

# A syntactic characterization of weakly Mal'tsev varieties

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# Structure of the talk

- 1 Internal structures
- 2 From Mal'tsev to weakly Mal'tsev categories
- 3 A syntactic characterization of weakly Mal'tsev varieties

# 1. Internal structures

A **reflexive graph** is given by

- a set  $C_0$  of objects,
- a set  $C_1$  of morphisms,
- a domain function  $d : C_1 \rightarrow C_0$ ,
- a codomain function  $c : C_1 \rightarrow C_0$ ,
- an identities function  $e : C_0 \rightarrow C_1$

such that

$$de(x) = x = ce(x).$$

(incidence axioms)

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such that

$$de(x) = x = ce(x). \quad (\text{incidence axioms})$$

## Definition

An **internal reflexive graph** in a category  $\mathcal{C}$  is given by a diagram

$$C_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} C_0$$

in  $\mathcal{C}$  such that

$$de = 1_{C_1} = ce. \quad (\text{incidence axioms})$$

A **(small) category** is given by

- a reflexive graph  $(C_0, C_1, d, c, e)$ ,
- a multiplication function  $m : C_{\rightarrow\rightarrow} := \{(f, g) \in C_1 \times C_1 \mid c(f) = d(g)\} \rightarrow C_1$

such that

$$dm(f, g) = d(f), \quad cm(f, g) = c(g), \quad \text{(incidence axioms)}$$

$$m(f, ec(f)) = f = m(de(f), f), \quad \text{(unitality axioms)}$$

$$m(m(f, g), h) = m(f, m(g, h)). \quad \text{(associativity axioms)}$$

## Definition

An **internal category** in a category  $\mathcal{C}$  with pullbacks is given by a diagram

$$C_{\rightarrow\rightarrow} \xrightarrow{m} C_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} C_0$$

in  $\mathcal{C}$ , where  $C_{\rightarrow\rightarrow}$  is given by the pullback

$$\begin{array}{ccc} C_{\rightarrow\rightarrow} & \xrightarrow{p_2} & C_1 \\ p_1 \downarrow & & \downarrow d \\ C_1 & \xrightarrow{c} & C_0, \end{array}$$

such that

- $(C_0, C_1, d, c, e)$  is an internal reflexive graph in  $\mathcal{C}$ ,
- the usual incidence, unitality and associativity axioms for  $m$  hold.

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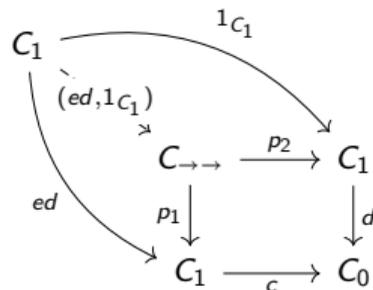
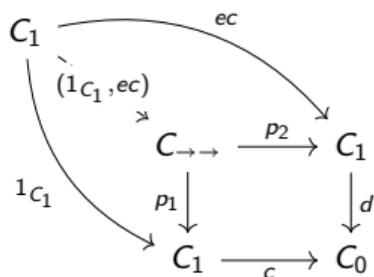
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- the usual incidence, unitality and associativity axioms for  $m$  hold.

$$m(1_{C_1}, ec) = 1_{C_1} = m(ed, 1_{C_1})$$

(unitality axioms)



A (small) category  $(C_0, C_1, d, c, e, m)$  is a **(small) groupoid** if there exists an inverses function

$$i : C_1 \rightarrow C_1$$

such that

$$di(f) = c(f),$$

$$ci(f) = d(f),$$

(incidence axioms)

$$m(f, i(f)) = ed(f),$$

$$m(i(f), f) = ec(f).$$

(inverse axioms)

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### Definition

An internal category  $(C_0, C_1, d, c, e, m)$  in a category  $\mathcal{C}$  with pullbacks is an **internal groupoid** if there exists a morphism

$$i : C_1 \rightarrow C_1$$

such that the usual incidence and inverse axioms for  $i$  hold.

$$C_{\rightarrow\rightarrow} \xrightarrow{m} C_1 \begin{array}{c} \overset{i}{\curvearrowright} \\ \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} \\ \rightarrow C_0 \end{array}$$

## 2. From Mal'tsev to weakly Mal'tsev categories

Proposition (Mal'tsev; 1954)

Let  $\mathbb{V}$  be a variety of universal algebras. TFAE:

