

(c) If $\square(\kappa)$ holds, then there is an n-Shelah group of size κ.
In fact, the same is true for weaker forms of $\square(\kappa)$, so an n-Shelah group may exist above a supercompact cardinal.

In all the above cases, our n is 10120.

Corollary (P – Rinot, 2023)

In L, for every regular uncountable cardinal κ, TFAE:

- *κ is not weakly compact;*
- *there is a Shelah group of size κ.*

Main result

Theorem (P – Rinot, 2023)

(a) *For every regular λ, there is an n-Shelah group of size λ^+.*

(b) *If there is a uniformly coherent κ-Souslin tree, then there is an n-Shelah group of size κ.*

(c) *If $\square(\kappa)$ holds, then there is an n-Shelah group of size κ.*
In fact, the same is true for weaker forms of $\square(\kappa)$, so an n-Shelah group may exist above a supercompact cardinal.

In all the above cases, our n is 10120.

Corollary (P – Rinot, 2023)

In L, for every regular uncountable cardinal κ, TFAE:

- *κ is not weakly compact;*
- *there is a Shelah group of size κ.*

The proof uses:

- Small cancellation theory, and
- Two forms of strong anti-Ramsey colorings.

Two notions of strong colorings

Let $\theta < \kappa$ denote a pair of infinite regular cardinals.

Definition (Erdős – Hajnal – Rado, 1965)

$\kappa \nrightarrow [\kappa]^2_\kappa$ asserts that there is a coloring $c : [\kappa]^2 \rightarrow \kappa$ such that, for every $\Gamma \in [\kappa]^\kappa$, $c``[\Gamma]^2$ is equal to κ.

Two notions of strong colorings

Let $\theta < \kappa$ denote a pair of infinite regular cardinals.

Definition (Erdős – Hajnal – Rado, 1965)

$\kappa \nrightarrow [\kappa]^2_\kappa$ asserts that there is a coloring $c : [\kappa]^2 \to \kappa$ such that, for every $\Gamma \in [\kappa]^\kappa$, $c``[\Gamma]^2$ is equal to κ.

Definition (Lambie-Hanson – Rinot, 2018)

$\mathrm{U}(\kappa, 2, \theta, 2)$ asserts that there is a coloring $d : [\kappa]^2 \to \theta$ such that, for every $\Gamma \in [\kappa]^\kappa$, $d``[\Gamma]^2$ is cofinal in θ.

Two notions of strong colorings

Let $\theta < \kappa$ denote a pair of infinite regular cardinals.

Definition (Erdős – Hajnal – Rado, 1965)

$\kappa \nrightarrow [\kappa]^2_\kappa$ asserts that there is a coloring $c : [\kappa]^2 \to \kappa$ such that, for every $\Gamma \in [\kappa]^\kappa$, $c``[\Gamma]^2$ is equal to κ.
i.e., for every $\xi < \kappa$, there are $\beta < \gamma$ in Γ such that $c(\beta, \gamma) = \xi$.

Definition (Lambie-Hanson – Rinot, 2018)

$\mathrm{U}(\kappa, 2, \theta, 2)$ asserts that there is a coloring $d : [\kappa]^2 \to \theta$ such that, for every $\Gamma \in [\kappa]^\kappa$, $d``[\Gamma]^2$ is cofinal in θ.
i.e., for every $i < \theta$, there are $\beta < \gamma$ in Γ such that $d(\beta, \gamma) > i$.

Two notions of strong colorings

Let $\theta < \kappa$ denote a pair of infinite regular cardinals.

Definition (Erdős – Hajnal – Rado, 1965)

$\kappa \nrightarrow [\kappa]^2_\kappa$ asserts that there is a coloring $c : [\kappa]^2 \rightarrow \kappa$ such that, for every $\Gamma \in [\kappa]^\kappa$, $c``[\Gamma]^2$ is equal to κ.
i.e., for every $\xi < \kappa$, there are $\beta < \gamma$ in Γ such that $c(\beta, \gamma) = \xi$.

