# Many ~~Some~~ Combinatorial Questions About Higher Baire Spaces

Tristan van der Vlugt (TU Wien)

*Vorst* ♔ ❄

Wien, Austria

July 17, 2025

The *classical Baire space* ${}^{\omega}\omega$ is the set of functions $f\colon \omega \to \omega$.
The *bounded* (or *product*) topology on ${}^{\omega}\omega$ is generated by basic open sets for each $s \in {}^{<\omega}\omega$ :

$$[s] = \{f \in {}^{\omega}\omega \mid s \subseteq f\}.$$

Let $\kappa$ be uncountable.

The *higher Baire space* ${}^{\kappa}\kappa$ is the set of functions $f\colon \kappa \to \kappa$.
The *bounded* topology on ${}^{\kappa}\kappa$ is generated by basic open sets for each $s \in {}^{<\kappa}\kappa$ :

$$[s] = \{f \in {}^{\kappa}\kappa \mid s \subseteq f\}.$$

If $\kappa$ is regular, this is also the $<\kappa$-box topology.

# Dominating Numbers

Given $f, g \in {}^\omega\omega$, we define $f \leq^* g$ if

$$\left|\{n \in \omega \mid f(n) > g(n)\}\right| < \aleph_0.$$

A set $B \subseteq {}^\omega\omega$ is *unbounded* if no $g$ exists with $f \leq^* g$ for all $f \in B$ . A set $D \subseteq {}^\omega\omega$ is *dominating* if every $f \in {}^\omega\omega$ has some $g \in D$ such that $f \leq^* g$.

Let $\mathfrak{b}$ be the least cardinality of an unbounded set, and let $\mathfrak{d}$ be the least cardinality of a dominating set.

Given $f, g \in {}^\kappa\kappa$, we define $f \leq^* g$ if

$$\left|\{\alpha \in \kappa \mid f(\alpha) > g(\alpha)\}\right| < \kappa.$$

A set $B \subseteq {}^\kappa\kappa$ is *unbounded* if no $g$ exists with $f \leq^* g$ for all $f \in B$ . A set $D \subseteq {}^\kappa\kappa$ is *dominating* if every $f \in {}^\kappa\kappa$ has some $g \in D$ such that $f \leq^* g$.

Let $\mathfrak{b}_\kappa$ be the least cardinality of an unbounded set, and let $\mathfrak{d}_\kappa$ be the least cardinality of a dominating set.

**Theorem**

For each *regular* uncountable $\kappa$ choose cardinals $\beta_\kappa$, $\delta_\kappa$, $\mu_\kappa$ such that:

$$\kappa^+ \le \mathrm{cf}(\beta_\kappa) = \beta_\kappa \le \delta_\kappa \le \mu_\kappa ,$$

$$\kappa^+ \le \mathrm{cf}(\mu_\kappa) ,$$

$$\forall \kappa, \kappa' (\kappa < \kappa' \to \mu_\kappa \le \mu_{\kappa'}) .$$

Then it is consistent that for all regular $\kappa$ we have each of $\mathfrak{b}_\kappa = \beta_\kappa$ and $\mathfrak{d}_\kappa = \delta_\kappa$ and $2^\kappa = \mu_\kappa$ .

---

Cummings and Shelah (1995). “Cardinal invariants above the continuum”.

Let $\kappa$ be regular uncountable. For $f, g \in {}^{\kappa}\kappa$, we define $f \leq^{\mathrm{cl}} g$ if

$$\{\alpha \in \kappa \mid f(\alpha) > g(\alpha)\} \text{ is nonstationary.}$$

Let $\mathfrak{b}_\kappa^{\mathrm{cl}}$ and $\mathfrak{d}_\kappa^{\mathrm{cl}}$ be the $\leq^{\mathrm{cl}}$-bounding and $\leq^{\mathrm{cl}}$-dominating numbers.

**Theorem**

$\mathfrak{b}_\kappa = \mathfrak{b}_\kappa^{\mathrm{cl}}$ and $\mathfrak{d}_\kappa^{\mathrm{cl}} \leq \mathfrak{d}_\kappa \leq \left(\mathfrak{d}_\kappa^{\mathrm{cl}}\right)^{\aleph_0}$, hence if $\kappa > \beth_\omega$ then $\mathfrak{d}_\kappa = \mathfrak{d}_\kappa^{\mathrm{cl}}$ .

**Question**

Is $\mathfrak{d}_\kappa = \mathfrak{d}_\kappa^{\mathrm{cl}}$ for $\kappa \leq \beth_\omega$?

---

Cummings and Shelah (1995). “Cardinal invariants above the continuum”.

Let $\kappa$, $\mu$ be cardinals. Given $f, g \in {}^{\mu}\kappa$, we define

$$f \leq_\lambda g \;\Leftrightarrow\; \Big|\{\alpha \in \mu \mid f(\alpha) > g(\alpha)\}\Big| < \lambda\,,$$

$$f \leq_{\mathrm{a}} g \;\Leftrightarrow\; \forall \alpha \in \mu(f(\alpha) \leq g(\alpha))\,,$$

$$f \leq_{\mathrm{b}} g \;\Leftrightarrow\; \exists \beta \in \mu\Big(\{\alpha \in \mu \mid f(\alpha) > g(\alpha)\} \subseteq \beta\Big)\,.$$

Let $\mathfrak{d}_{\mu_\kappa} := \mathfrak{d}^{\kappa}_{\mu_\kappa}$ and $\mathfrak{d}^{\mathrm{a}}_{\mu_\kappa}$ and $\mathfrak{d}^{\mathrm{b}}_{\mu_\kappa}$ be their dominating numbers.

**Theorems**

If $\kappa \leq \mathrm{cf}(\mu)$, then $\mathfrak{d}_{\mu_\kappa} = \mathfrak{d}^{\mathrm{a}}_{\mu_\kappa} = \mathfrak{d}^{\mathrm{b}}_{\mu_\kappa}$ .

If $\kappa$ is regular and $\kappa < \mu$, then $\mathfrak{d}_{\mu_{\kappa^+}} \leq \mathfrak{d}_{\mu_\kappa}$ .

---

Brendle (2022). “The higher Cichoń diagram in the degenerate case”.

**Question**

Let $\kappa < \lambda \leq \mu$, with $\kappa$, $\lambda$ regular. Is $\mathfrak{d}_{\mu_\lambda} \leq \mathfrak{d}_{\mu_\kappa}$ ?

Is $\mathfrak{d}_\mu \leq \mathfrak{d}_{\mu_\kappa}$ for regular $\mu$ ?

