

A Few Words on Metric Ultraproducts of Groups¹

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Ultraproduct

Let

- 1 $\mathcal{G} = (G_m)_{m \in \mathbb{N}}$ be a family of groups (viewed as first-order structures);
- 2 \mathcal{U} be a non-principal ultrafilter on \mathbb{N} .

Definition

The *ultraproduct* of \mathcal{G} is defined as

$$\mathcal{G}^* = \prod_{m \in \mathbb{N}} G_m / \sim$$

where

$$(g_m) \sim (h_m) \iff \{m \in \mathbb{N} : g_m = h_m\} \in \mathcal{U}$$

Ultraproducts are fundamental tools. One can derive the Compactness Theorem using ultraproducts, and they have many applications in mathematics (algebra, topology).

Generalization: metric ultraproduct

A *metric group* $(G, \|\cdot\|)$ is a group with a norm $\|\cdot\|: G \rightarrow [0, \infty]$ such that $\|gh\| \leq \|g\| + \|h\|$, $\|g^{-1}\| = \|g\|$, $\|g\| = 0 \Leftrightarrow g = e$, and $\|g\| = \|hgh^{-1}\|$ (**bi-invariance**).

A norm induces a metric $\|\cdot\| \rightsquigarrow d(g, h) = \|gh^{-1}\|$ and conversely $d(\cdot, \cdot) \rightsquigarrow \|g\| = d(g, e)$.

Definition

Let $\mathcal{G} = (G_m, \|\cdot\|_m)_{m \in \mathbb{N}}$ be a family of metric groups, and \mathcal{U} a non-principal ultrafilter on \mathbb{N} . The *metric ultraproduct* is

$$\mathcal{G}_{\text{met}}^* = \prod_{m \in \mathbb{N}}^{\text{met}} G_m = G_{\text{fin}} / N_{\mathcal{U}}$$

where

$$G_{\text{fin}} = \left\{ (g_m) \in \prod_{m \in \mathbb{N}} G_m : \sup_{m \in \mathbb{N}} \|g_m\|_m < \infty \right\} \text{ and } N_{\mathcal{U}} = \left\{ (g_m) : \lim_{m \rightarrow \mathcal{U}} \|g_m\|_m = 0 \right\}$$

(the infinitesimal subgroup $N_{\mathcal{U}}$ is a normal subgroup of G_{fin}).

$\mathcal{G}_{\text{met}}^*$ is a topological space; the topology is induced by the canonical bi-invariant norm:

$$\|\cdot\|: \mathcal{G}_{\text{met}}^* \rightarrow [0, \infty] \text{ defined by } \|(g_m)/N_{\mathcal{U}}\| = \lim_{m \rightarrow \mathcal{U}} \|g_m\|_m.$$

Examples of bi-invariant norms on groups

Examples of bounded and unbounded norms:

- Discrete norm: $\|g\| := \begin{cases} 1 & : g \neq e \\ 0 & : g = e \end{cases}$
- Hamming norm on S_n : for $\sigma \in S_n$, $\|\sigma\|_H := |\{i : \sigma(i) \neq i\}|$
- Rank norm on $GL_n(F)$ (F : field): $\|g\|_r := \text{rank}(g - I)$
- Conjugacy length (pseudonorm) on a finite group G :

$$\|g\|_\ell := \frac{\log |g^G|}{\log |G|},$$

This is a norm if $Z(G) = \{e\}$.

- Invariant word norm on a group G : let $S = S^{-1} \subseteq G$ be a normal subset (i.e., $s \in S \implies s^x = x^{-1}sx \in S$)

$$\|g\|_S = \min\{n : g \text{ is a product of } n \text{ elements from } S\}.$$

Motivation: Gottschalk conjecture, “1-1” \Rightarrow “onto”

Let A be a finite set and G be a group. G acts on

$$A^G = \{f: G \rightarrow A\}$$

by homeomorphisms

$$g \cdot f(x) = f(g^{-1}x), \text{ for } g, x \in G, f \in A^G.$$

This action is called the *Bernoulli shift*.

Definition

A *cellular automaton* is a function $T: A^G \rightarrow A^G$, which is continuous and G -equivariant:
 $T(g \cdot f) = g \cdot T(f)$.

Conjecture (W. Gottschalk, '72)

If a cellular automaton $T: A^G \rightarrow A^G$ is “1-1” (injective), then T is “onto” (surjective).

Theorem (Ax–Grothendieck theorem)

If $P: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a polynomial mapping and P is “1-1”, then P is “onto”.

Proof: Application of the Compactness Theorem to an elementary statement regarding polynomials over finite fields.

