Cardinal invariants of products of ideals

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The Vienna Oracle of Set Theory

1. The σ-ideal $K_{\mathscr{I}}$
2. Additivity
3. Uniformity

The σ-ideal $K_{\mathscr{I}}$

Let $\mathscr{I}$ be an ideal on ω. For a function $\phi\colon \omega \to \mathscr{I}$, let:

$$K_\phi := \{x \in \omega^\omega : \forall^\infty n < \omega\ x(n) \in \phi(n)\}.$$

$K_{\mathscr{I}} :=$ the (σ-)ideal generated by $\{K_\phi : \phi \in \mathscr{I}^\omega\}$.

($=$ the σ-ideal generated by the sets of the form $\prod_{n<\omega} I_n \in \mathscr{I}^\omega$.)

For example, when $\mathscr{I} = \mathrm{Fin}$ is the finite ideal, $K_{\mathscr{I}}$ is the σ-ideal generated by compact sets in ω^ω.

In this talk we focus on $\mathrm{add}(K_{\mathscr{I}})$ and $\mathrm{non}(K_{\mathscr{I}})$.

additivity

The additivity of the σ-ideal turns out to have a relationship between the $*$-additivity of the ideal $\mathscr{I}$. Recall:

$$\mathrm{add}^*(\mathscr{I}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathscr{I}, \forall X \in \mathscr{I}\ \exists A \in \mathcal{A}\ (A \not\subseteq^* X)\}$$

However, $\mathrm{add}^*(\mathscr{I}) = \omega$ holds if (and only if) $\mathscr{I}$ is not a P-ideal ($\mathscr{I}$ is a P-ideal $:\Leftrightarrow \forall \mathcal{A} \in [\mathscr{I}]^\omega\ \ \exists X \in \mathscr{I}\ \forall A \in \mathcal{A}\ (A \subseteq^* X)$).
Thus we look at the ω-version number:

$$\mathrm{add}^*_\omega(\mathscr{I}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathscr{I}, \forall \mathcal{X} \in [\mathscr{I}]^\omega\ \exists A \in \mathcal{A}\ \forall X \in \mathcal{X}\ (A \not\subseteq^* X)\}$$

Note that $\mathrm{add}^*_\omega(\mathscr{I}) \geq \omega_1$ and $\mathrm{add}^*(\mathscr{I}) = \mathrm{add}^*_\omega(\mathscr{I})$ if (and only if) $\mathscr{I}$ is a P-ideal.

Remark

For an ultrafilter $\mathcal{U}$, Brendle and Shelah [BS99(Sh:642)] introduced the number $\mathfrak{p}'(\mathcal{U})$, which is the same as add^*_ω(the dual ideal of $\mathcal{U}$).

additivity

(Recall:)

$$\mathrm{add}^*_\omega(\mathscr{I}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathscr{I}, \forall \mathcal{X} \in [\mathscr{I}]^\omega\ \exists A \in \mathcal{A}\ \forall X \in \mathcal{X}\ (A \not\subseteq^* X)\}$$

We find the following relationship between $\mathrm{add}^*_\omega(\mathscr{I})$ and $\mathrm{add}(K_{\mathscr{I}})$:

Theorem (Cieślak, Gappo, Martínez-Celis and Y.)

$\min\{\mathrm{add}^*_\omega(\mathscr{I}), \mathfrak{b}\} \le \mathrm{add}(K_{\mathscr{I}}) \le \mathrm{add}^*_\omega(\mathscr{I})$.

For many concrete ideals $\mathscr{I}$, the values of $\mathrm{add}^*_\omega(\mathscr{I})$ are $\omega_1, \mathrm{add}(\mathcal{N}), \mathrm{add}(\mathcal{M})$, or $\mathfrak{b}$. Since all of them are $\le \mathfrak{b}$, we have $\mathrm{add}(K_{\mathscr{I}}) = \mathrm{add}^*_\omega(\mathscr{I})$ for such ideals. Consequently, we ask:

Question

Does ZFC prove $\mathrm{add}^*_\omega(\mathscr{I}) \le \mathfrak{b}$ (for $\mathscr{I}$ in a certain good class)?

1. The σ-ideal $K_{\mathscr{I}}$
2. Additivity
3. Uniformity

Basic properties on $\mathrm{non}(K_{\mathscr{I}})$

Let us move on to $\mathrm{non}(K_{\mathscr{I}})$ and see their basic properties first. First of all, since $\mathrm{Fin} \subseteq \mathscr{I}$ and $\omega \notin \mathscr{I}$, it follows that:

Lemma

$\mathfrak{b} \leq \mathrm{non}(K_{\mathscr{I}}) \leq \mathrm{non}(\mathcal{M})$.

Cardona, Gavalová, Mejía, Repický and Šupina [CGMRS24] studied cardinal invariants associated with slaloms $\phi \colon \omega \to \mathcal{P}(\omega)$ in a general framework. Using their notation of slalom numbers, $\mathrm{non}(K_{\mathscr{I}}) = \mathfrak{sl}_{\mathfrak{t}}^{\perp}(\mathscr{I}, \mathrm{Fin})$. Thanks to their work, we particularly have:

Fact

$\mathscr{I} \leq_K \mathscr{J}$ implies $\mathrm{non}(K_{\mathscr{I}}) \leq \mathrm{non}(\mathcal{K}_{\mathscr{J}})$ for any ideals $\mathscr{I}$ and $\mathscr{J}$, where $\leq_K$ denotes the Katětov-order among ideals on ω.

