Every effect algebra can be made into a total algebra

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In our previous paper [2] we introduced the concept of a basic algebra. The name ‘basic algebra’ is used because these algebras capture common features of many known structures such as Boolean algebras, orthomodular lattices, MV-algebras or lattice effect algebras. In [2] we paid special attention to lattice effect algebras, which were originally defined as partial algebras $(E, +, 0, 1)$, but the presence of the join operation allows one to replace partial $+$ by total $\oplus$. The intent of the present paper is to establish similar results for general effect algebras in the context of commutative directoids.

A **commutative directoid** [6] is a commutative, idempotent groupoid $(A, \sqcup)$ satisfying the equation $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$. For instance, every semilattice is a commutative directoid. It can easily be seen that the stipulation

$$x \leq y \quad \text{if and only if} \quad x \sqcup y = y \tag{1}$$

defines a partial order on A such that, for every $x, y \in A$, $x \sqcup y$ is an upper bound of $\{x, y\}$. Thus the poset $(A, \leq)$ is upwards directed. Conversely, we may associate a commutative directoid to an arbitrary upwards directed set by letting $x \sqcup y = y \sqcup x$ be some upper bound of $\{x, y\}$, such that whenever x, y are comparable, then $x \sqcup y = y \sqcup x$ is the greater of x, y.

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An **antitone involution** on a poset $(P, \leq)$ is a mapping $\beta : P \to P$ such that, for all $x, y \in P$, (i) $x \leq y \Rightarrow \beta(y) \leq \beta(x)$, and (ii) $\beta(\beta(x)) = x$.

By a **commutative directoid with sectional antitone involutions** we shall mean a system $(A, \sqcup, (\beta_a)_{a \in A}, 0, 1)$ where (i) $(A, \sqcup)$ is a commutative directoid with a least element 0 and a greatest element 1, and (ii) every section $[a)$ is equipped with an antitone involution β_a.

In particular, if $(A, \sqcup)$ is a semilattice, then the underlying poset is a lattice in which $\beta_0(\beta_0(x) \sqcup \beta_0(y))$ is the infimum of $\{x, y\}$, and hence we may say that $(A, \sqcup, (\beta_a)_{a \in A}, 0, 1)$ is a **lattice with sectional antitone involutions**.

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A **weak basic algebra** is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities and quasi-identity (where 1 is an abbreviation for $\neg 0$):

$$x \oplus 0 = x, \tag{W1}$$

$$\neg\neg x = x, \tag{W2}$$

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x, \tag{W3}$$

$$x \oplus (\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus z) = 1, \tag{W4}$$

$$\neg x \oplus (y \oplus x) = 1, \tag{W5}$$

$$\neg x \oplus y = 1 \ \& \ \neg y \oplus z = 1 \ \Rightarrow \ \neg(\neg z \oplus x) \oplus (\neg y \oplus x) = 1. \tag{W6}$$

If $(A, \oplus, \neg, 0)$ is a weak basic algebra and if we put

$$x \sqcup y = \neg(\neg x \oplus y) \oplus y,$$

then $(A, \sqcup)$ is a commutative directoid with a least element 0 and a greatest element 1, such that the underlying order $\leq$ is given by

$$x \leq y \quad \text{if and only if} \quad x \sqcup y = y \quad \text{if and only if} \quad \neg x \oplus y = 1,$$

and for each $a \in A$, $x \mapsto \neg x \oplus a$ is an antitone involution on $[a) = \{x \in A \mid a \leq x\}$.

Conversely, if $(A, \sqcup, (\beta_a)_{a \in A}, 0, 1)$ is a commutative directoid with sectional antitone involutions, then we can define $\oplus$ and $\neg$ as $x \oplus y = \beta_y(\beta_0(x) \sqcup y)$ and $\neg x = \beta_0(x)$, respectively, and $(A, \oplus, \neg, 0)$ becomes a weak basic algebra in which $x \sqcup y = \neg(\neg x \oplus y) \oplus y$ and $\beta_a(x) = \neg x \oplus a$.

In every weak basic algebra, in addition to the 'join-like' operation $\sqcup$, we can introduce the dual 'meet-like' operation $\sqcap$ by

$$x \sqcap y = \neg(\neg x \sqcup \neg y).$$

Then we have $x \leq y$ if and only if $x \sqcap y = x$, and the structure $(A, \sqcup, \sqcap)$ is a λ**-lattice** in the sense of [8], i.e., both $(A, \sqcup)$ and $(A, \sqcap)$ are commutative directoids and the absorption laws $x \sqcup (x \sqcap y) = x = x \sqcap (x \sqcup y)$ are satisfied.