**Theorem**

Consistently $\mathfrak{d}_{\mu_\kappa} < 2^\mu$ for $\kappa$ regular uncountable.

**Question**

Is $\mathfrak{d}_{\mu_\kappa} = 2^\mu$ when $\max\{\kappa, 2^{<\kappa}\} < \mu$ for regular $\kappa$? Is $\mathfrak{d}_{\omega_{1\omega}} = 2^{\aleph_1}$ ?

**Theorem**

If $\mu$ is inaccessible and $\kappa < \mu$, then $\mathfrak{d}_{\mu_\kappa} = 2^\mu$.

---

Brendle (2022). “The higher Cichoń diagram in the degenerate case”.

Jech and Přikrý (1979). “Ideals over Uncountable Sets: Application of Almost Disjoint Functions and Generic Ultrapowers”.

vdV. (2025). “Cardinal Characteristics on Bounded Generalised Baire Spaces”.

Let $\kappa$ be regular uncountable. For $f, g \in {}^\kappa\kappa$, we define $f \neq^\infty g$ if

$$\left|\{\alpha \in \kappa \mid f(\alpha) = g(\alpha)\}\right| < \kappa.$$

Let $\mathfrak{d}_\kappa(\neq^\infty)$ and $\mathfrak{b}_\kappa(\neq^\infty)$ be the least cardinality of respectively $\neq^\infty$-dominating and $\neq^\infty$-unbounded sets.

**Theorem**

If $\kappa$ is successor, then $\mathfrak{b}_\kappa(\neq^\infty) = \mathfrak{b}_\kappa$ .
If $\kappa$ is successor and $2^{<\kappa} = \kappa$, then $\mathfrak{d}_\kappa(\neq^\infty) = \mathfrak{d}_\kappa$ .

**Question**

If $\kappa$ is successor and $\kappa < 2^{<\kappa}$, is $\mathfrak{d}_\kappa(\neq^\infty) = \mathfrak{d}_\kappa$ ?

Hyttinen (2006). “Cardinal Invariants and Eventually Different Functions”.

Matet and Shelah (preprint). “Positive partition relations for $P_\kappa(\lambda)$”.

# The Meagre Ideal

## Higher Meagre Sets



A $\kappa$-union of nowhere dense subsets of ${}^{\kappa}\kappa$ is called $\kappa$-*meagre*.
Let $\mathcal{M}_\kappa$ be the set of $\kappa$-meagre subsets of ${}^{\kappa}\kappa$.

We write $\mathrm{non}(\mathcal{M}_\kappa)$ for the least size of $X \subseteq {}^{\kappa}\kappa$ with $X \notin \mathcal{M}_\kappa$.
We write $\mathrm{cov}(\mathcal{M}_\kappa)$ for the least size of $\mathcal{X} \subseteq \mathcal{M}_\kappa$ with $\bigcup \mathcal{X} = {}^{\kappa}\kappa$.

$(\omega)$ **Theorem**

$\mathrm{non}(\mathcal{M}) = \mathfrak{b}(\not=^\infty)$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}(\not=^\infty)$.

**Theorem**

$\mathrm{non}(\mathcal{M}_\kappa) \geq \mathfrak{b}_\kappa(\not=^\infty)$ and $\mathrm{cov}(\mathcal{M}_\kappa) \leq \mathfrak{d}_\kappa(\not=^\infty)$ for all regular $\kappa$, with equality for inaccessible $\kappa$.

---

Bartoszyński (1987). “Combinatorial aspects of measure and category”.

Landver (1992). “Baire numbers, uncountable Cohen sets and perfect-set forcing”.

Blass, Hyttinen, and Zhang (preprint). “Mad families and their neighbors”.

**Corollary**

$\mathrm{cov}(\mathcal{M}_\kappa) \leq \mathfrak{d}_\kappa$ and $\mathfrak{b}_\kappa \leq \mathrm{non}(\mathcal{M}_\kappa)$.

**Theorem**

If $\kappa$ is supercompact, then $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{d}_\kappa$ can be forced.

**Question**

Is $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{d}_\kappa$ consistent for inaccessible $\kappa$?

Is $\mathrm{non}(\mathcal{M}_\kappa) > \mathfrak{b}_\kappa$ consistent for inaccessible $\kappa$?

**Question**

How about simultaneous consistency of these inequalities?

Stronger yet, of $\mathfrak{b}_\kappa < \mathrm{non}(\mathcal{M}_\kappa) < \mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{d}_\kappa$ ?

---


Shelah (2020). “On $\mathrm{CON}(\mathfrak{d}_\lambda > \mathrm{cov}_\lambda(\mathrm{meagre}))$”.

vdV. (2025). “The Horizontal Direction”.

# The Accessible Case



**Theorem**

If $\kappa < 2^{<\kappa}$ (or if $\kappa$ is singular), then $\mathrm{cov}(\mathcal{M}_\kappa) = \mathrm{cf}(\kappa)^+$.

**Theorem**

$2^{<\kappa} \leq \mathrm{non}(\mathcal{M}_\kappa)$.

**Corollary**

$\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{d}_\kappa(\cancel{=\infty}) = \mathfrak{b}_\kappa(\cancel{=\infty}) < 2^{<\kappa} \leq \mathrm{non}(\mathcal{M}_\kappa)$ is consistent for accessible $\kappa$.

**Question**

Are $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{d}_\kappa$ or $\mathfrak{b}_\kappa < \mathrm{non}(\mathcal{M}_\kappa)$ consistent for any successor $\kappa = 2^{<\kappa}$ ?

Landver (1992). "Baire numbers, uncountable Cohen sets and perfect-set forcing".

Blass, Hyttinen, and Zhang (preprint). "Mad families and their neighbors".

Matet and Shelah (preprint). "Positive partition relations for $P_\kappa(\lambda)$".

Let $\kappa$ be regular, $I = \langle i_\alpha \mid \alpha \in \kappa \rangle$ enumerate a club with $i_0 = 0$ and $f, g \in {}^\kappa 2$. We call $(g, I)$ a *chopped* $\kappa$*-sequence* and we say that $f$ *matches* $(g, I)$ if there is $A \in [\kappa]^\kappa$ such that $f$ and $g$ agree on $\bigcup_{\alpha \in A}[i_\alpha, i_{\alpha+1})$. Let $\mathcal{M}_\kappa^{\mathrm{c}}$ consist of all $X \subseteq {}^\kappa 2$ for which there is $(g, I)$ not matched by any $f \in X$.