Connection with $\mathrm{non}^*_\omega(\mathscr{I})$

Let us see the connection with the $*$-uniformity of $\mathscr{I}$. Recall:
$\mathrm{non}^*(\mathscr{I}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega,\ \forall X \in \mathscr{I}\ \exists A \in \mathcal{A}\ |A \cap X| < \omega)\}$
Again we look at the ω-version:

$$\mathrm{non}^*_\omega(\mathscr{I}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega, \forall \mathcal{X} \in [\mathscr{I}]^\omega\ \exists A \in \mathcal{A}\ \forall X \in \mathcal{X}\ |A \cap X| < \omega\}$$

Theorem (Cieślak, Gappo, Martínez-Celis and Y.)

$\mathrm{non}(K_{\mathscr{I}}) \leq \max\{\mathfrak{b}, \mathrm{non}^*_\omega(\mathscr{I})\}$.

Remark

Šupina [Šup23] proved the following dual inequality in a topological way:

$$\min\{\mathrm{cov}^*(\mathscr{I}), \mathfrak{d}\} \leq \mathrm{cov}(K_{\mathscr{I}}).$$

Recall:
$\mathrm{cov}^*(\mathscr{I}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathscr{I}, \forall X \in [\omega]^\omega\ \exists A \in \mathcal{A}\ |A \cap X| = \omega)\}$,
so the ω-version of $\mathrm{cov}^*(\mathscr{I})$ would be the same.

Optimality of $\mathfrak{b} \leq \mathrm{non}(K_{\mathscr{I}}) \leq \max\{\mathfrak{b}, \mathrm{non}^*_\omega(\mathscr{I})\}$

For some specific ideal, $\mathrm{non}(K_{\mathscr{I}})$ is equal to either of the upper or lower bounds of $\mathfrak{b} \leq \mathrm{non}(K_{\mathscr{I}}) \leq \max\{\mathfrak{b}, \mathrm{non}^*_\omega(\mathscr{I})\}$. Recall:

- $\mathrm{Fin} \times \mathrm{Fin} := \{A \subseteq \omega \times \omega : \forall^\infty n < \omega \, |(A)_n| < \omega\}$ is the Fubini product of two Fin's, where $(A)_n := \{m < \omega : (n, m) \in A\}$ denotes the n-th vertical section of A.
- $\mathcal{S}$ is Solecki's ideal: defined on the countable set $\Omega := \{U \in \mathrm{Clopen}(2^\omega) : \mathrm{Leb}(U) = \frac{1}{2}\}$ and generated by subsets $A \subseteq \Omega$ with non-empty intersection.

Theorem (Cieślak, Gappo, Martínez-Celis and Y.)

- $\mathrm{non}(K_{\mathrm{Fin} \times \mathrm{Fin}}) = \mathfrak{b}$.
- $\mathrm{non}(K_{\mathcal{S}}) = \max\{\mathfrak{b}, \mathrm{non}^*_\omega(\mathcal{S})\}$.

Short remark on $\mathrm{non}^*_\omega(\mathcal{S})$

We compute the value of $\mathrm{non}^*_\omega(\mathcal{S})$. Hrušák, Meza-Alcántara and Minami [HMM10] proved $\mathrm{cov}^*(\mathcal{S}) = \mathrm{non}(\mathcal{N})$ and we obtain the following "ω-versioned" dual equality:

Theorem (Cieślak, Gappo, Martínez-Celis and Y.)

$$\mathrm{non}^*_\omega(\mathcal{S}) = \mathrm{cov}_\omega(\mathcal{N})$$
$$:= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{N}, \forall A \in [\mathbb{R}]^\omega\ \exists N \in \mathcal{F}\ (A \subseteq N)\}$$

$\mathrm{cov}_\omega(\mathcal{N})$ itself seems interesting:

Lemma

- $\mathrm{cov}(\mathcal{N}) \leq \mathrm{cov}_\omega(\mathcal{N}) \leq \mathrm{non}(\mathcal{M})$.
- $\mathrm{cf}(\mathrm{cov}_\omega(\mathcal{N})) \geq \omega_1$.

In particular, $\mathrm{cov}(\mathcal{N}) < \mathrm{cov}_\omega(\mathcal{N})$ is consistent since $\mathrm{cov}(\mathcal{N})$ may have countable cofinality, proved by Shelah [She00(Sh:592)].