Conversely, if $(A, \sqcup, (\beta_a)_{a \in A}, 0, 1)$ is a commutative directoid with sectional antitone involutions, then we can define $\oplus$ and $\neg$ as $x \oplus y = \beta_y(\beta_0(x) \sqcup y)$ and $\neg x = \beta_0(x)$, respectively, and $(A, \oplus, \neg, 0)$ becomes a weak basic algebra in which $x \sqcup y = \neg(\neg x \oplus y) \oplus y$ and $\beta_a(x) = \neg x \oplus a$.
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Every basic algebra is a weak basic algebra and the above assignment between weak basic algebras and commutative directoids with sectional antitone involutions, restricted to basic algebras, furnishes a one-to-one correspondence between basic algebras and lattices with sectional antitone involutions.

We know that weak basic algebras form a variety.

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Proposition 1

An algebra $\mathbf{A} = (A, \oplus, \neg, 0)$ satisfying (W1)—(W4) is a weak basic algebra if and only if it satisfies the identity

$$\neg(\neg((x \sqcup y) \sqcup z) \oplus x) \oplus (\neg y \oplus x) = 1. \tag{2}$$

Another central concept is that of an effect algebra, introduced by Foulis and Bennett [4]. We recall that an **effect algebra** is a system $(E,+,0,1)$ where $0,1$ are distinguished elements of E and $+$ is a partial binary operation on E such that

(EA1) $x+y=y+x$ if one side is defined,

(EA2) $(x+y)+z=x+(y+z)$ if one side is defined,

(EA3) for every $x\in E$ there exists a unique $x'\in E$ with $x'+x=1$,

(EA4) if $x+1$ is defined then $x=0$.

Every effect algebra bears a natural partial order given by

$$x\leq y \quad \text{if and only if} \quad y=x+z \text{ for some } z\in E.$$

The poset $(E,\leq)$ is bounded, 0 is the bottom element and 1 is the top element. If, moreover, $(E,\leq)$ is a lattice, then $(E,+,0,1)$ is called a **lattice effect algebra**.

In every effect algebra, a partial subtraction $-$ can be defined as follows:

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Now, we focus on the relationships between effect algebras and weak basic algebras.

Theorem 1

Let $\mathbf{A} = (A, \oplus, \neg, 0)$ be a weak basic algebra. Define the partial addition $+$ on A as follows: $x + y$ is defined if and only if $x \leq \neg y$, and in this case $x + y = x \oplus y$. Then $\mathcal{E}(\mathbf{A}) = (A, +, 0, 1)$ is an effect algebra if and only if $\mathbf{A}$ satisfies the quasi-identity

$$x \leq \neg y \ \& \ x \oplus y \leq \neg z \ \Rightarrow \ (x \oplus y) \oplus z = x \oplus (z \oplus y). \tag{3}$$

Moreover, over weak basic algebras, (3) is equivalent to the identity

$$(x \oplus y) \oplus (\neg(x \oplus y) \sqcap z) = (x \sqcap \neg y) \oplus ((\neg(x \oplus y) \sqcap z) \oplus y). \tag{4}$$

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Corollary 1

[2] Let $\mathbf{A} = (A, \oplus, \neg, 0)$ be a basic algebra and let $\mathcal{E}(\mathbf{A}) = (A, +, 0, 1)$ be as in Theorem 1. Then $\mathcal{E}(\mathbf{A})$ is a lattice effect algebra if and only if $\mathbf{A}$ satisfies the quasi-identity (3).

In case of basic algebras, $\mathbf{A}$ can be retrieved from $\mathcal{E}(\mathbf{A})$ ([2], see below). However, as the following example shows, this is not true for weak basic algebras.

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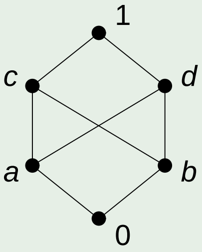
Example

Let $(A, \leq)$ be the poset



and let the sections $[0) = A$, $[a)$ and $[b)$ be equipped with the following antitone involutions:

$$\beta_0 : 0 \mapsto 1, 1 \mapsto 0, a \mapsto d, d \mapsto a, b \mapsto c, c \mapsto b,$$
$$\beta_a : a \mapsto 1, 1 \mapsto a, c \mapsto c, d \mapsto d,$$
$$\beta_b : b \mapsto 1, 1 \mapsto b, c \mapsto d, d \mapsto c;$$

the other sections admit unique antitone involutions.

