On Expansions of Linear Orders and NIP

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Conference in honor of Saharon Shelah's 80th birthday

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Outline

1. A question on expansions of linear orders
2. Motivation from classification theory for NIP

ω-categorical expansions of DLO

We are interested in **ω-categorical expansions** of dense linear orders (DLO).

Some basic examples

- $(M, \leq, P)$ where $P \subseteq M$ is *dense-co-dense*
- $(M, \leq, R)$ the *ordered random graph*, where $R \subseteq M^2$ defines a random graph.

Generic Construction

Those are examples of a more general construction.

Fact (Generic Linear Order)

*Let M be a structure with **trivial algebraic closure**: $\mathrm{acl}(A) = A$ for all A.*

There exists a unique (up to elementary equivalence) expansion by $\leq$ such that every infinite parameter-definable set is $\leq$-dense.

This construction preserves ω-categoricity and NIP.

Observation

In this construction, every definable **closed** subset of M^n (in the topology induced by $\leq$) is already definable from $\leq$ alone.

Adding New Closed Sets

Question

Which ω-categorical expansions add new closed 0-definable sets?

Examples

- **Convex predicate**: $(M, \leq, P)$, $P \subseteq M$ convex, no endpoints.
  $P \subseteq M$ is a new closed definable set.
- **Convex equivalence relation**: $(M, \leq, E)$
  $E \subseteq M^2$ is a new closed definable set.
- **Convex C-relation**: $(M, \leq, C)$
  $C \subseteq M^3$ is a new closed definable set.

Convex C-Relations

Definition (Collection of Balls)

A **collection of balls** $\mathcal{B}$ is a set of non-empty convex subsets of M, called *balls*, such that any two balls are either:

- disjoint, or
- one is contained in the other.

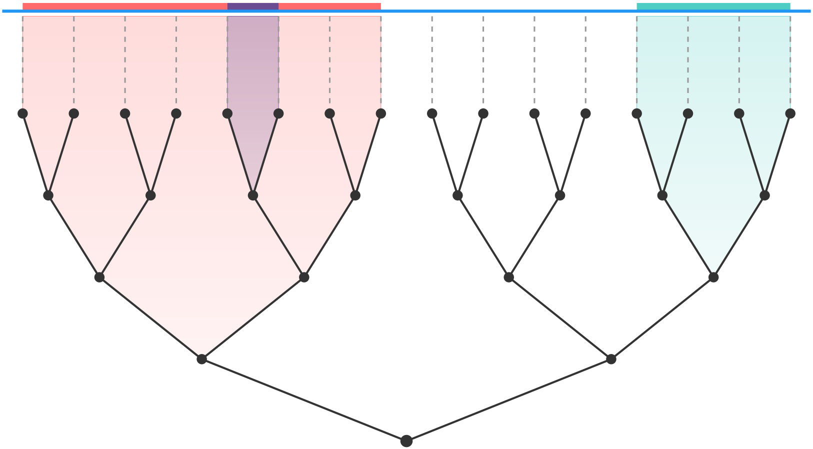
The structure is *non-trivial* if some ball is neither a singleton nor all of M.

C-Relation

Given $\mathcal{B}$, define:

$$C(x; y, z) \iff \text{there is a ball containing } y, z \text{ but not } x$$

C-Structure Visualization



The Main Question

Question

Let $(M, \leq, \ldots)$ be an ω-categorical expansion of DLO (optionally NIP).

Does $(P_3) \Rightarrow (P_n)$ for all n?

where

Property (P_n)

Every closed 0-definable subset of M^n is 0-definable from the order alone.

Proposition

The result holds if there is no parameter-definable convex C-structure whose tree contains an infinite binary tree.

From Shelah's classification theory: for T superstable, given a regular type p, we get a pregeometry on realizations of p, and from that we get a dimension.
Dimensions of the various regular types give invariants for models M of T.

What about for NIP?

Intuition:

$$\boxed{\text{NIP} = \text{Stable} + \text{Linear Orders}}$$

Theorem (S.)

Let T be NIP, unstable. Then there exists:

- *a finite set A,*
- *a relation $R(x, y)$ type-definable over A,*

such that R defines a strict quasi-order with an infinite chain.

Corollary

Every unstable ω-categorical NIP structure interprets an infinite linear order.

Hope

One can develop a classification theory for (subclasses of) NIP, similar to the one for superstable, where dimensions will be replaced by linear orders.

One difficulty is to understand the induced structure on the linear orders obtained above.

Ranked structure

Assume M has *no trees* in a strong sense:

- There is a notion of **rank** on definable sets, which measures the maximal depth of a chain of definable equivalence relations with infinitely many classes.
- Formally: M is *rosy* of finite rank.

Theorem (S.)

*Given $n < \omega$, there are **finitely many** structures (up to bi-definability) which are:*

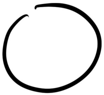
- *ω-categorical, NIP;*
- *rank 1, primitive;*
- *at most n 4-types.*

One can give an explicit list.

Joint work with Onshuus: Higher rank structures exhibit *coordinatization* and admit *distal expansions*.

arity 2 dimension 1

(+ ↔)



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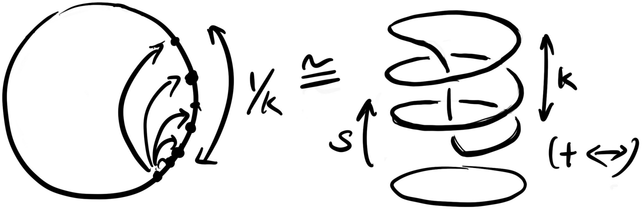
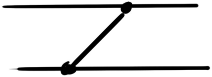
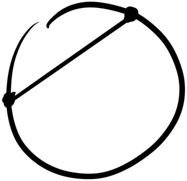


arity 2	dimension 2
	$(M, \leq_1, \leq_2)$
	$(M, \leq_1, \leq_2, R)$
	+ finite covers

⋮

Thank you!

Happy 80th Birthday, Saharon!