- 1  $\mathbb{V}$  is congruence permutable, i.e., for any algebra  $A \in \mathbb{V}$  and  $R, S \in \text{Cong}(A)$ , it holds that

$$R \circ S = S \circ R.$$

- 2 There exists a ternary term  $p \in F(x, y, z)$  in the algebraic theory of  $\mathbb{V}$  that satisfies the identities

$$p(x, x, y) = y \quad \text{and} \quad p(x, y, y) = x.$$

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### Examples of congruence permutable varieties

- Grp ( $p(x, y, z) = x \cdot y^{-1} \cdot z$ ),

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Definition (Carboni, Pedicchio, Pirovano; 1992)

A **Mal'tsev category** is a finitely complete category  $\mathcal{C}$  in which any of the following equivalent conditions hold:

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- The dual of any elementary topos: Set<sup>op</sup>

## Proposition (Bourn; 1996)

Let  $\mathcal{C}$  be a finitely complete category. TFAE:

- 1  $\mathcal{C}$  is a Mal'tsev category.
- 2 Given split epimorphisms  $f, g$  in  $\mathcal{C}$  with respective splittings  $r, s$ , i.e.,  $fr = 1_C = gs$ , then, in the pullback of  $f$  and  $g$ , the induced pullback injections  $e_1, e_2$  are jointly extremally epimorphic,

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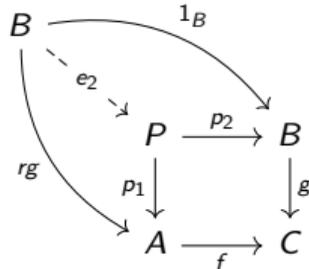
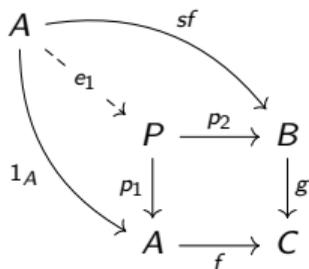
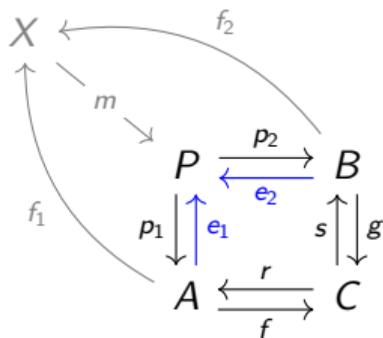
$$\begin{array}{ccccc}
 A & & \xrightarrow{sf} & & B \\
 \downarrow 1_A & \dashrightarrow e_1 & \searrow & \xrightarrow{p_2} & \downarrow g \\
 & P & & & \\
 & \downarrow p_1 & & & \\
 & A & \xrightarrow{f} & & C
 \end{array}$$

$$\begin{array}{ccccc}
 B & & \xrightarrow{1_B} & & B \\
 \downarrow rg & \dashrightarrow e_2 & \searrow & \xrightarrow{p_2} & \downarrow g \\
 & P & & & \\
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## Proposition

Any reflexive graph in a Mal'tsev category  $\mathcal{C}$  allows at most one internal category structure.

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## Proof

Let

$$C_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} C_0$$

be a reflexive graph in  $\mathcal{C}$ , i.e.  $d, c$  are split epimorphisms with common splitting  $e$ .

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$$\begin{array}{ccc} C \rightarrow & \xrightarrow{p_2} & C_1 \\ p_1 \downarrow & \xleftarrow{e_2} & \uparrow e \\ C_1 & \xleftarrow{e} & C_0 \\ & \xrightarrow{c} & \downarrow d \end{array}$$

are jointly extremally epimorphic.

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$$\begin{array}{ccccc}
 & & & & C_1 \\
 & & & & \uparrow \\
 C_1 & \xleftarrow{m} & C \rightrightarrows & \xrightarrow{p_2} & C_1 \\
 & & \uparrow e_1 & \xleftarrow{e_2} & \uparrow e \\
 & & C_1 & \xleftarrow{e} & C_0 \\
 & & \downarrow p_1 & \xrightarrow{c} & \downarrow d \\
 & & & & C_0
 \end{array}$$

$1_{C_1}$  (curved arrow from  $C_1$  to  $C_1$ )  
 $1_{C_1}$  (curved arrow from  $C_0$  to  $C_1$ )

are jointly extremally epimorphic.

The unitality axioms are exactly  $me_1 = 1_{C_1}$  and  $me_2 = 1_{C_1}$ .

## Remark

- In a Mal'tsev category, a reflexive graph as above is an internal category if and only if the kernel congruences of  $d, c$  centralize each other.