On the asymptotic density zero ideal $\mathcal{Z}$

Let $\mathcal{Z}$ denote the asymptotic density zero ideal:

$$\mathcal{Z} := \left\{ A \subseteq \omega : \frac{|A \cap n|}{n} \xrightarrow{n \to \infty} 0 \right\}.$$

Due to Pawlikowski [Paw00], the following holds, where $\mathcal{E}$ denotes the σ-ideal on the reals generated by closed null sets:

$$\mathrm{non}(K_{\mathcal{Z}}) \leq \max\{\mathfrak{b}, \mathrm{non}(\mathcal{E})\}.$$

We introduce a forcing notion which increases $\mathrm{non}(K_{\mathcal{Z}})$ and keeps $\mathfrak{b}$ small (more technically, “it has ultrafilter-limits”), and by iterating the poset we obtain:

Theorem (Cieślak, Gappo, Martínez-Celis and Y.)

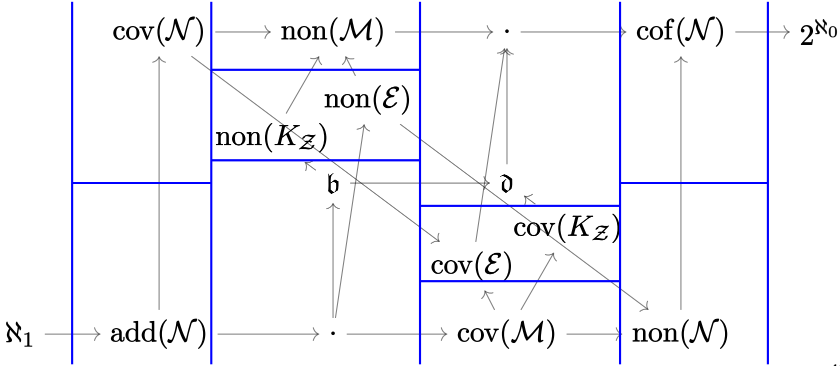
$\mathfrak{b} < \mathrm{non}(K_{\mathcal{Z}})$ is consistent.

Cichoń's maximum with $\mathrm{non}(K_{\mathcal{Z}})$ and $\mathrm{cov}(K_{\mathcal{Z}})$

Moreover, by using the methods from Cardona, Mejía, Uribe-Zapata [CMU24] and Yamazoe [Yam24] to keep $\mathrm{non}(\mathcal{E})$ small through our forcing iteration, we have (recall $\mathrm{non}(K_{\mathcal{Z}}) \leq \max\{\mathfrak{b}, \mathrm{non}(\mathcal{E})\}$):

Theorem (Cieślak, Gappo, Martínez-Celis and Y.)

$\mathrm{non}(K_{\mathcal{Z}})$ and $\mathrm{cov}(K_{\mathcal{Z}})$ can be added to Cichoń's maximum.



Question and table

Question

How much can we extend the class of ideals $\mathscr{I}$ such that $\mathrm{non}(K_{\mathscr{I}})$ and $\mathrm{cov}(K_{\mathscr{I}})$ can be added to a model of Cichoń's maximum?

We conclude this talk with the following table.

ideal	$\mathrm{add}^*_\omega(\mathscr{I})$	$\mathrm{non}^*_\omega(\mathscr{I})$	$\mathrm{non}(K_{\mathscr{I}})$
$\mathcal{R}$	ω_1	ω_1	$\mathfrak{b}$
$\mathcal{S}$	ω_1	$\mathrm{cov}_\omega(\mathcal{N})$	$\max\{\mathfrak{b}, \mathrm{cov}_\omega(\mathcal{N})\}$
nwd	$\mathrm{add}(\mathcal{M})$	$\mathrm{non}(\mathcal{M})$	?
conv	ω_1	ω_1	$\mathfrak{b}$
$\mathrm{Fin} \times \mathrm{Fin}$	$\mathfrak{b}$	$\mathfrak{d}$	$\mathfrak{b}$
$\mathcal{ED}$	ω_1	$\mathrm{cov}(\mathcal{M})$	$\mathfrak{b}$
$\mathcal{ED}_{\mathrm{fin}}$	ω_1	?	?
$\mathcal{I}_{\frac{1}{n}}$	$\mathrm{add}(\mathcal{N})$	$\mathrm{non}^*(\mathcal{I}_{\frac{1}{n}})$	?
$\mathcal{Z}$	$\mathrm{add}(\mathcal{N})$	$\mathrm{non}^*(\mathcal{Z})$	?

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Appendix

- $\mathcal{R}$ is the random graph ideal: generated by homogeneous sets for the random graph (Rado graph).
- $\mathtt{nwd} := \{A \subseteq \mathbb{Q} : A$ is nowhere dense in $\mathbb{Q}\}$.
- $\mathtt{conv}$ is the ideal generated by sequences in $\mathbb{Q} \cap [0,1]$ convergent in $[0,1]$.
- $\mathcal{ED} := \{A \subseteq \omega \times \omega : \exists k < \omega\, \forall^\infty n < \omega\, |(A)_n| \leq k\}$.
- $\mathcal{ED}_{\mathrm{fin}} := \mathcal{ED} \restriction \Delta$ where $\Delta := \{(n,m) \in \omega \times \omega : m \leq n\}$.
- $\mathcal{I}_{\frac{1}{n}}$ is the summable ideal: consists of $A \subseteq \omega$ s.t. $\sum_{n \in A} \frac{1}{n}$ is finite.