Definition (Lambie-Hanson – Rinot, 2018)

$\mathrm{U}(\kappa, 2, \theta, 2)$ asserts that there is a coloring $d : [\kappa]^2 \rightarrow \theta$ such that, for every $\Gamma \in [\kappa]^\kappa$, $d``[\Gamma]^2$ is cofinal in θ.
i.e., for every $i < \theta$, there are $\beta < \gamma$ in Γ such that $d(\beta, \gamma) > i$.

Mutually active strong colorings

For every $\Gamma \in [\kappa]^\kappa$, there exists a club $D \subseteq \kappa$ such that, for all:

- $\xi \in \delta \in D$,
- $i < \theta$,
- $\gamma \in \Gamma \setminus \delta$,

there exists $\beta \in \Gamma \cap \delta$ such that $c(\beta, \gamma) = \xi$ and $d(\beta, \gamma) > i$.

The coloring hypothesis

Theorem (P – Rinot, 2023)

Suppose that:

(1) $\theta < \kappa$ *is a pair of infinite regular cardinals*

(2) $c : [\kappa]^2 \to \kappa$ *is a witness for* $\kappa \nrightarrow [\kappa]^2_\kappa$

(3) $d : [\kappa]^2 \to \theta$ *is a* *subadditive* *witness for* $\mathrm{U}(\kappa, 2, \theta, 2)$

(4) *c and d are mutually active*

Then there is a 10120-Shelah group of cardinality κ *with a malnormal filtration.*

The coloring hypothesis

Theorem (P – Rinot, 2023)

Suppose that:

(1) $\theta < \kappa$ *is a pair of infinite regular cardinals*

(2) $c : [\kappa]^2 \to \kappa$ *is a witness for* $\kappa \nrightarrow [\kappa]^2_\kappa$

(3) $d : [\kappa]^2 \to \theta$ *is a* *subadditive* *witness for* $\mathrm{U}(\kappa, 2, \theta, 2)$

(4) c *and* d *are mutually active*

Then there is a 10120-Shelah group of cardinality κ *with a malnormal filtration.*

Definition

d is subadditive iff writing $D^\gamma_{<i} := \{\alpha < \gamma \mid d(\alpha, \gamma) < i\}$, it is the case that the filtrations $\langle D^\gamma_{<i} \mid \ i < \theta\rangle$ $(\gamma \in \kappa)$ cohere in the following sense:

The coloring hypothesis

Theorem (P – Rinot, 2023)

Suppose that:

(1) $\theta < \kappa$ *is a pair of infinite regular cardinals*

(2) $c : [\kappa]^2 \to \kappa$ *is a witness for* $\kappa \nrightarrow [\kappa]^2_\kappa$

(3) $d : [\kappa]^2 \to \theta$ *is a* *subadditive* *witness for* $\mathrm{U}(\kappa, 2, \theta, 2)$

(4) c *and* d *are mutually active*

Then there is a 10120-Shelah group of cardinality κ *with a malnormal filtration.*

Definition

d is subadditive iff writing $D^\gamma_{<i} := \{\alpha < \gamma \mid d(\alpha, \gamma) < i\}$, it is the case that the filtrations $\langle D^\gamma_{<i} \mid i < \theta\rangle$ $(\gamma \in \kappa)$ cohere in the following sense:

$$\beta \in D^\gamma_{<i} \implies D^\beta_{<i} = D^\gamma_{<i} \cap \beta.$$

Overview of the construction

The group G will be generated by κ-many generators $\langle x_\alpha \mid \alpha < \kappa \rangle$.

Overview of the construction

The group G will be generated by κ-many generators $\langle x_\alpha \mid \alpha < \kappa \rangle$.
For $A \subseteq \kappa$, we denote by G_A the group generated by $\{x_\alpha \mid \alpha \in A\}$.

Overview of the construction

The group G will be generated by κ-many generators $\langle x_\alpha \mid \alpha < \kappa\rangle$.
For $A \subseteq \kappa$, we denote by G_A the group generated by $\{x_\alpha \mid \alpha \in A\}$.