**Theorem**

$\mathcal{M}_\kappa^{\mathrm{c}} \subseteq \mathcal{M}_\kappa$, with equality iff $\kappa$ is inaccessible.

**Question**

Are $\mathrm{cov}(\mathcal{M}_\kappa) < \mathrm{cov}(\mathcal{M}_\kappa^{\mathrm{c}})$ or $\mathrm{non}(\mathcal{M}_\kappa^{\mathrm{c}}) < \mathrm{non}(\mathcal{M}_\kappa)$ consistent with $\kappa$ accessible?

---

Blass, Hyttinen, and Zhang (preprint). “Mad families and their neighbors”.

For $(g, I)$ a *chopped $\kappa$-sequence*, we say that $f$ *stat-matches* $(g, I)$ if there is *stationary* $A \in [\kappa]^\kappa$ such that $f$ and $g$ agree on $\bigcup_{\alpha \in A}[i_\alpha, i_{\alpha+1})$. Let $\mathcal{M}_\kappa^{\mathrm{s}}$ consist of all $X \subseteq {}^\kappa 2$ for which there is $(g, I)$ not stat-matched by any $f \in X$.

**Theorem**

$\mathcal{M}_\kappa \nsubseteq \mathcal{M}_\kappa^{\mathrm{s}}$.

If we assume $\diamondsuit_\kappa^*$, then $\mathcal{M}_\kappa^{\mathrm{s}} \subseteq \mathcal{M}_\kappa$.

**Question**

What more can we say about $\mathcal{M}_\kappa^{\mathrm{s}}$ ?

Blass, Hyttinen, and Zhang (preprint). “Mad families and their neighbors”.

# Cofinality and Additivity



Let $\mathrm{cof}(\mathcal{M}_\kappa)$ be the least size of a $\subseteq$-cofinal subset of $\mathcal{M}_\kappa$.
Let $\mathrm{add}(\mathcal{M}_\kappa)$ be the least size of $X \subseteq \mathcal{M}_\kappa$ with $\bigcup X \notin \mathcal{M}_\kappa$.

**Theorem**

$\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathrm{cov}(\mathcal{M}_\kappa), \mathfrak{b}_\kappa\}$.

$\max\{\mathrm{non}(\mathcal{M}_\kappa), \mathfrak{d}_\kappa\} \leq \mathrm{cof}(\mathcal{M}_\kappa)$, with equality if $2^{<\kappa} = \kappa$.

**Theorem**

$\max\left\{\mathrm{non}(\mathcal{M}_\kappa), \mathfrak{d}_{\mu_\kappa}\right\} \leq \mathrm{cof}(\mathcal{M}_\kappa)$ where $\mu = |2^{<\kappa}|$.

**Question**

Is the above inequality consistently strict?

---

Brendle, Brooke-Taylor, Friedman, and Montoya (2018). “Cichoń’s diagram for uncountable cardinals”.

Brendle (2022). “The higher Cichoń diagram in the degenerate case”.

# Localisation

For $h \in {}^\kappa\kappa$, an $h$-*slalom* $\varphi$ is an element of $\prod_{\alpha\in\kappa}[\kappa]^{\leq|h(\alpha)|}$.
Hence, $\operatorname{dom}(\varphi) = \kappa$ and $\varphi(\alpha) \subseteq \kappa$ with $|\varphi(\alpha)| \leq |h(\alpha)|$.
Given $f \in {}^\kappa\kappa$, we write $f \in^* \varphi$ if

$$\left|\{\alpha \in \kappa \mid f(\alpha) \notin \varphi(\alpha)\}\right| < \kappa.$$

Let $\mathfrak{d}^h_\kappa(\in^*)$ be the least $\in^*$-dominating subset of $\prod_{\alpha\in\kappa}[\kappa]^{\leq|h(\alpha)|}$.
Let $\mathfrak{b}^h_\kappa(\in^*)$ be the least $\in^*$-unbounded subset of ${}^\kappa\kappa$.

$(\omega)$ **Theorem**

If $h\colon \omega \to \omega \setminus \{0, 1\}$ has $h(n) \to \infty$ as $n \to \infty$, then
$\mathfrak{d}^h_\omega(\in^*) = \operatorname{cof}(\mathcal{N})$ and $\mathfrak{b}^h_\omega(\in^*) = \operatorname{add}(\mathcal{N})$.

---

Bartoszyński (1987). “Combinatorial aspects of measure and category”.

**Theorem**

Let $\kappa$ be inaccessible and $h_0 : \alpha \mapsto |\alpha|$ and $h_1 : \alpha \mapsto 2^{|\alpha|}$.
In the $\kappa$-Sacks model $\mathfrak{d}^{h_1}_\kappa(\in^*) < \mathfrak{d}^{h_0}_\kappa(\in^*)$.

**Question**

Is $\mathfrak{b}^{h_0}_\kappa(\in^*) < \mathfrak{b}^{h_1}_\kappa(\in^*)$ consistent for some $h_0, h_1 \in {}^\kappa\kappa$ ?

**Theorem**

Let $\kappa$ be inaccessible, $\kappa = 2^{<\kappa}$ and assume $\diamondsuit_\kappa$ holds, then there are $h_\xi \in {}^\kappa\kappa$ for each $\xi \in \kappa^+$ and a forcing notion that forces that all $\mathfrak{d}^{h_\xi}_\kappa(\in^*)$ are mutually distinct cardinalities.

---

Brendle, Brooke-Taylor, Friedman, and Montoya (2018). “Cichoń’s diagram for uncountable cardinals”.

vdV. (2024). “Separating many localisation cardinals on the generalised Baire space”.

# Bounded Localisation and Anti-Localisation



If $b \in {}^{\kappa}\kappa$, a $(b,h)$-*slalom* is an element of $\prod_{\alpha\in\kappa}[b(\alpha)]^{\leq|h(\alpha)|}$.
For a slalom $\varphi$ we also define $f \in^{\infty} \varphi$ if

$$\Big|\{\alpha \in \kappa \mid f(\alpha) \in \varphi(\alpha)\}\Big| = \kappa.$$

Consider $\mathfrak{d}_{\kappa}^{b,h}(\in^*)$, $\mathfrak{b}_{\kappa}^{b,h}(\in^*)$, $\mathfrak{d}_{\kappa}^{b,h}(\not\ni^{\infty})$, and $\mathfrak{b}_{\kappa}^{b,h}(\not\ni^{\infty})$.