Example

There are three possible ways in which we can associate a commutative directoid to $(A, \leq)$ and, consequently, there are three weak basic algebras with the underlying poset $(A, \leq)$:
(i) For $a \sqcup_1 b = c$ we get $\mathbf{A}_1 = (A, \oplus_1, \neg, 0)$ where

$\oplus_1$	0	a	b	c	d	1	$\neg$
0	0	a	b	c	d	1	1
a	a	d	c	c	1	1	d
b	b	c	d	1	d	1	c
c	c	c	1	1	1	1	b
d	d	1	d	1	1	1	a
1	1	1	1	1	1	1	0

Example

(ii) For $a \sqcup_2 b = d$ we get $\mathbf{A}_2 = (A, \oplus_2, \neg, 0)$ where

$\oplus_2$	0	a	b	c	d	1	$\neg$
0	0	a	b	c	d	1	1
a	a	d	c	c	1	1	d
b	b	c	d	1	d	1	c
c	c	d	1	1	1	1	b
d	d	1	c	1	1	1	a
1	1	1	1	1	1	1	0

Example

(iii) For $a \sqcup_3 b = 1$ we get $\mathbf{A}_3 = (A, \oplus_3, \neg, 0)$ where

$\oplus_3$	0	a	b	c	d	1	$\neg$
0	0	a	b	c	d	1	1
a	a	d	c	c	1	1	d
b	b	c	d	1	d	1	c
c	c	a	1	1	1	1	b
d	d	1	b	1	1	1	a
1	1	1	1	1	1	1	0

Example

All these weak basic algebras induce the same effect algebra $\mathcal{E}(\mathbf{A}_1) = \mathcal{E}(\mathbf{A}_2) = \mathcal{E}(\mathbf{A}_3) = (A, +, 0, 1)$ where

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	d	c	.	1	.
b	b	c	d	1	.	.
c	c	.	1	.	.	.
d	d	1	.	.	.	.
1	1	.	.	.	.	.

Let $\mathbf{E} = (E, +, 0, 1)$ be an effect algebra. Since the underlying poset $(E, \leq)$ is bounded, it can be organized into a commutative directoid $(E, \sqcup)$. We shall simply say that the pair $(\mathbf{E}, \sqcup)$ is an effect algebra with an associated commutative directoid.

Theorem 2

Let $(\mathbf{E}, \sqcup)$ be an effect algebra $\mathbf{E} = (E, +, 0, 1)$ with an associated commutative directoid. Define

$$x \oplus y = (x' \sqcup y)' + y \quad \text{and} \quad \neg x = x'.$$

Then $\mathcal{B}(\mathbf{E}, \sqcup) = (E, \oplus, \neg, 0)$ is a weak basic algebra satisfying (3). Moreover, $\mathcal{E}(\mathcal{B}(\mathbf{E}, \sqcup))$, the effect algebra assigned to $\mathcal{B}(\mathbf{E}, \sqcup)$ by Theorem 1, is just $\mathbf{E}$.

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Example

Let **E** be the effect algebra we have obtained in Example 1. If we put $a \sqcup_1 b = c$ then $\mathcal{B}(\mathbf{E}, \sqcup_1)$ is just the weak basic algebra $\mathbf{A}_1$ from Example 1. Analogously, if $a \sqcup_2 b = d$ then $\mathcal{B}(\mathbf{E}, \sqcup_2) = \mathbf{A}_2$, and for $a \sqcup_3 b = 1$ we have $\mathcal{B}(\mathbf{E}, \sqcup_3) = \mathbf{A}_3$.

There is a one-to-one correspondence between weak basic algebras satisfying (3) (respectively, (4)) and pairs $(\mathbf{E}, \sqcup)$ where $\mathbf{E} = (E, +, 0, 1)$ is an effect algebra with an associated commutative directoid $(E, \sqcup)$. Namely, the assignment

$$\mathbf{A} \mapsto (\mathcal{E}(\mathbf{A}), \sqcup),$$

where $\mathcal{E}(\mathbf{A})$ is as in Theorem 1 and $x \sqcup y = \neg(\neg x \oplus y) \oplus y$, is a bijection the inverse of which is

$$(\mathbf{E}, \sqcup) \mapsto \mathcal{B}(\mathbf{E}, \sqcup),$$

where $\mathcal{B}(\mathbf{E}, \sqcup)$ is defined in Theorem 2.

