Tilting as a bi-interpretation

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[1]A report on joint work with Silvain Rideau-Kikuchi and Pierre Simon, arXiv:2505.01321

Ax-Kochen-Ershov theorem

Theorem (Ax-Kochen, Ershov)

Let $\mathcal{U}$ be a non-principal ultrafilter on the set of prime numbers. Then

$$\lim_{p \to \mathcal{U}} \mathrm{Th}_{\mathcal{L}_{\mathrm{ring}}}(\mathbb{Q}_p) = \lim_{p \to \mathcal{U}} \mathrm{Th}_{\mathcal{L}_{\mathrm{ring}}}(\mathbb{F}_p((t)))$$

- $\mathbb{F}_p((t))$ is the field of Laurent series $\sum_{j \geq N} a_j t^j$ for some integer N with $a_j \in \mathbb{F}_p = \{0, 1, \ldots, p-1\}$ the field of p elements
- $\mathbb{Q}_p$ is the field of p-adic numbers $\sum_{j \geq N} a_j p^j$ for some integer N with $a_j \in \{0, 1, \ldots, p-1\} \subseteq \mathbb{Z}$ using the p-adic norm $|\sum_{j \geq N} a_j p^j|_p = p^{-N}$ if $a_N \neq 0$

Aside: Ax-Kochen and CH

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VIVE LA DIFFÉRENCE II. THE AX–KOCHEN ISOMORPHISM THEOREM

BY

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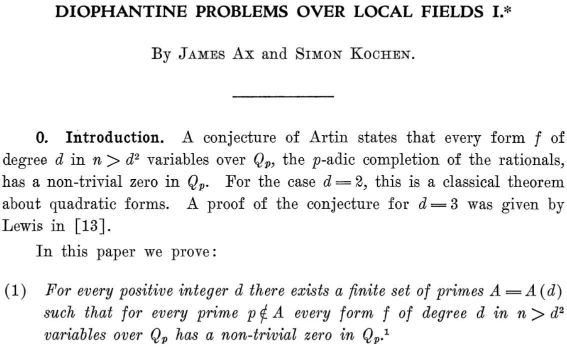
ABSTRACT

We show in §1 that the Ax–Kochen isomorphism theorem [AK] requires the continuum hypothesis. Most of the applications of this theorem are insensitive to set theoretic considerations. (A probable exception is the work of Moloney [Mo].) In §2 we give an unrelated result on cuts in models of Peano arithmetic which answers a question on the ideal structure of countable ultraproducts of $\mathbb{Z}$ posed in [LLS]. In §1 we also answer a question of Keisler regarding Scott complete ultrapowers of $\mathbb{R}$ (see 1.18).

Ax-Kochen-Ershov transfer

More concretely, AKE implies that if some property of valued fields can be expressed in first-order logic, then it holds for power series fields over finite fields of large characteristic if and only if it is true of the p-adics for large p.

Ax-Kochen asymptotic Artin conjecture


DIOPHANTINE PROBLEMS OVER LOCAL FIELDS I.*

By James Ax and Simon Kochen.

0. Introduction. A conjecture of Artin states that every form f of degree d in $n > d^2$ variables over Q_p, the p-adic completion of the rationals, has a non-trivial zero in Q_p. For the case $d = 2$, this is a classical theorem about quadratic forms. A proof of the conjecture for $d = 3$ was given by Lewis in [13].

In this paper we prove:

(1) *For every positive integer d there exists a finite set of primes $A = A(d)$ such that for every prime $p \notin A$ every form f of degree d in $n > d^2$ variables over Q_p has a non-trivial zero in Q_p.*[1]

Ax-Kochen-Ershov for the Langland's Program

Cluckers, Hales, and Loeser (2011) use the Ax-Kochen-Ershov theorem to transfer each instance of the Fundamental Lemma of the Langland's Program to $\mathbb{Q}_p$ for $p \gg 0$.