$(\omega)$ **Theorem**

There is a forcing extension in which there exists $\langle (b_\xi, h_\xi) \mid \xi \in 2^{\aleph_0} \rangle$ such that $\square_{\omega}^{b_\xi,h_\xi}(\circ) \neq \square_{\omega}^{b_{\xi'},h_{\xi'}}(\circ)$ for all $\xi \neq \xi'$ and $\square \in \{\mathfrak{b}, \mathfrak{d}\}$ and $\circ \in \{\in^*, \not\ni^{\infty}\}$.

Cardona, Klausner, and Mejía (2021). “Continuum Many Different Things: Localisation, Anti-Localisation and Yorioka Ideals”.

Goldstern and Shelah (1993). “Many simple cardinal invariants”.

**Theorem**

For $\kappa$ inaccessible, there is $\langle (b_\xi, h_\xi) \mid \xi \in \kappa \rangle$ such that for any $A \in [\kappa]^{<\omega}$ it is consistent that $\mathfrak{d}_\kappa^{b_\xi, h_\xi}(\not\ni^\infty) \neq \mathfrak{d}_\kappa^{b_{\xi'}, h_{\xi'}}(\not\ni^\infty)$ for all distinct $\xi, \xi' \in A$.

**Question**

Can we get a similar result for $\mathfrak{b}_\kappa^{b_\xi, h_\xi}(\not\ni^\infty)$?

**Question**

Are there consistently $2^\kappa$ many distinct such cardinal characteristics? What about (unbounded) $\mathfrak{d}_\kappa^h(\in^*)$ ?

---

vdV. (2025). “Cardinal Characteristics on Bounded Generalised Baire Spaces”.

# Random Forcing

Random forcing is c.c.c. and ${}^{\omega}\omega$-bounding.

Shelah asked if there exists a forcing notion that is a $<\kappa$-closed $<\kappa^+$-c.c. forcing notion that is ${}^{\kappa}\kappa$-bounding.

There are several solutions:

- Friedman and Laguzzi (2017). *“A null ideal for inaccessibles”*.
- Cohen and Shelah (2019). *“Generalizing random real forcing for inaccessible cardinals”*.
- Shelah (2017). *“A parallel to the null ideal for inaccessible $\lambda$: Part I”*.
  Baumhauer, Goldstern, and Shelah (2020). *“The Higher Cichoń Diagram”*.

A set $S \subseteq \kappa$ is *nowhere stationary (nwst)* if $S \cap \alpha$ is not stationary (in $\alpha$) for any limit $\alpha \leq \kappa$ with $\mathrm{cf}(\alpha) > \aleph_0$. Let $\mathscr{I}$ denote the class of inaccessible cardinals.

We recursively define $\mathbb{Q}_\kappa$ and an ideal $\mathcal{N}_\kappa$ for each $\kappa \in \mathscr{I}$. For a nwst set $S \subseteq \mathscr{I} \cap \kappa$ and function $N \in \prod_{\lambda \in S} \mathcal{N}_\lambda$, we define a tree $T_N \subseteq {}^{<\kappa}2$ by recursion on the levels of $T_N$:

- Let $\varnothing \in T_N$.
- For $s \in {}^{\alpha}2$ with $\alpha \notin S$, let $s \in T_N$ iff $s \restriction \xi \in T_N$ for all $\xi < \alpha$.
- For $s \in {}^{\lambda}2$ with $\lambda \in S$, let $s \in T_N$ iff $s \notin N(\lambda)$.

Define $\mathbb{Q}_\kappa$ as all trees $T \subseteq {}^{<\kappa}2$ such that there is $N$ as above and $s \in T_N$ for which $T = (T_N)_s$. We order $\mathbb{Q}_\kappa$ by inclusion, and define $\mathcal{N}_\kappa$ as the ideal $\mathcal{I}_{\mathbb{Q}_\kappa}$.

# Properties of $\mathbb{Q}_\kappa$

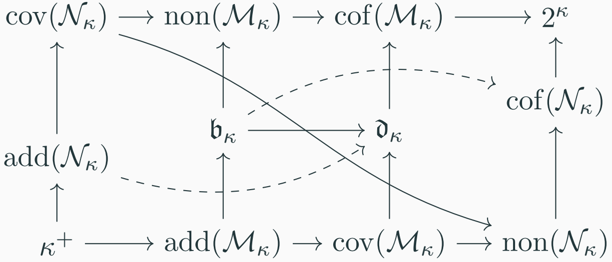
If $\kappa \in \mathscr{I}$ is *not* $1$-inaccessible ($=$ a limit of inaccessibles), then $\mathbb{Q}_\kappa$ is forcing equivalent to $\mathbb{C}_\kappa$. Hence $\mathcal{N}_\kappa = \mathcal{M}_\kappa$.

If $\kappa$ is $1$-inaccessible, then $\mathcal{N}_\kappa \neq \mathcal{M}_\kappa$ and furthermore:

- $\mathbb{Q}_\kappa$ is strategically $<\kappa$-closed and $<\kappa^+$-c.c.
- If $\kappa$ is *weakly compact*, then $\mathbb{Q}_\kappa$ is ${}^\kappa\kappa$-bounding.
- There are $A \in \mathcal{N}_\kappa$ and $B \in \mathcal{M}_\kappa$ with ${}^\kappa 2 = A \cup B$,
- If $\mathbb{P}$ is $(\kappa, <\kappa)$-centred and $<\kappa$-distributive, then $\mathbb{P}$ does not add a $\mathbb{Q}_\kappa$-generic.

However, Fubini's theorem fails for $\mathcal{N}_\kappa$, consequently $\mathrm{cov}(\mathcal{N}_\kappa) \leq \mathrm{non}(\mathcal{N}_\kappa)$.

Let $\kappa$ be an inaccessible limit of inaccessibles.




The dashed arrows require $\kappa$ to be Mahlo.

**Question**

Can the use of Mahlo be weakened?

**Question**

Are $\mathrm{add}(\mathcal{N}_\kappa) \leq \mathrm{add}(\mathcal{M}_\kappa)$ and $\mathrm{cof}(\mathcal{M}_\kappa) \leq \mathrm{cof}(\mathcal{N}_\kappa)$?

Easier: are $\mathrm{add}(\mathcal{N}_\kappa) \leq \mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa \leq \mathrm{cof}(\mathcal{N}_\kappa)$?

**Question**

Is $\mathrm{add}(\mathcal{N}_\kappa) = \mathfrak{b}^h_\kappa(\in^*)$ or $\mathrm{cof}(\mathcal{N}_\kappa) = \mathfrak{d}^h_\kappa(\in^*)$ for some $h \in {}^\kappa\kappa$ ?