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- In a Mal'tsev category, a reflexive graph as above is an internal category if and only the kernel congruences of  $d, c$  centralize each other.
- If  $\mathcal{C}$  is a congruence permutable variety with Mal'tsev term  $p(x, y, z)$ , and  $[\text{Eq}(d), \text{Eq}(c)] = \Delta_{C_1}$ , then, for any  $f, g \in C_1$  with  $c(f) = d(g)$ ,

$$m(f, g) = p(f, ec(f), g).$$

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### Remark

If  $\mathcal{C}$  is congruence permutable variety with Mal'tsev term  $p(x, y, z)$ , then the inverses morphism  $i : C_1 \rightarrow C_1$  for an internal category is given by

$$i(f) = p(ed(f), f, ec(f)).$$

There are categories such as  $\text{DistrLatt}$ ,  $\text{CancCommMon}$ ,  $\text{Top}^{\text{op}}$  in which:

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Definition (Martins-Ferreira; 2008)

A **weakly Mal'tsev category** is a category that has all pullbacks of split epimorphisms along split epimorphisms and in which the pullback injections  $e_1, e_2$  in a pullback

$$\begin{array}{ccc} P & \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow{e_2} \end{array} & B \\ \begin{array}{c} \downarrow p_1 \\ \uparrow e_1 \end{array} & & \begin{array}{c} \uparrow s \\ \downarrow g \end{array} \\ A & \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{f} \end{array} & C \end{array}$$

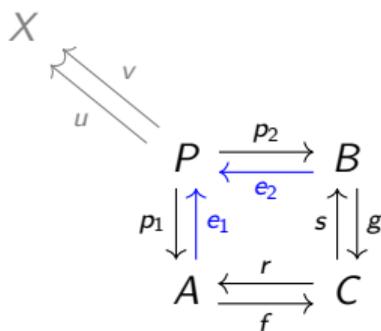
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as above are jointly epimorphic, i.e., if  $ue_1 = ve_1$  and  $ue_2 = ve_2$ , then  $u = v$ ,

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### Remark

In the weakly Mal'tsev category  $\text{CancCommMon}$ , the diagram

$$C_1 := \{(n, m) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid n \leq m\} \begin{array}{c} \xrightarrow{d(n,m)=n} \\ \xleftarrow{e(n)=(n,n)} \\ \xrightarrow{c(n,m)=m} \end{array} \mathbb{N}_0 =: C_0$$

yields an internal category which is not an internal groupoid.

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Proposition (Janelidze, Martins-Ferreira; 2012)

Let  $\mathcal{C}$  be a finitely complete category. TFAE:

- 1  $\mathcal{C}$  is a weakly Mal'tsev category.
- 2 Any reflexive strong relation is a congruence.
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### 3. A syntactic characterization of weakly Mal'tsev varieties

#### Proposition

Let  $\mathbb{V}$  be a variety of universal algebras. TFAE:

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- 3 For any pullback

$$\begin{array}{ccc} P & \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow{e_2} \end{array} & B \\ \begin{array}{c} \downarrow p_1 \\ \uparrow e_1 \end{array} & & \begin{array}{c} \uparrow s \\ \downarrow g \end{array} \\ A & \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{f} \end{array} & C \end{array}$$

as above in  $\mathbb{V}$ , the pullback injections  $e_1, e_2$  are jointly extremally epimorphic.

Proof of "3.  $\Rightarrow$  2."

For any pullback

$$\begin{array}{ccc} P & \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow{e_2} \end{array} & B \\ p_1 \downarrow \uparrow e_1 & & s \uparrow \downarrow g \\ A & \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{f} \end{array} & C \end{array}$$

as above in  $\mathbb{V}$ , the pullback injections  $e_1, e_2$  are jointly extremally epimorphic.

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 P & \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow{e_2} \end{array} & F(x, y) \\
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 F(x, y) & \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{f} \end{array} & F(x)
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are jointly extremally epimorphic, where  $f, r, s$  are the unique morphisms such that  $f(x) = x = f(y)$ ,  $r(x) = x$  and  $s(x) = y$ , respectively.

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$\Leftrightarrow$  The element  $(y, x) \in P = \{(t, t') \in F(x, y) \times F(x, y) \mid t(x, x) = t'(x, x)\}$  lies in  $\text{Im}([e_1, e_2])$ .

### Lemma

It holds that

$$\text{Im}([e_1, e_2]) = \{(p(x, x, y), p(x, y, y)) \mid p \in F(x, y, z)\}.$$

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" $\subseteq$ ": The elements  $(x, x) = e_2(x)$ ,  $(x, y) = e_1(x) = e_2(y)$  and  $(y, y) = e_1(y)$  lie in  $\text{Im}([e_1, e_2])$ .