Let $\mathbf{E} = (E, +, 0, 1)$ be an effect algebra. When constructing $(\mathbf{E}, \sqcup)$, we did not take care of existing suprema so far. This means that $\mathcal{B}(\mathbf{E}, \sqcup)$ need not be a basic algebra even though $\mathbf{E}$ is a lattice effect algebra. The situation can be improved if we define $\sqcup$ in such a way that the following condition holds:

$$\text{If } \sup\{x, y\} \text{ exists, then } x \sqcup y = y \sqcup x = \sup\{x, y\}. \qquad \text{(S)}$$

Corollary 2

Let $(\mathbf{E}, \sqcup)$ be an effect algebra with an associated commutative directoid that satisfies the condition (S). Then $\mathcal{B}(\mathbf{E}, \sqcup)$ is a weak basic algebra, and if $\mathbf{E}$ is a lattice effect algebra, then $\mathcal{B}(\mathbf{E}, \sqcup)$ is a basic algebra.

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$$x \leftrightarrow y \quad \text{if and only if} \quad \text{there is } z \text{ such that } x, y \leq z \text{ and } z - x \leq y. \tag{5}$$

For *lattice* effect algebras we proved in [2] that $x \leftrightarrow y$ if and only if $x \oplus y = y \oplus x$ in the derived basic algebra. In general we have:

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Let $(\mathbf{E}, \sqcup)$ and $\mathcal{B}(\mathbf{E}, \sqcup)$ be as in Theorem 2. For every $x, y \in E$, if $x \oplus y = y \oplus x$, then $x \leftrightarrow y$.

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Proposition 2

Let $(\mathbf{E}, \sqcup)$ and $\mathcal{B}(\mathbf{E}, \sqcup)$ be as in Theorem 2. For every $x, y \in E$, if $x \oplus y = y \oplus x$, then $x \leftrightarrow y$.

The reverse implication fails to be true. Let **E** be the effect algebra from Examples 1 and 2. It can be easily seen that every two elements are compatible, while the addition in $\mathbf{A}_2$ and $\mathbf{A}_3$ is not commutative (for instance, $a \leftrightarrow c$, but $a \oplus_i c \neq c \oplus_i a$ for $i = 2, 3$).

In order to overcome this disadvantage, we define the 'join-like' operation $\sqcup$ in an effect algebra $\mathbf{E} = (E, +, 0, 1)$ in the following way:

$$\text{If } x \leftrightarrow y, \text{ then } x \sqcup y = y \sqcup x = z \text{ where } z \geq x, y \text{ and } z - x \leq y. \tag{C}$$

We can prove that in *every* effect algebra $\mathbf{E} = (E, +, 0, 1)$, the operation $\sqcup$ can always be defined in such a way that it obeys the requirements of the condition (C).

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Theorem 3

Let $(\mathbf{E}, \sqcup)$ be an effect algebra with an associated commutative directoid satisfying condition (C). Then $\mathcal{B}(\mathbf{E}, \sqcup)$ is a weak basic algebra such that, for all $x, y \in E$, the following are equivalent:

(i) $x \leftrightarrow y$,

(ii) $(x \sqcup y) - y = x - (x \sqcap y)$,

(iii) $x \oplus y = y \oplus x$.

By a **block** of a weak basic algebra $(A, \oplus, \neg, 0)$ we mean a subset B of A which is maximal with respect to the property that $x \oplus y = y \oplus x$ for all $x, y \in B$. It is evident that every element of A is contained in a block.

Theorem 4

Let $(\mathbf{E}, \sqcup)$ be an effect algebra with an associated commutative directoid satisfying the condition (C). Assume that for all $x, y, z \in E$, if $x \leftrightarrow y$, $x \leftrightarrow z$ and $y + z$ is defined, then $x \leftrightarrow y + z$. Then a block B of $\mathcal{B}(\mathbf{E}, \sqcup)$ is a subalgebra of $\mathcal{B}(\mathbf{E}, \sqcup)$ if and only if $x \sqcup y \in B$ for all $x, y \in B$.