**Question**

How to add many $\mathbb{Q}_\kappa$-generics in a ${}^\kappa\kappa$-bounding way?

**Question**

Generally, how can the ${}^\kappa\kappa$-bounding property be preserved under iterated forcing?

# Maximal Almost Disjoint Families

We call $\mathcal{A} \subseteq [\kappa]^\kappa$ *almost disjoint* (AD) if $|A \cap B| < \kappa$ for all distinct $A, B \in \mathcal{A}$. If $\mathcal{A} \subseteq [\kappa]^\kappa$ is AD and no $\mathcal{A} \subsetneq \mathcal{A}' \subseteq [\kappa]^\kappa$ is AD, we call $\mathcal{A}$ *maximal* AD (MAD).

Define $\mathfrak{a}_\kappa$ as the least size of a MAD family on $[\kappa]^\kappa$.

**Theorem**

$\mathfrak{b}_\kappa \leq \mathfrak{a}_\kappa$ for all regular $\kappa$.

**Question**

Is $\mathfrak{b}_\kappa < \mathfrak{a}_\kappa$ consistent for all regular $\kappa$?

$(\omega)$ **Theorem**

It is consistent that $\mathfrak{b} \leq \mathfrak{d} < \mathfrak{a}$.

---

Shelah (2004). “Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and FS linearly ordered iterated forcing”.

**Theorem**

For $\kappa$ supercompact, it is consistent that $\mathfrak{b}_\kappa < \mathfrak{d}_\kappa < \mathfrak{a}_\kappa$.

$(\omega)$ **Question** *Roitman*

Is $\mathfrak{d} < \mathfrak{a}$ consistent with $\mathfrak{d} = \omega_1$?

**Theorem**

Let $\kappa$ be regular, then $\mathfrak{d}_\kappa = \kappa^+$ implies $\mathfrak{a}_\kappa = \kappa^+$.

In fact, $\mathfrak{b}_\kappa = \kappa^+$ implies $\mathfrak{a}_\kappa = \kappa^+$.

---

Raghavan and Shelah (in preparation). "Boolean Ultrapowers and Iterated Forcing".

Blass, Hyttinen, and Zhang (preprint). "Mad families and their neighbors".

Raghavan and Shelah (2019). "Two Results on Cardinal Invariants at Uncountable Cardinals".

Define $\mathrm{spec}(\mathfrak{a}_\kappa) = \{|A| \mid A \in [\kappa]^\kappa$ is MAD$\}$.

We call a set of cardinals $B$ a $\kappa$-*Blass spectrum* if

- $\kappa^+ \in B$ and $B \cap \kappa^+ = \varnothing$,
- $\sup(B \cap \lambda) \in B$ for all $\lambda$,
- $\lambda^+ \in B$ for all $\lambda \in B$ with $\mathrm{cf}(\lambda) \leq \kappa$,
- $\lambda \in B$ for all $\lambda \geq \kappa^+$ with $\lambda \leq |B|$.

**Theorem**

Assume GCH, let $\kappa$ be regular and $B$ a $\kappa$-Blass spectrum, then $\mathrm{spec}(\mathfrak{a}_\kappa) = B$ holds in a cofinality-preserving extension.

Blass (1993). “Simple Cardinal Characteristics of the Continuum”.

Fischer (2015). “Maximal Cofinitary Groups Revisited”.

$(\omega)$ **Theorem**

Let $B$ be a set of uncountable cardinals, closed under singular limits such that $\sup(B) = \sup(B)^{\aleph_0} = \sup(B)^{<\min(B)} \in B$ and $\min(B)^{<\min(B)} = \min(B)$, then there is a c.c.c. forcing that forces $\mathrm{spec}(\mathfrak{a}) = B$.

Note that if $B$ is as above, then $\min(B)$ is regular, thus the above theorem is not optimal (see slide #28).

**Question**

Can we generalise the consistent values for $\mathrm{spec}(\mathfrak{a}_\kappa)$ beyond Blass spectra?

---

Shelah and Spinas (2015). "MAD Spectra".

**Theorem**

Assume GCH. Let $C = \{\aleph_0\} \cup \{\kappa^+ \mid \kappa \text{ is regular}\}$. If $B_\kappa$ is a $\kappa$-Blass spectrum for each $\kappa \in C$, then there is a cardinal-preserving forcing notion that forces $\mathrm{spec}(\mathfrak{a}_\kappa) = B_\kappa$ for all $\kappa \in C$.

**Theorem**

Assume GCH. Let $E$ be an *Easton function* such that $\lambda, \kappa \in \mathrm{dom}(E)$ and $\lambda < \kappa$ implies $E(\lambda) \leq \kappa^+$, then $\forall \kappa \in E(\mathrm{spec}(\mathfrak{a}_\kappa) = \{\kappa^+, E(\kappa)\})$ is consistent.

**Question**

Can we weaken the assumptions in these Theorems?

---

Bağ, Fischer, and Friedman (submitted). "Global MAD Spectra".

$(\omega)$ **Theorem**

It is consistent that $\mathfrak{a}$ has countable cofinality.

**Question**

Is it consistent that $\mathfrak{a}_\kappa$ is singular?

**Question**

Is $\mathfrak{a}_\kappa \neq \mathfrak{a}_\lambda$ consistent with $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda) < \kappa < \lambda$ ?

**Question**

Is $\mathfrak{a}_\kappa \neq \mathfrak{a}_\lambda$ for consistent for $\aleph_0 < \mathrm{cf}(\lambda) = \kappa < \lambda$ ?

---

Brendle (2003). "The almost-disjointness number may have countable cofinality".

Brendle (2013). "Some problems concerning mad families".

**Theorem**

Let $\kappa$ be singular, then $\kappa \in \mathrm{spec}(\mathfrak{a}_\kappa)$ if $\lambda^{\mathrm{cf}(\kappa)} < \kappa$ for all $\lambda < \kappa$.

Let $\mathcal{C}_\kappa$ collect all cofinal sequences of regular cardinals in $\kappa$. For each $s = \langle \kappa_\xi \mid \xi \in \mathrm{cf}(\kappa)\rangle \in \mathcal{C}_\kappa$ let $\mathfrak{b}^s_\kappa$ be the least size of a $\leq^*$-unbounded set in $\prod_{\xi\in\mathrm{cf}(\kappa)} \kappa_\xi$. Let $\mathfrak{b}^{\sup}_\kappa = \sup\{\mathfrak{b}^s_\kappa \mid s \in \mathcal{C}_\kappa\}$.