Perfectoid transfer

PERFECTOID SPACES

by Peter SCHOLZE

ABSTRACT

We introduce a certain class of so-called perfectoid rings and spaces, which give a natural framework for Faltings' almost purity theorem, and for which there is a natural tilting operation which exchanges characteristic 0 and characteristic p. We deduce the weight-monodromy conjecture in certain cases by reduction to equal characteristic.

Perfectoid fields

A perfectoid field (of residue characteristic p) is a complete (ultra) normed field $(K, |\cdot|)$ with ring of integers $\mathcal{O} := \{x \in K : |x| \leq 1\}$ for which

- $\mathcal{O}/p\mathcal{O}$ is semiperfect: $x \mapsto x^p$ is onto and
- $|K|$ is dense in $[0, \infty)$.

Examples include:

- Any complete perfect normed field of characteristic p, e.g. $\widehat{\mathbb{F}_p((t))^{\frac{1}{p^\infty}}}$
- Any complete algebraically closed field field of residue characteristic p.
- $\widehat{\mathbb{Q}_p(\{\zeta_{p^n}\}_{n \in \mathbb{Z}_+})}$ where ζ_{p^n} is a primitive $p^{n\text{th}}$ root of unity.
- $\widehat{\mathbb{Q}_p(\{\sqrt[p^n]{p}\}_{n \in \mathbb{Z}_+})}$

$\mathbb{Q}_p$ is not perfectoid as $|\mathbb{Q}_p| = \{0\} \cup p^{\mathbb{Z}}$ is not dense in $[0, \infty)$.

Tilting

Given a perfectoid field K, the multiplicative monoid

$$K^\flat := \varprojlim \left(\cdots \longrightarrow K \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} \cdots \longrightarrow K \xrightarrow{x \mapsto x^p} K \right)$$
$$= \{(x_n)_{n=0}^\infty \in K^\omega : x_{n+1}^p = x_n \text{ for all } n \in \omega\}$$

has the structure of a complete perfect normed field of characteristic p where addition is defined by the formula

$$(x_n)_{n=0}^\infty + (y_n)_{n=0}^\infty = (\lim_{m \to \infty} (x_{n+m} + y_{n+m})^{p^m})_{n=0}^\infty$$

Examples of tilts

- If K is a perfectoid field of characteristic p, $K^\flat \cong K$.
- $\left(\widehat{\mathbb{Q}_p(\{\sqrt[p^n]{p}\}_{n \in \mathbb{Z}_+})}\right)^\flat \cong \widehat{\mathbb{F}_p((t))^{\frac{1}{p^\infty}}}$
- $\left(\widehat{\mathbb{Q}_p(\{\zeta_{p^n}\}_{n \in \mathbb{Z}_+})}\right)^\flat \cong \widehat{\mathbb{F}_p((t))^{\frac{1}{p^\infty}}}$

Logical riddles

- No equality of theories could explain Scholze's transfer of theorems from $K^\flat$ to K. What notion of logical equivalence could explain this and make it possible to transfer other theorems?
- Scholze's proof involves theorems on the equivalence of certain associated categories (e.g. of adic spaces) between K and $K^\flat$. How can this be when it is possible to have $K_1 \ncong K_2$ but $K_1^\flat \cong K_2^\flat$?
- How can we explain those equivalences of associated categories via mathematical logic?

Jahnke-Kartas answer

Theorem (Jahnke-Kartas, JAMS 2025)

If K is a perfectoid field and $K^* \succeq K$ is an $\aleph_1$-saturated elementary extension, then there is a coarsening w of the valuation on K^* and a natural elementary embedding of $K^\flat$ into the residue field of K^* with respect to w.

Continuous logic theories of valued fields: subtle dependence on formalism

Our answers to the riddles will be that tilting is part of a bi-interpretation in continuous logic. However, in the natural continuous logic for normed fields introduced by Ben Yaacov, the valuation ring is not definable which make the untilting interpretations (likely) undefinable as well.