- $\langle G_\gamma \mid \gamma < \kappa\rangle$ will form a malnormal filtration of G;

Overview of the construction

The group G will be generated by κ-many generators $\langle x_\alpha \mid \alpha < \kappa \rangle$.
For $A \subseteq \kappa$, we denote by G_A the group generated by $\{x_\alpha \mid \alpha \in A\}$.

- $\langle G_\gamma \mid \gamma < \kappa \rangle$ will form a malnormal filtration of G;
- $\langle G_{D^\gamma_{<i}} \mid i < \theta \rangle$ will form a filtration of G_γ;

Overview of the construction

The group G will be generated by κ-many generators $\langle x_\alpha \mid \alpha < \kappa\rangle$.
For $A \subseteq \kappa$, we denote by G_A the group generated by $\{x_\alpha \mid \alpha \in A\}$.

- $\langle G_\gamma \mid \gamma < \kappa\rangle$ will form a malnormal filtration of G;
- $\langle G_{D^\gamma_{<i}} \mid i < \theta\rangle$ will form a filtration of G_γ;
- $G_{D^\gamma_{<i} \cap D^\delta_{<j}}$ will coincide with $G_{D^\gamma_{<i}} \cap G_{D^\delta_{<j}}$.

Overview of the construction

The group G will be generated by κ-many generators $\langle x_\alpha \mid \alpha < \kappa \rangle$.
For $A \subseteq \kappa$, we denote by G_A the group generated by $\{x_\alpha \mid \alpha \in A\}$.

- $\langle G_\gamma \mid \gamma < \kappa \rangle$ will form a malnormal filtration of G;
- $\langle G_{D^\gamma_{<i}} \mid i < \theta \rangle$ will form a filtration of G_γ;
- $G_{D^\gamma_{<i} \cap D^\delta_{<j}}$ will coincide with $G_{D^\gamma_{<i}} \cap G_{D^\delta_{<j}}$.

Definition

For every $g \in G$, (γ_g, i_g) stands for the left-lexicographically least pair $(\gamma, i) \in \kappa \times \theta$ such that $g \in G_{D^\gamma_{<i+1} \cup \{\gamma\}}$. (so in particular, $g \in G_{\gamma_g \cup \{\gamma_g\}} \setminus G_{\gamma_g}$)

Overview of the construction

The group G will be generated by κ-many generators $\langle x_\alpha \mid \alpha < \kappa\rangle$.
For $A \subseteq \kappa$, we denote by G_A the group generated by $\{x_\alpha \mid \alpha \in A\}$.

- $\langle G_\gamma \mid \gamma < \kappa\rangle$ will form a malnormal filtration of G;
- $\langle G_{D^\gamma_{<i}} \mid i < \theta\rangle$ will form a filtration of G_γ;
- $G_{D^\gamma_{<i}\cap D^\delta_{<j}}$ will coincide with $G_{D^\gamma_{<i}} \cap G_{D^\delta_{<j}}$.

Definition

For every $g \in G$, (γ_g, i_g) stands for the left-lexicographically least pair $(\gamma, i) \in \kappa \times \theta$ such that $g \in G_{D^\gamma_{<i+1}\cup\{\gamma\}}$. (so in particular, $g \in G_{\gamma_g\cup\{\gamma_g\}} \setminus G_{\gamma_g}$)

The group G is constructed in a two-dimensional recursion, where we gradually determine the relations for the subgroups $\langle G_{D^\gamma_{<i}} \mid \gamma < \kappa,\ i < \theta\rangle$.

Overview of the construction

The group G will be generated by κ-many generators $\langle x_\alpha \mid \alpha < \kappa\rangle$.
For $A \subseteq \kappa$, we denote by G_A the group generated by $\{x_\alpha \mid \alpha \in A\}$.

- $\langle G_\gamma \mid \gamma < \kappa\rangle$ will form a malnormal filtration of G;
- $\langle G_{D^\gamma_{<i}} \mid i < \theta\rangle$ will form a filtration of G_γ;
- $G_{D^\gamma_{<i} \cap D^\delta_{<j}}$ will coincide with $G_{D^\gamma_{<i}} \cap G_{D^\delta_{<j}}$.