**Theorem**

Let $\kappa$ be singular, then $\min\{\mathfrak{b}_{\mathrm{cf}(\kappa)}, \mathfrak{b}^{\sup}_\kappa\} \leq \mathfrak{a}_\kappa$, and thus consistently $\kappa < \mathfrak{a}_\kappa$. Also, if $\mathfrak{a}_\kappa \leq \lambda < \mathfrak{b}^{\sup}_\kappa$, then $\lambda \in \mathrm{spec}(\mathfrak{a}_\kappa)$.

Erdős and Hechler (1973). “On maximal almost-disjoint families over singular cardinals”.

Kojman, Kubiś, and Shelah (2004). “On two problems of Erdős and Hechler: New methods in singular madness”.

**Theorem**

If $\mathrm{cf}(\kappa) = \aleph_0 < \kappa$ then $\mathfrak{a}_\kappa < \mathfrak{a}$ is consistent.

**Question**

Is $\mathfrak{a}_\kappa = \kappa$ consistent for singular $\kappa$?

**Question**

Is $\mathfrak{b}_{\mathrm{cf}(\kappa)} \neq \mathfrak{a}_\kappa$ consistent for singular $\kappa$?

---

Brendle (in preparation). “Mad families on Singular cardinals”.

Montoya (submitted). “Maximal Almost Disjoint Families at Singular Cardinals”.

# Other Maximal Combinatorial Families

# Sets of Functions



Let $\mathrm{S}(\kappa)$ be the set of bijections in ${}^{\kappa}\kappa$ and let $\mathrm{S}_{<\kappa}(\kappa) = \{\mathrm{id}_\kappa\} \cup \{f \in \mathrm{S}(\kappa) \mid f \neq^\infty \mathrm{id}_\kappa\}$.

- $\mathfrak{a}^{\mathrm{e}}_\kappa$ is the least size of a maximal AD subset of ${}^{\kappa}\kappa$.
- $\mathfrak{a}^{\mathrm{p}}_\kappa$ is the least size of a maximal AD subset of $\mathrm{S}(\kappa)$.
- $\mathfrak{a}^{\mathrm{g}}_\kappa$ is the least size of a maximal AD subgroup of $\mathrm{S}_{<\kappa}(\kappa)$.

**Theorem**

For $\kappa$ regular, $\mathfrak{b}_\kappa \leq \min\{\mathfrak{a}^{\mathrm{e}}_\kappa, \mathfrak{a}^{\mathrm{p}}_\kappa, \mathfrak{a}^{\mathrm{g}}_\kappa\}$.

**Question**

Are $\mathfrak{a}^{\mathrm{e}}_\kappa$, $\mathfrak{a}^{\mathrm{p}}_\kappa$ and $\mathfrak{a}^{\mathrm{g}}_\kappa$ consistently different?

Is $\mathfrak{a}_\kappa < \max\{\mathfrak{a}^{\mathrm{e}}_\kappa, \mathfrak{a}^{\mathrm{p}}_\kappa, \mathfrak{a}^{\mathrm{g}}_\kappa\}$ consistent?

---

Blass, Hyttinen, and Zhang (preprint). “Mad families and their neighbors”.

$(\omega)$ **Theorem**

$\mathrm{non}(\mathcal{M}) \leq \min\{\mathfrak{a}_{\mathrm{e}}, \mathfrak{a}_{\mathrm{p}}, \mathfrak{a}_{\mathrm{g}}\}$.

**Theorem**

For $\kappa$ regular, $\mathfrak{b}_\kappa(\cancel{=\infty}) \leq \mathfrak{a}^{\mathrm{e}}_\kappa$. Hence $\mathrm{non}(\mathcal{M}_\kappa) \leq \mathfrak{a}^{\mathrm{e}}_\kappa$ holds for inaccessible $\kappa$.

**Question**

For $\kappa$ regular, is $\mathfrak{b}_\kappa(\cancel{=\infty}) \leq \min\{\mathfrak{a}^{\mathrm{p}}_\kappa, \mathfrak{a}^{\mathrm{g}}_\kappa\}$ ?

---

Brendle, Spinas, and Zhang (2000). “Uniformity of the Meager Ideal and Maximal Cofinitary Groups”.

Blass, Hyttinen, and Zhang (preprint). “Mad families and their neighbors”.

Call $\mathcal{A} \subseteq [\kappa]^\kappa$ an *independent* family if for every disjoint $\mathcal{B}, \mathcal{C} \in [\mathcal{A}]^{<\omega}$ we have $|\bigcap \mathcal{B} \setminus \bigcup \mathcal{C}| = \kappa$. Call $\mathcal{A} \subseteq [\kappa]^\kappa$ *strongly* independent if $|\bigcap \mathcal{B} \setminus \bigcup \mathcal{C}| = \kappa$ for all disjoint $\mathcal{B}, \mathcal{C} \in [\mathcal{A}]^{<\kappa}$.

Let $\mathfrak{i}_\kappa$ be the least size of a maximal independent family in $[\kappa]^\kappa$.

**Theorem**

If $\kappa$ is regular, $2^\kappa \in \operatorname{spec}(\mathfrak{i}_\kappa)$ and $\mathfrak{d}_\kappa \leq \mathfrak{i}_\kappa$.
Assume GCH and let $\kappa$ be measurable, then there is a forcing extension in which $\kappa^+ = \mathfrak{i}_\kappa < 2^\kappa$.

**Question**

Does the consistency of $\mathfrak{i}_\kappa < 2^\kappa$ imply $\kappa$ is measurable?
Is $\kappa^+ < \mathfrak{i}_\kappa < 2^\kappa$ consistent for measurable $\kappa$?

---

Fischer and Montoya (2022). “Higher Independence”.

**Theorem**

The existence of a maximal strongly independent family on $[\omega_1]^{\omega_1}$ is equiconsistent a measurable cardinal.

**Theorem**

Assume GCH and a proper class of measurables, then a measurable-preserving (class) forcing forces the existence of a maximal strongly independent family for each measurable.

**Question**

Is the least size of a maximal strongly independent family in $[\kappa]^\kappa$ related to $\mathfrak{i}_\kappa$?

Kunen (1983). “Maximal $\sigma$-Independent Families”.

Ryan-Smith (preprint). “Proper classes of maximal $\theta$-independent families from large cardinals”.