We formulate MVF, the theory of metric valued fields, in a metric expansion of the language of rings in which we consider

- valued fields (K, v) with a a norm $|\cdot|$ so that $v(x) \leq v(y) \rightarrow |x| \geq |y|$, i.e. up to reversing additive and multiplicative notation, v is a refinement of the norm, and
- the universe of our structure to be the valuation ring $\mathcal{O}_{K,v} = \{x \in K : v(x) \geq 0\}$ rather than K itself.

Naming the predicate $D(x, y) := \inf_z |y - zx|$ the theory of algebraically closed metric valued fields has quantifier elimination.

The theory of perfectoid fields

For a real number $\alpha \in [0,1)$ we define $\mathrm{PERF}_{|p|=\alpha}$ to be the theory of perfectoid fields in which $|p| = \alpha$. Relative to MVF it is axiomatized by

- $|p| = \alpha$,
- $\inf_x \max\{|x| \dot{-} \alpha^{1/p}, \alpha^{1/p} \dot{-} |x|\} = 0$,
- $\sup_x \inf_y \inf_z |x - y^p - pz| = 0$.

Bi-interpretation theorems

- Tilting gives a quantifier-free interpretation of $\mathrm{PERF}_{|p|=0}$ in $\mathrm{PERF}_{|p|=\alpha}$ for any $\alpha \in [0,1)$.
- There is an expansion of $\mathrm{PERF}_{|p|=0}$ by constants ξ, $\mathrm{PERF}_{|p|=0,\xi}$ so that tilting is half of a biinterpretation between $\mathrm{PERF}_{|p|=\alpha}$ (for $\alpha > 0$) and $\mathrm{PERF}_{|p|=0,\xi}$. The constant ξ names a point on the so-called Fargues-Fontaine curve and the (quantifier-free) interpretation of $\mathrm{PERF}_{|p|=\alpha}$ in $\mathrm{PERF}_{|p|=0,\xi}$ comes from a p-adic Hodge theory construction of Fargues-Fontaine.
- Expanding further with constants $\varpi = (\varpi_n)_{n=1}^{\infty}$ where the intention is that $\varpi_{n+1}^p = \varpi_n$ and $\alpha^{\frac{1}{p}} \leq |\varpi_1| < \alpha$ when $|p| = \alpha > 0$, all of this is mediated by bi-interpretations with a third theory $\mathrm{TPERF}_{\xi,\varpi}$ of truncated perfectoid fields in which the models take the form $\mathcal{O}_{K,v}/p\mathcal{O}_{K,v}$ where $(K,v) \models \mathrm{PERF}_{|p|=\alpha}$ with $\alpha > 0$.

Consequence: Fontaine-Wintenberger Theorem

As a general proposition, one sees that a quantifier-free interpretation which is part of a bi-interpretation takes existentially closed structures to existentially closed structures.

Since existentially closed metric valued fields are exactly the algebraically closed metric valued fields, we have

Corollary

A perfectoid field K is algebraically closed if and only if $K^\flat$ is algebraically closed.

It then follows from standard Galois theory that

Corollary (Fontaine-Wintenberger Theorem)

For K a perfectoid field there is a natural isomorphism $\mathrm{Gal}(K^{\mathrm{alg}}/K) \cong \mathrm{Gal}((K^\flat)^{\mathrm{alg}}/K^\flat)$.

Consequence: Approximation Theorem

From the definition of an interpretation $I : \widetilde{N} \to N$ of the structure N in the structure M, we see that any definable set $X \subseteq N^x$ pulls back to a definable set in M. When I is part of a biinterpretation, it is also true that the push forward of a definable set from M is definable in N.

We will write Ω for the definable set relative to PERF in the variables $x = (x_i)_{i=0}^{\infty}$ defined by $x_{n+1}^p = x_n$ for $n \in \omega$. So, for $K \models$ PERF, $\Omega(K)$ is the universe of $\mathcal{O}_K^\flat = \mathcal{O}_{K^\flat}$. The projection to the 0^{th} cooordinate of Ω gives us a function $\sharp : K^\flat \to K$.