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Hence, for  $p \in F(x, y, z)$ ,

$$p((x, x), (x, y), (y, y)) = (p(x, x, y), p(x, y, y))$$

also lies in  $\text{Im}([e_1, e_2])$ .

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" $\supseteq$ ": Use that  $F(x, y) + F(x, y)$  can be described as a quotient of  $T(F(x, y) \dot{\cup} F(x, y)) \dots$

## Strategy for weakly Mal'tsev varieties

$\mathbb{V}$  is a weakly Mal'tsev category.

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are jointly epimorphic.

$\Leftrightarrow$  The morphism  $[e_1, e_2] : F(x, y) \rightarrow F(x, y) \rightarrow P$  is an epimorphism. Equivalently, its cokernel projections  $q_1, q_2 : P \rightarrow Q$  are equal.

$$F(x, y) + F(x, y) \xrightarrow{[e_1, e_2]} P \begin{array}{c} \xrightarrow{q_1} \\ \xrightarrow{q_2} \end{array} Q$$

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$\Leftrightarrow$  It holds that

$$q_1(y, x) = q_2(y, x).$$

- We construct  $q_1, q_2$  by means of the commutative diagram

$$\begin{array}{ccc}
 F(x, y) + F(x, y) & \xrightarrow{[e_1, e_2]} & P \\
 [e_1, e_2] \downarrow & & \downarrow \iota_2 \\
 P & \xrightarrow{\iota_1} & P + P \\
 & \searrow q_1 & \searrow q \\
 & & Q
 \end{array}$$

$q_2$

where  $q : P + P \rightarrow Q$  is the coequalizer of  $\iota_1[e_1, e_2], \iota_2[e_1, e_2]$ .

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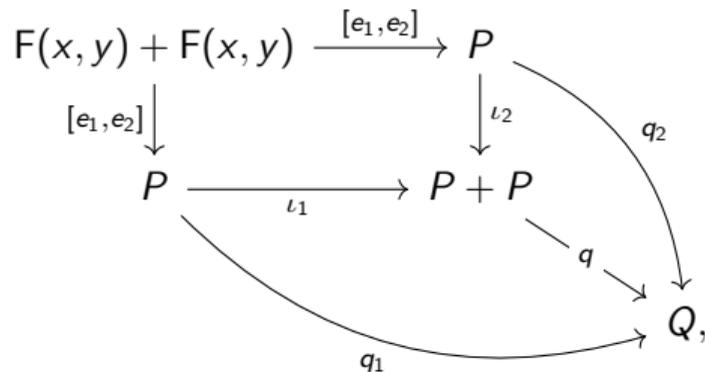
$q_2$  (curved arrow from  $P$  to  $Q$ )

where  $q : P + P \rightarrow Q$  is the coequalizer of  $\iota_1[e_1, e_2], \iota_2[e_1, e_2]$ .

This means that we describe  $Q$  as the quotient of  $P + P$  by the congruence  $C$  which is generated by the subset

$$\{(\iota_1[e_1, e_2](X), \iota_2[e_1, e_2](X)) \mid X \in F(x, y) + F(x, y)\}.$$

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- It holds that  $q_1(y, x) = q_2(y, x)$  if and only if  $(\iota_1(y, x), \iota_2(y, x)) \in C$ .

- Thanks to the above lemma, the congruence  $C$  is generated by the subset

$$S := \{(\iota_1(p(x, x, y), p(x, y, y)), \iota_2(p(x, x, y), p(x, y, y))) \mid p \in F(x, y, z)\}.$$

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- 1 the reflexive closure,
- 2 the symmetric closure,
- 3 the closure under operations,
- 4 the transitive closure of  $S$ .

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- We construct  $C$  by taking
  - 1 the reflexive closure,
  - 2 the symmetric closure,
  - 3 the closure under operations,
  - 4 the transitive closure of  $S$ .
- Using the description of  $P + P$  as a quotient of  $F(P \dot{\cup} P)$ , the condition  $(\iota_1(y, x), \iota_2(y, x)) \in C$  implies the existence of terms as in:

## Theorem (E., Jacqmin, Martins-Ferreira; 2024)

Let  $\mathbb{V}$  be a variety of universal algebras. TFAE:

- 1  $\mathbb{V}$  is a weakly Mal'tsev category.
- 2 In the algebraic theory of  $\mathbb{V}$ , there exist