A family $\mathcal{A} \subseteq [\kappa]^\kappa$ has the $\kappa$-*intersection property* ($\kappa$-IP) if $|\bigcap \mathcal{B}| = \kappa$ for all $\mathcal{B} \in [\mathcal{A}]^{<\kappa}$. A *pseudointersection* of $\mathcal{A}$ is a set $B \in [\kappa]^\kappa$ such that $B \subseteq^* A$ for all $A \in \mathcal{A}$, where $B \subseteq^* A$ iff $B \setminus b \subseteq A$ for some $b \in [B]^{<\kappa}$. A $\kappa$-*tower* is a maximal family with the $\kappa$-IP that is well-ordered by $\subseteq^*$.

Let $\mathfrak{p}_\kappa$ be the least size of a family $\mathcal{A}$ with the $\kappa$-IP, but without a pseudointersection. Let $\mathfrak{t}_\kappa$ be the least size of a $\kappa$-tower.

Define $\mathfrak{p}_\kappa^{\mathrm{cl}}$ and $\mathfrak{t}_\kappa^{\mathrm{cl}}$ as above, but restricting to *club* sets in $[\kappa]^\kappa$.

**Theorem**

$\mathfrak{p}_\kappa^{\mathrm{cl}} = \mathfrak{t}_\kappa^{\mathrm{cl}} = \mathfrak{b}_\kappa$.

---

Schilhan (2019). “Generalised pseudointersections”.

$(\omega)$ **Theorem**

$\mathfrak{p} = \mathfrak{t}$.

**Theorems**

Let $\kappa$ be regular uncountable and $\kappa = \kappa^{<\kappa}$.

If $\mathfrak{p}_\kappa = \kappa^+$ or if $\mathrm{cf}(2^\kappa) \leq \kappa^{++}$, then $\mathfrak{p}_\kappa = \mathfrak{t}_\kappa$.

Furthermore, either $\mathfrak{p}_\kappa = \mathfrak{t}_\kappa$ or there is a club-supported $(\mathfrak{p}_\kappa, \lambda)$-gap of slaloms for some $\lambda < \mathfrak{p}_\kappa$.

**Question**

Is $\mathfrak{p}_\kappa = \mathfrak{t}_\kappa$ ?

---

Malliaris and Shelah (2016). "Cofinality spectrum theorems in model theory, set theory, and general topology".

Garti (preprint). "Pity on $\lambda$".

Fischer, Montoya, Schilhan, and Soukup (2022). "Towers and gaps at uncountable cardinals".

# Splitting Families

For $X, Y \in [\kappa]^\kappa$, we say $X$ splits $Y$ if $|Y \cap X| = |Y \setminus X| = \kappa$. Let $\mathfrak{s}_\kappa$ be the least size of a family $\mathcal{S} \subseteq [\kappa]^\kappa$ for which each $Y \in [\kappa]^\kappa$ is split by some $X \in \mathcal{S}$. Let $\mathfrak{r}_\kappa$ be the least size of a family $\mathcal{R} \subseteq [\kappa]^\kappa$ such that no $X \in [\kappa]^\kappa$ splits all $Y \in \mathcal{R}$.

**Theorem**

$\kappa \leq \mathfrak{s}_\kappa$ if and only if $\kappa$ is inaccessible.

**Theorem**

$\kappa^+ \leq \mathfrak{s}_\kappa$ if and only if $\kappa$ is weakly compact.

---

Motoyoshi (1992). "On the cardinalities of splitting families of uncountable regular cardinals".

Suzuki (1998). "About splitting numbers".

Zapletal (1997). "Splitting Number at Uncountable Cardinals".

**Theorem** *Kamo, '90s*

If $\kappa$ is supercompact, there is a forcing extension in which $\kappa$ is supercompact and $\kappa^{++} \leq \mathfrak{s}_\kappa$.

**Theorem**

If $\kappa^{++} \leq \mathfrak{s}_\kappa$ for $\kappa$ regular, then $\kappa$ is measurable with $o(\kappa) = \kappa^{++}$ in some inner model.

**Theorem**

Let $\kappa < \kappa^+ < \lambda$ be regular (with $\lambda \neq \mu^+$ for singular $\mu$).
If GCH holds and $o(\kappa) = \lambda$, then $\mathfrak{s}_\kappa = \lambda$ is consistent.
If $0^{\P}$ fails and $\mathfrak{s}_\kappa \geq \lambda$, then $o(\kappa) \geq \lambda$ holds in an inner model.

---

Zapletal (1997). "Splitting Number at Uncountable Cardinals".

Ben-Neria and Gitik (2015). "On the splitting number at regular cardinals".

**Question**

Let $\kappa^+ < \lambda$. Is $\mathfrak{s}_\kappa = \lambda$ equiconsistent with the existence of a measurable $\kappa$ with $o(\kappa) = \lambda$?

$(\omega)$ **Theorem**

It is consistent that $\mathfrak{s}$ is singular (of uncountable cofinality).

**Question**

Can $\mathfrak{s}_\kappa$ be singular for regular uncountable $\kappa$?

---

Ben-Neria and Gitik (2015). “On the splitting number at regular cardinals”.

Dow and Shelah (2018). “On the cofinality of the splitting number”.

($\omega$) **Theorem**

It is consistent that $\mathfrak{b} = \mathfrak{r} = \mathfrak{u} < \mathfrak{s}$.

**Theorem**

If $\kappa$ is regular uncountable, then $\mathfrak{s}_\kappa \leq \mathfrak{b}_\kappa$.

**Theorem**

If $\kappa > \beth_\omega$ be regular, then $\mathfrak{d}_\kappa \leq \mathfrak{r}_\kappa$. So, $\mathfrak{s}_\kappa \leq \mathfrak{b}_\kappa \leq \mathfrak{d}_\kappa \leq \mathfrak{r}_\kappa \leq \mathfrak{u}_\kappa$.

**Question**

Is $\mathfrak{d}_\kappa \leq \mathfrak{r}_\kappa$ for all regular uncountable $\kappa$?

---

Blass and Shelah (1987). “There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed”.

Raghavan and Shelah (2017). “Two inequalities between cardinal invariants”.

Raghavan and Shelah (2019). “Two Results on Cardinal Invariants at Uncountable Cardinals”.

Define $\mathfrak{s}_\kappa^{\mathrm{cl}}$ and $\mathfrak{r}_\kappa^{\mathrm{cl}}$ by weakening "$X$ splits $Y$" to "$X$ *stationarily* splits $Y$", that is, $Y \cap X$ and $Y \setminus X$ are both stationary.