Definition

For every $g \in G$, (γ_g, i_g) stands for the left-lexicographically least pair $(\gamma, i) \in \kappa \times \theta$ such that $g \in G_{D^\gamma_{<i+1} \cup \{\gamma\}}$. (so in particular, $g \in G_{\gamma_g \cup \{\gamma_g\}} \setminus G_{\gamma_g}$)

The group G is constructed in a two-dimensional recursion, where we gradually determine the relations for the subgroups $\langle G_{D^\gamma_{<i}} \mid \gamma < \kappa,\ i < \theta\rangle$. The subgroup $G_{D^\gamma_{<i+1} \cup \{\gamma\}}$ is an amalgamation of the subgroups $G_{D^\gamma_{<i} \cup \{\gamma\}}$ and $G_{D^\gamma_{<i+1}}$ over $G_{D^\gamma_{<i}}$.

Overview of the construction

The group G will be generated by κ-many generators $\langle x_\alpha \mid \alpha < \kappa\rangle$.
For $A \subseteq \kappa$, we denote by G_A the group generated by $\{x_\alpha \mid \alpha \in A\}$.

- $\langle G_\gamma \mid \gamma < \kappa\rangle$ will form a malnormal filtration of G;
- $\langle G_{D^\gamma_{<i}} \mid i < \theta\rangle$ will form a filtration of G_γ;
- $G_{D^\gamma_{<i} \cap D^\delta_{<j}}$ will coincide with $G_{D^\gamma_{<i}} \cap G_{D^\delta_{<j}}$.

Definition

For every $g \in G$, (γ_g, i_g) stands for the left-lexicographically least pair $(\gamma, i) \in \kappa \times \theta$ such that $g \in G_{D^\gamma_{<i+1} \cup \{\gamma\}}$. (so in particular, $g \in G_{\gamma_g \cup \{\gamma_g\}} \setminus G_{\gamma_g}$)

The group G is constructed in a two-dimensional recursion, where we gradually determine the relations for the subgroups $\langle G_{D^\gamma_{<i}} \mid \gamma < \kappa,\ i < \theta\rangle$. The subgroup $G_{D^\gamma_{<i+1} \cup \{\gamma\}}$ is an amalgamation of the subgroups $G_{D^\gamma_{<i} \cup \{\gamma\}}$ and $G_{D^\gamma_{<i+1}}$ over $G_{D^\gamma_{<i}}$. Small cancellation theory is applied to equate words involving $x \in G_{D^\gamma_{<i} \cup \{\gamma\}} \setminus G_{D^\gamma_{<i}}$, $y \in G_{D^\gamma_{<i}}$ and $z \in G_{D^\gamma_{<i+1}} \setminus G_{D^\gamma_{<i}}$ with group elements of $G_{D^\gamma_{<i}}$, based on an interpretation of $c(\gamma_z, \gamma)$.

Overview of the construction

The group G will be generated by κ-many generators $\langle x_\alpha \mid \alpha < \kappa\rangle$.
For $A \subseteq \kappa$, we denote by G_A the group generated by $\{x_\alpha \mid \alpha \in A\}$.

- $\langle G_\gamma \mid \gamma < \kappa\rangle$ will form a malnormal filtration of G;
- $\langle G_{D^\gamma_{<i}} \mid i < \theta\rangle$ will form a filtration of G_γ;
- $G_{D^\gamma_{<i} \cap D^\delta_{<j}}$ will coincide with $G_{D^\gamma_{<i}} \cap G_{D^\delta_{<j}}$.