Corollary

Let K be a perfectoid field and $X \subseteq K^n$ some definable set. Define $X^\flat := \{x \in (K^\flat)^n : \sharp(x) \in X\}$. For every $\epsilon > 0$ there is a subset $Y_\epsilon \subseteq (K^\flat)^n$ which is quantifier-free definable in the first-order language of valued fields for which Y_ϵ and $X^\flat$ have the same ϵ-neighborhood.

Consequence: Equivalence of adic spaces via type spaces

Let $K \models \mathrm{PERF}_{|p|=\alpha}$ be perfectoid and let $A \le \mathcal{O}_K$ be a perfectoid substructure. For every type-A-definable set $X \subseteq \Omega^n$, let $\widetilde{X} \subseteq (\mathcal{O}_K^\flat)^n$ denote the corresponding type-$A^\flat$-definable set.

Proposition

The type spaces $S_X(A)$ are $S_{\widetilde{X}}(A^\flat)$ are homeomorphic. Moreover, the homeomorphism is functorial and induces an equivalence of categories:

$$\left\{ \begin{array}{c} \textit{type-}A\textit{-definable subsets} \\ \textit{of (cartesian powers of) } \Omega(K) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{type-}A^\flat\textit{-definable subsets} \\ \textit{of (cartesian powers of) } \mathcal{O}_{K^\flat} \end{array} \right\}.$$

Proposition

There is a continuous bijection from $S_X(A)$ to the Huber adic space of continuous valuations associated to X. The logic topology on $S_X(A)$ is the constructible topology associated to the adic space.

Question: Equivalence of adic spaces with the "right" topology on type spaces?

Putting these propositions together, we see that tilting gives an equivalence of categories between adic spaces over a perfectoid field K and adic spaces over its tilt $K^\flat$, when we give these spaces their constructible topologies.

Scholze's theorem, drawing on the almost mathematics of Faltings and Gabber-Ramero, gives the equivalence with the correct topologies.

Question

What general results about bi-interpretations could explain the equivalence with the correct topology? Our guess is that we should work with positive continuous logic.

Question: Refined Approximation Theorem?

Scholze's transfer theorem is based on a more refined approximation lemma for algebraic equations: if $f \in \mathcal{O}_K[x_1, \ldots, x_n]$ is a polynomial over the valuation ring of a perfectoid field, $X = \{a \in \mathcal{O}_K^n : f(a) = 0\}$, and $\epsilon > 0$, then there is a polynomial $g_\epsilon \in \mathcal{O}_{K^\flat}[x_1, \ldots, x_n]$ so that the ϵ-neighborhood of $X^\flat$ contains the zero set of g.

Our approximation lemma applied without further computation gives that the ϵ-neighborhood of $X^\flat$ is the ϵ-neighborhood of a quantifier-free, first-order definable set.

Question

If $Y \subseteq \mathcal{O}_K^n$ is an algebraic set (i.e. defined by the vanishing of polynomial equations) and $\epsilon > 0$ must there be some algebraic set Z contained in the ϵ-neighborhood of $Y^\flat$ with $\dim Y = \dim Z$?

A positive answer would allow for a transfer of the Weight Monodromy Conjecture in general.

A birthday present: classify the untilts

- We have seen that it is possible for nonisomorphic perfectoid fields $K_1 \ncong K_2$ to have $K_1^\flat \cong K_2^\flat$.
- Kedlaya and Temkin show that this is even possible when the tilt is algebraically closed: there is some $L \ncong \mathbb{C}_p = \widehat{\mathbb{Q}_p^{\mathrm{alg}}}$ with $L^\flat \cong \mathbb{C}_p^\flat$.
- The set of untilts with additional structure of a perfectoid field of characteristic p is parameterized by a definable set (the degree one points of the Fargues-Fontaine curve).
- Question: For K a complete, nontrivially normed, algebraically closed field of characteristic p, how complicated is $\{L : L^\flat \cong K\}/\cong$?
- Is the expected complexity in the answer to that last question a reflection of a nonstructure principle? Unlike Ben Yaacov's theory of algebraically closed metric valued fields, our theory is unstable.