**Theorem**

$\kappa \leq \mathfrak{r}_\kappa^{\mathrm{cl}}$ and $\kappa^+ < \mathfrak{r}_\kappa^{\mathrm{cl}} < 2^\kappa$ is consistent.

**Question**

Is $\mathfrak{r}_\kappa^{\mathrm{cl}} = \kappa$ consistent?

---

Schürz (preprint). "Cardinal characteristics on $\kappa$ modulo non-stationary".

**Theorems**

$\kappa \leq \mathfrak{s}_\kappa^{\mathrm{cl}}$ if and only if $\kappa$ is inaccessible.

$\kappa^+ \leq \mathfrak{s}_\kappa^{\mathrm{cl}}$ if and only if $\kappa$ is ineffable.

$\kappa^{++} \leq \mathfrak{s}_\kappa^{\mathrm{cl}}$ implies $\kappa$ is measurable and in some inner model $o(\kappa) = \kappa^{++}$ holds.

**Question**

How do $\mathfrak{s}_\kappa^{\mathrm{cl}}$ and $\mathfrak{r}_\kappa^{\mathrm{cl}}$ relate to $\mathfrak{s}_\kappa$ and $\mathfrak{r}_\kappa$?

**Question**

Are $\mathfrak{b}_\kappa < \mathfrak{s}_\kappa^{\mathrm{cl}}$ and $\mathfrak{r}_\kappa^{\mathrm{cl}} < \mathfrak{d}_\kappa$ consistent?

---

Schürz (preprint). “Cardinal characteristics on $\kappa$ modulo non-stationary”.

# Ultrafilters

Let $\mathfrak{u}_\kappa$ be the least size of a base for a uniform ultrafilter on $\kappa$.

**Theorem**

If $\kappa > \beth_\omega$ is regular then $\mathfrak{d}_\kappa \leq \mathfrak{r}_\kappa \leq \mathfrak{u}_\kappa$.

**Theorem** *Carlson, '80s; Woodin*

Let $\kappa$ be supercompact, $\kappa < \mathrm{cf}(\lambda)$, then there is a cardinal preserving extension such that $\mathfrak{u}_\kappa = \kappa^+ < 2^\kappa = \lambda$.

**Theorem**

Let $\kappa$ be supercompact, then there is a forcing extension with a cardinal $\lambda > \mathrm{cf}(\lambda) = \kappa$ (that is a limit of measurable cardinals), such that $\mathfrak{u}_\lambda = \lambda^+ < 2^\lambda$ with $2^\lambda$ arbitrarily large.

---

Džamonja and Shelah (2003). "Universal graphs at the successor of a singular cardinal".

Garti and Shelah (2014). "Partition calculus and cardinal invariants".

Garti and Shelah (2012). "The Ultrafilter Number for Singular Cardinals".

**Question**

What is the consistency strength of $\mathfrak{u}_\kappa < 2^\kappa$ for measurable $\kappa$?

**Question** *Kunen, '70s*

Is $\mathfrak{u}_{\omega_1} < 2^{\aleph_1}$ consistent?

$\kappa$-Mathias forcing $\mathbb{M}_\kappa^{\mathcal{F}}$ (guided by a filter $\mathcal{F}$ on $\kappa$) is the forcing notion with conditions $(s, F)$, where $s \in [\kappa]^{<\kappa}$ and $F \in \mathcal{F}$ and $s \subseteq \min(F)$, ordered by $(s', F') \leq (s, F)$ if $s \subseteq s'$ and $F' \cup s' \setminus s \subseteq F$.

**Theorem**

If $\kappa = \kappa^{<\kappa}$ is regular uncountable, $\mathbb{M}_\kappa^{\mathcal{F}}$ adds a $\kappa$-Cohen generic.

---

Khomskii, Koelbing, Laguzzi, and Wohofsky (2022). "Laver trees in the generalized Baire space".

**Theorem**

(Modulo some $\varepsilon$) Every Laver-like forcing on ${}^{\kappa}\kappa$ adds a $\kappa$-Cohen generic.

**Question**

Does every $<\kappa$-closed forcing notion that adds a dominating $\kappa$-real also add a $\kappa$-Cohen generic?

A $\kappa$-complete ultrafilter on $\kappa$ is *Canjar* if $\mathbb{M}_\kappa^{\mathcal{F}}$ does not add a dominating $\kappa$-real.

**Question**

Does there exist a $\kappa$-complete Canjar ultrafilter on (measurable) $\kappa$?

Khomskii, Koelbing, Laguzzi, and Wohofsky (2022). "Laver trees in the generalized Baire space".

**Theorem**

Let $\kappa$ be supercompact and $\kappa < \lambda \leq \mu$ where $\lambda$ is regular and $\mu^\kappa = \mu$. Then $\mathfrak{u}_\kappa = \lambda \leq 2^\kappa = \mu$ holds in a forcing extension with the same cardinals.

In the above model, also $\mathfrak{b}_\kappa$, $\mathfrak{d}_\kappa$, $\mathfrak{a}_\kappa$, $\mathfrak{s}_\kappa$, $\mathfrak{r}_\kappa$, the cardinal invariants of $\mathcal{M}_\kappa$ and (with assumptions) $\mathfrak{p}_\kappa$ and $\mathfrak{t}_\kappa$ are all $\lambda$.

**Question**

Is $\mathfrak{i}_\kappa = \lambda$ in the above model?

**Question**

How to obtain the consistency of, e.g., $\mathfrak{u}_\kappa < \mathfrak{a}_\kappa$, $\mathfrak{i}_\kappa$ etc.

---

Brooke-Taylor, Fischer, Friedman, and Montoya (2017). "Cardinal characteristics at $\kappa$ in a small $\mathfrak{u}(\kappa)$ model".

**Theorem**

Assume there is a measurable cardinal $\mu$ and $\kappa = 2^{<\kappa} < \mu$, then there exists a forcing extension where $\mathfrak{u}_{2^\kappa} < 2^{2^\kappa}$.

**Theorem**

Assume there is a supercompact cardinal, then there is a forcing extension in which $\mathfrak{u}_{\aleph_{\omega+1}} < 2^{\aleph_{\omega+1}}$.

**Question**

Is $\mathfrak{u}_\kappa < \kappa^{++} = 2^\kappa$ consistent for accessible $\kappa$?

**Question**

What is the consistency strength of the above results?

---

Raghavan and Shelah (2020). "Small Ultrafilter Number".

The most recent version of these slides can probably be found at: https://tvdvlugt.nl/hbsquestions.pdf