Definition

For every $g \in G$, (γ_g, i_g) stands for the left-lexicographically least pair $(\gamma, i) \in \kappa \times \theta$ such that $g \in G_{D^\gamma_{<i+1} \cup \{\gamma\}}$. (so in particular, $g \in G_{\gamma_g \cup \{\gamma_g\}} \setminus G_{\gamma_g}$)

The group G is constructed in a two-dimensional recursion, where we gradually determine the relations for the subgroups $\langle G_{D^\gamma_{<i}} \mid \gamma < \kappa,\ i < \theta\rangle$. The subgroup $G_{D^\gamma_{<i+1} \cup \{\gamma\}}$ is an amalgamation of the subgroups $G_{D^\gamma_{<i} \cup \{\gamma\}}$ and $G_{D^\gamma_{<i+1}}$ over $G_{D^\gamma_{<i}}$. Small cancellation theory is applied to equate words involving $x \in G_{D^\gamma_{<i} \cup \{\gamma\}} \setminus G_{D^\gamma_{<i}}$, $y \in G_{D^\gamma_{<i}}$ and $z \in G_{D^\gamma_{<i+1}} \setminus G_{D^\gamma_{<i}}$ with group elements of $G_{D^\gamma_{<i}}$, based on an interpretation of $c(\gamma_z, \gamma)$.
We want to equate $p(xyz, xyxyz)$ with some prescribed $h \in G_{D^\gamma_{<i}}$ for as many triplets (x, y, z) as possible:

Overview of the construction

The group G will be generated by κ-many generators $\langle x_\alpha \mid \alpha < \kappa \rangle$.
For $A \subseteq \kappa$, we denote by G_A the group generated by $\{x_\alpha \mid \alpha \in A\}$.

- $\langle G_\gamma \mid \gamma < \kappa \rangle$ will form a malnormal filtration of G;
- $\langle G_{D^\gamma_{<i}} \mid i < \theta \rangle$ will form a filtration of G_γ;
- $G_{D^\gamma_{<i} \cap D^\delta_{<j}}$ will coincide with $G_{D^\gamma_{<i}} \cap G_{D^\delta_{<j}}$.

Definition

For every $g \in G$, (γ_g, i_g) stands for the left-lexicographically least pair $(\gamma, i) \in \kappa \times \theta$ such that $g \in G_{D^\gamma_{<i+1} \cup \{\gamma\}}$. (so in particular, $g \in G_{\gamma_g \cup \{\gamma_g\}} \setminus G_{\gamma_g}$)

The group G is constructed in a two-dimensional recursion, where we gradually determine the relations for the subgroups $\langle G_{D^\gamma_{<i}} \mid \gamma < \kappa,\ i < \theta \rangle$. The subgroup $G_{D^\gamma_{<i+1} \cup \{\gamma\}}$ is an amalgamation of the subgroups $G_{D^\gamma_{<i} \cup \{\gamma\}}$ and $G_{D^\gamma_{<i+1}}$ over $G_{D^\gamma_{<i}}$. Small cancellation theory is applied to equate words involving $x \in G_{D^\gamma_{<i} \cup \{\gamma\}} \setminus G_{D^\gamma_{<i}}$, $y \in G_{D^\gamma_{<i}}$ and $z \in G_{D^\gamma_{<i+1}} \setminus G_{D^\gamma_{<i}}$ with group elements of $G_{D^\gamma_{<i}}$, based on an interpretation of $c(\gamma_z, \gamma)$.
We want to equate $p(xyz, xyxyz)$ with some prescribed $h \in G_{D^\gamma_{<i}}$ for as many triplets (x, y, z) as possible: employing small cancellation theory and preservation theorems for that.

Overview of the construction

The group G will be generated by κ-many generators $\langle x_\alpha \mid \alpha < \kappa \rangle$.
For $A \subseteq \kappa$, we denote by G_A the group generated by $\{x_\alpha \mid \alpha \in A\}$.

- $\langle G_\gamma \mid \gamma < \kappa \rangle$ will form a malnormal filtration of G;
- $\langle G_{D^\gamma_{<i}} \mid i < \theta \rangle$ will form a filtration of G_γ;
- $G_{D^\gamma_{<i} \cap D^\delta_{<j}}$ will coincide with $G_{D^\gamma_{<i}} \cap G_{D^\delta_{<j}}$.