- $k, m, N \in \mathbb{N}_0$ ,

- binary terms

$$f_1, g_1, \dots, f_k, g_k,$$

- ternary terms

$$p_1, \dots, p_m,$$

- $(1 + k + m)$ -ary terms

$$\eta_1^{(1)}, \eta_2^{(1)}, \varepsilon_1^{(1)}, \varepsilon_2^{(1)}, \dots, \eta_1^{(N+1)}, \eta_2^{(N+1)}, \varepsilon_1^{(N+1)}, \varepsilon_2^{(N+1)},$$

- $(2(1 + k + m) + 2)$ -ary terms

$$\sigma_1, \dots, \sigma_{N+1},$$

- $2(1 + k + 2m)$ -ary terms

$$s_1, \dots, s_N$$

satisfying the identities



$$\begin{aligned}u &= \sigma_1(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(1)}(u, \vec{v}, \vec{w}), \eta_2^{(1)}(u', \vec{v}', \vec{w}')), \\ \sigma_i(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \varepsilon_1^{(i)}(u, \vec{v}, \vec{w}), \varepsilon_2^{(i)}(u', \vec{v}', \vec{w}')) &= s_i(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w}'), \\ s_i(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w}) &= \sigma_{i+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(i)}(u, \vec{v}, \vec{w}), \eta_2^{(i)}(u', \vec{v}', \vec{w}')), \\ \sigma_{N+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \varepsilon_1^{(N+1)}(u, \vec{v}, \vec{w}), \varepsilon_2^{(N+1)}(u', \vec{v}', \vec{w}')) &= u',\end{aligned}$$

where  $\vec{v} = (v_1, \dots, v_k)$  and  $\vec{w} = (w_1, \dots, w_m)$  and analogously for  $\vec{v}'$  and  $\vec{w}'$ ;

- $$\begin{aligned}
 u &= \sigma_1(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(1)}(u, \vec{v}, \vec{w}), \eta_2^{(1)}(u', \vec{v}', \vec{w}')), \\
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 s_i(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w}) &= \sigma_{i+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(i)}(u, \vec{v}, \vec{w}), \eta_2^{(i)}(u', \vec{v}', \vec{w}')), \\
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- $$\begin{aligned}
 &\eta_\alpha^{(i)}(y, f_1(x, y), \dots, f_k(x, y), p_1(x, x, y), \dots, p_m(x, x, y)) \\
 &= \varepsilon_\alpha^{(i)}(y, f_1(x, y), \dots, f_k(x, y), p_1(x, x, y), \dots, p_m(x, x, y)), \\
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 \end{aligned}$$

for all  $\alpha \in \{1, 2\}$  and  $i \in \{1, \dots, N + 1\}$ ;

$$\begin{aligned}
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&\eta_\alpha^{(i)}(x, g_1(x, y), \dots, g_k(x, y), p_1(x, y, y), \dots, p_m(x, y, y)) \\
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\end{aligned}$$

for all  $\alpha \in \{1, 2\}$  and  $i \in \{1, \dots, N + 1\}$ ;

$$f_i(x, x) = g_i(x, x)$$

for all  $i \in \{1, \dots, k\}$ .

### Example of congruence permutable varieties

If  $\mathbb{V}$  is a congruence permutable variety with Mal'tsev term  $p \in F(x, y, z)$  satisfying  $p(x, x, y) = y$  and  $p(x, y, y) = y$ , then

- $k$  can be chosen to be 0: no binary terms necessary,
- $m$  can be chosen to be 1:  $p_1(x, y, z) = p(x, y, z)$ ,
- $N$  can be chosen to be 1: all the terms  $\eta_\alpha^{(i)}, \varepsilon_\alpha^{(i)}, \sigma_i, s_j$  can be chosen to be projections.

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### Example of varieties of distributive lattices

If  $\mathbb{V}$  is a variety of distributive lattices, then

- $k$  can be chosen to be 0: no binary terms necessary,
- $m$  can be chosen to be 3:  $p_1(x, y, z) = x$ ,  $p_2(x, y, z) = y$ ,  $p_3(x, y, z) = z$ ,
- $N$  can be chosen to be 5.

Theorem (E., Jacqmin, Martins-Ferreira; 2024)

Let  $\mathbb{V}$  be a variety of universal algebras. TFAE:

- 1 Any reflexive regular relation in  $\mathbb{V}$  is symmetric.
- 2 There are terms as in the above theorem that satisfy the same identities except

$$f_i(x, x) = g_i(x, x)$$

for all  $i \in \{1, \dots, k\}$ .

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