The condition that $x \leftrightarrow y$ and $x \leftrightarrow z$ together yield $x \leftrightarrow y + z$ (if $y + z$ exists) holds in lattice effect algebras, however, the next example shows that this additional assumption in Theorem 4 cannot be omitted:

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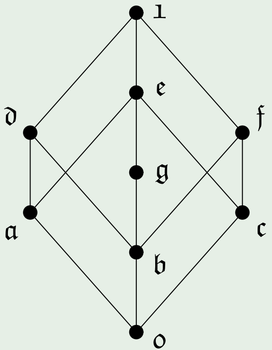
Example

Let E be the set consisting of the following pairs of integers: $\mathfrak{o}=(0,0)$, $\mathfrak{a}=(1,2)$, $\mathfrak{b}=(1,1)$, $\mathfrak{c}=(2,1)$, $\mathfrak{d}=(2,3)$, $\mathfrak{e}=(3,3)$, $\mathfrak{f}=(3,2)$, $\mathfrak{g}=(2,2)$ and $\imath=(4,4)$. If we equip E with $+$ defined as the restriction to E of the usual pointwise addition, then $\mathbf{E}=(E,+,0)$ becomes an effect algebra in which $(x,y)'=(4-x,4-y)$. The underlying poset of **E** is as follows (notice that $(x,y)\leq(u,v)$ if and only if $(x,y)=(u,v)$, or $x<u \;\&\; y<v$):



Example

It is obvious that $\mathfrak{f} \leftrightarrow \mathfrak{b}$, but $\mathfrak{f}$ is *not* compatible with $\mathfrak{g} = \mathfrak{b} + \mathfrak{b}$. Indeed, the only common upper bound of $\mathfrak{f}, \mathfrak{g}$ is $\mathfrak{1}$, and $\mathfrak{1} - \mathfrak{f} = \mathfrak{a} \not\leq \mathfrak{g}$ as well as $\mathfrak{1} - \mathfrak{g} = \mathfrak{g} \not\leq \mathfrak{f}$, thus $\mathfrak{f} \not\leftrightarrow \mathfrak{g}$ by (5).
In accordance with the conditions (S) and (C), we put $\mathfrak{a} \sqcup \mathfrak{b} = \mathfrak{d} (= \mathfrak{a} + \mathfrak{b})$ and $\mathfrak{b} \sqcup \mathfrak{c} = \mathfrak{f} (= \mathfrak{b} + \mathfrak{c})$; in the other cases $\sqcup$ coincides with sup. A direct inspection shows that $E \setminus \{\mathfrak{g}\}$ is a block of the assigned weak basic algebra $\mathcal{B}(\mathbf{E}, \sqcup)$ (see the table below) which is closed under $\sqcup$, but it is not closed under $\oplus$ as $\mathfrak{b} + \mathfrak{b} = \mathfrak{g}$. On the other hand, $\{\mathfrak{o}, \mathfrak{b}, \mathfrak{e}, \mathfrak{g}, \mathfrak{1}\}$ is both a block and a subalgebra of $\mathcal{B}(\mathbf{E}, \sqcup)$.

Example 3 (3/3)

Example

$\oplus$	$\mathfrak{0}$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{d}$	$\mathfrak{e}$	$\mathfrak{f}$	$\mathfrak{g}$	$\mathfrak{1}$	$\neg$
$\mathfrak{0}$	$\mathfrak{0}$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{d}$	$\mathfrak{e}$	$\mathfrak{f}$	$\mathfrak{g}$	$\mathfrak{1}$	$\mathfrak{1}$
$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{d}$	$\mathfrak{e}$	$\mathfrak{d}$	$\mathfrak{e}$	$\mathfrak{1}$	$\mathfrak{g}$	$\mathfrak{1}$	$\mathfrak{f}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{d}$	$\mathfrak{g}$	$\mathfrak{f}$	$\mathfrak{d}$	$\mathfrak{1}$	$\mathfrak{f}$	$\mathfrak{e}$	$\mathfrak{1}$	$\mathfrak{e}$
$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{e}$	$\mathfrak{f}$	$\mathfrak{c}$	$\mathfrak{1}$	$\mathfrak{e}$	$\mathfrak{f}$	$\mathfrak{g}$	$\mathfrak{1}$	$\mathfrak{d}$
$\mathfrak{d}$	$\mathfrak{d}$	$\mathfrak{d}$	$\mathfrak{d}$	$\mathfrak{1}$	$\mathfrak{d}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{e}$	$\mathfrak{1}$	$\mathfrak{c}$
$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{1}$	$\mathfrak{e}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{b}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{1}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{f}$	$\mathfrak{e}$	$\mathfrak{1}$	$\mathfrak{a}$
$\mathfrak{g}$	$\mathfrak{g}$	$\mathfrak{d}$	$\mathfrak{e}$	$\mathfrak{f}$	$\mathfrak{d}$	$\mathfrak{1}$	$\mathfrak{f}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{g}$
$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{1}$	$\mathfrak{0}$

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