Definition

For every $g \in G$, (γ_g, i_g) stands for the left-lexicographically least pair $(\gamma, i) \in \kappa \times \theta$ such that $g \in G_{D^\gamma_{<i+1} \cup \{\gamma\}}$. (so in particular, $g \in G_{\gamma_g \cup \{\gamma_g\}} \setminus G_{\gamma_g}$)

The group G is constructed in a two-dimensional recursion, where we gradually determine the relations for the subgroups $\langle G_{D^\gamma_{<i}} \mid \gamma < \kappa,\ i < \theta \rangle$. The subgroup $G_{D^\gamma_{<i+1} \cup \{\gamma\}}$ is an amalgamation of the subgroups $G_{D^\gamma_{<i} \cup \{\gamma\}}$ and $G_{D^\gamma_{<i+1}}$ over $G_{D^\gamma_{<i}}$. Small cancellation theory is applied to equate words involving $x \in G_{D^\gamma_{<i} \cup \{\gamma\}} \setminus G_{D^\gamma_{<i}}$, $y \in G_{D^\gamma_{<i}}$ and $z \in G_{D^\gamma_{<i+1}} \setminus G_{D^\gamma_{<i}}$ with group elements of $G_{D^\gamma_{<i}}$, based on an interpretation of $c(\gamma_z, \gamma)$.
Biggest challenge boils down to the task of ensuring that if $z, z' \in G_{D^\gamma_{<i+1}} \setminus G_{D^\gamma_{<i}}$ are such that $\gamma_z = \alpha < \gamma_{z'} = \alpha' \in D^\gamma_{<i+1} \setminus D^\gamma_{<i}$ (e.g. $z = x_\alpha$, $z' = x_{\alpha'}$), then z and z' are independent over $G_{D^\gamma_{<i}}$.

An excerpt from the paper

Looking at Definition 5.14, we see that:

$$
\begin{array}{lllllll}
a_{\sigma*} & = & a & = & z_\alpha & = & a \\
t_{\sigma*} & = & t & = & t & = & t \\
h_{\sigma*} & = & \pi_0(c(\alpha_a,\gamma)) & = & \pi_0(\xi) & = & h \\
y_{\sigma*,0} & = & \pi_1(c(\alpha_a,\gamma)) & = & \pi_1(\xi) & = & y_0 \\
y_{\sigma*,1} & = & \pi_2(c(\alpha_a,\gamma)) & = & \pi_2(\xi) & = & y_1 \\
z_{\sigma*} & = & \pi_3(c(\alpha_a,\gamma)) & = & \pi_3(\xi) & = & z \\
\varepsilon_{\sigma*} & = & \pi_4(c(\alpha_a,\gamma)) & = & \pi_4(\xi) & = & \varepsilon \\
b_{\sigma*} & = & y_{\sigma*,0}\cdot t^{\varepsilon_{\sigma*}}\cdot y_{\sigma*,1}\cdot z_{\sigma*} & = & z_\gamma\cdot z & = & b \\
b'_{\sigma*} & = & b_{\sigma*}\cdot b_{\sigma*} & = & z_\gamma\cdot z\cdot z_\gamma\cdot z & = & b' \\
K_{\sigma*} & = & G_{D^\gamma_{\leq i}}\cap G_{(\alpha_a+1)} & = & G_{D^\gamma_{\leq i}}\cap G_{(\alpha+1)} & = & G_{D^\gamma_{\leq i}\cap(\alpha+1)}
\end{array}
$$

Table 1. Evaluations

An high-level explanation of why it works

Given $X \subseteq G$ of full size, we may thin it out to ensure that $x \mapsto i_x$ is constant over X, say it is j.

An high-level explanation of why it works

Given $X \subseteq G$ of full size, we may thin it out to ensure that $x \mapsto i_x$ is constant over X, say it is j.
Let $h \in G$, and we shall find $x, y, z \in X$ with $h = p(xyz, xyxyz)$.

An high-level explanation of why it works

Given $X \subseteq G$ of full size, we may thin it out to ensure that $x \mapsto i_x$ is constant over X, say it is j.
Let $h \in G$, and we shall find $x, y, z \in X$ with $h = p(xyz, xyxyz)$.
Fix an elementary submodel $M \prec H_{\kappa^+}$ that knows about everything and $\delta := M \cap \kappa$ is $< \kappa$. In particular, $h \in G_\delta = G \cap M$.

An high-level explanation of why it works

Given $X \subseteq G$ of full size, we may thin it out to ensure that $x \mapsto i_x$ is constant over X, say it is j.
Let $h \in G$, and we shall find $x, y, z \in X$ with $h = p(xyz, xyxyz)$.
Fix an elementary submodel $M \prec H_{\kappa^+}$ that knows about everything and $\delta := M \cap \kappa$ is $< \kappa$. In particular, $h \in G_\delta = G \cap M$.
Pick $x \in X \setminus M$, so that $\gamma_x \geq \delta$. We denote by $\bar{x}$ a certain residue of x that lies in M.

Given $X \subseteq G$ of full size, we may thin it out to ensure that $x \mapsto i_x$ is constant over X, say it is j.
Let $h \in G$, and we shall find $x, y, z \in X$ with $h = p(xyz, xyxyz)$.
Fix an elementary submodel $M \prec H_{\kappa^+}$ that knows about everything and $\delta := M \cap \kappa$ is $< \kappa$. In particular, $h \in G_\delta = G \cap M$.
Pick $x \in X \setminus M$, so that $\gamma_x \geq \delta$. We denote by $\bar{x}$ a certain residue of x that lies in M. Meanwhile, pick $y \in X \cap M$ with $d(\gamma_y, \gamma_x) > j$.

An high-level explanation of why it works

Given $X \subseteq G$ of full size, we may thin it out to ensure that $x \mapsto i_x$ is constant over X, say it is j.
Let $h \in G$, and we shall find $x, y, z \in X$ with $h = p(xyz, xyxyz)$.
Fix an elementary submodel $M \prec H_{\kappa^+}$ that knows about everything and $\delta := M \cap \kappa$ is $< \kappa$. In particular, $h \in G_\delta = G \cap M$.
Pick $x \in X \setminus M$, so that $\gamma_x \geq \delta$. We denote by $\bar{x}$ a certain residue of x that lies in M. Meanwhile, pick $y \in X \cap M$ with $d(\gamma_y, \gamma_x) > j$.
Finally, pick $z \in X \cap M$ such that:

(1) $\gamma_z > \max\{\gamma_y, \gamma_{\bar{x}}\}$,

(2) $i := d(\gamma_z, \gamma_x)$ is large enough (this implies $z \in G_{D^{\gamma_x}_{<i+1}} \setminus G_{D^{\gamma_x}_{<i}}$),

(3) $\xi := c(\gamma_z, \gamma_x)$ is a code for $(h, \bar{x}, y)$ (the code belongs to M).

An high-level explanation of why it works

Given $X \subseteq G$ of full size, we may thin it out to ensure that $x \mapsto i_x$ is constant over X, say it is j.
Let $h \in G$, and we shall find $x, y, z \in X$ with $h = p(xyz, xyxyz)$.
Fix an elementary submodel $M \prec H_{\kappa^+}$ that knows about everything and $\delta := M \cap \kappa$ is $< \kappa$. In particular, $h \in G_\delta = G \cap M$.
Pick $x \in X \setminus M$, so that $\gamma_x \geq \delta$. We denote by $\bar{x}$ a certain residue of x that lies in M. Meanwhile, pick $y \in X \cap M$ with $d(\gamma_y, \gamma_x) > j$.
Finally, pick $z \in X \cap M$ such that:

(1) $\gamma_z > \max\{\gamma_y, \gamma_{\bar{x}}\}$,

(2) $i := d(\gamma_z, \gamma_x)$ is large enough (this implies $z \in G_{D^{\gamma_x}_{<i+1}} \setminus G_{D^{\gamma_x}_{<i}}$),

(3) $\xi := c(\gamma_z, \gamma_x)$ is a code for $(h, \bar{x}, y)$ (the code belongs to M).

Since x belongs to $G_{D^{\gamma_x+1}_{<i+1}}$, our particular amalgamation procedure together with (1)–(3) ensure that $G_{D^{\gamma_x+1}_{<i+1}} \models h = p(xyz, xyxyz)$. □

Thank you for your attention!