Metric ultraproduct and Gottschalk conjecture

Let

$$\mathcal{S} = \prod_{m \in \mathbb{N}}^{\text{met}} \left(\mathcal{S}_m, \frac{1}{m} \|\cdot\|_H \right)$$

be a metric ultraproduct of permutation groups \mathcal{S}_m , with the normalized Hamming norm.

Definition

- A group G is *sofic* if G is isomorphic to a subgroup of a metric ultraproduct of permutation groups with the normalized Hamming norm: $G < \mathcal{S}$.
- \mathcal{S} is called a *universal sofic group*.

Theorem (M. Gromov '99)

The Gottschalk conjecture holds for sofic groups: If a cellular automaton $T: A^S \rightarrow A^S$ is "1-1", then T is "onto".

Major open problem in group theory: Are all finitely generated groups sofic?

Aim

Our aim is to understand the (first-order metric) logical structure of metric ultraproducts in general.

Compactness theorem for ultraproduct

Suppose $\mathcal{G}^* = \prod_{m \in \mathbb{N}} G_m / \sim$ is an ultraproduct of a family of groups $\mathcal{G} = (G_m)_{m \in \mathbb{N}}$.

A subset $X \subseteq \mathcal{G}^*$ is *internal* if there is a collection of sets $\{X_n\}_{n \in \mathbb{N}}$, $X_n \subseteq G_n$ such that

$$X = \frac{X_0 \times X_1 \times X_2 \times \cdots}{\sim}.$$

Theorem (The standard compactness theorem)

Let $(X_m)_{m \in \mathbb{N}}$ be a sequence of internal subsets of \mathcal{G}^* . The following conditions are equivalent:

- 1 $\mathcal{G}^* = \bigcup_{m \in \mathbb{N}} X_m$
- 2 There is an $N \in \mathbb{N}$ such that

$$\mathcal{G}^* = X_0 \cup \dots \cup X_N$$

Compactness theorem for metric ultraproduct

The standard theorem is **not true** for metric ultraproducts because it does not take into account the metric. To obtain the true theorem, we need to make some modifications.

Suppose $\mathcal{G}_{\text{met}}^* = \prod_{m \in \mathbb{N}}^{\text{met}} G_m$ and assume that $\sup_{m \in \mathbb{N}} \{\|\cdot\|_m\} < \infty$.

A subset $X \subseteq \mathcal{G}_{\text{met}}^*$ is *metrically internal* if there is a collection of sets $\{X_n\}_{n \in \mathbb{N}}$, $X_n \subseteq G_n$ such that

$$X = \frac{X_0 \times X_1 \times X_2 \times \cdots}{N_{\mathcal{U}}}.$$

Theorem (J. G., K. Majcher, M. Ziegler, Journal of Algebra 663 (2025): 171-209)

Let $(X_m)_{m \in \mathbb{N}}$ be a sequence of metrically internal subsets of $\mathcal{G}_{\text{met}}^*$. The following conditions are equivalent:

- 1 $\mathcal{G}_{\text{met}}^* = \bigcup_{m \in \mathbb{N}} X_m$
- 2 For any sequence $(\varepsilon_0, \varepsilon_1, \dots) \subset \mathbb{R}_{>0}$ there is an $N \in \mathbb{N}$ such that

$$\mathcal{G}_{\text{met}}^* = X_0 \cdot \mathcal{B}(\varepsilon_0) \cup \dots \cup X_N \cdot \mathcal{B}(\varepsilon_N),$$

where $\mathcal{B}(\varepsilon) = \{g \in \mathcal{G}_{\text{met}}^* : \|g\| < \varepsilon\}$.

Corollary (using Baire category theorem)

Let $(X_n)_{n \in \mathbb{N}}$ be an increasing sequence of internal subsets of $\mathcal{G}_{\text{met}}^*$. If $\mathcal{G}_{\text{met}}^* = \bigcup_{m \in \mathbb{N}} X_m$, then there is an $N \in \mathbb{N}$ such that $\mathcal{G}_{\text{met}}^* = X_N \cdot X_N$.

Application: ultraproducts as simple groups

A group G is *simple* if G has no nontrivial proper normal subgroups.

Question

When is a (discrete) ultraproduct of groups simple?

In the case of the discrete norm, we have the following criterion.

Fact (well known)

Let $\mathcal{G}^* = \prod_{m \in \mathbb{N}} G_m / \sim$ be an ultraproduct of a family of groups $\mathcal{G} = (G_m)_{m \in \mathbb{N}}$. The following conditions are equivalent:

- 1 \mathcal{G}^* is a simple group,
- 2 there is an $N \in \mathbb{N}$, such that for \mathcal{U} -almost all $m \in \mathbb{N}$, G_m is N -uniformly simple; that is

$$G_m = \left(g^{G_m} \cup g^{-1G_m} \right)^{\leq N} \text{ holds}$$

for all $g \in G_m, g \neq e$, where $g^G = \{h^{-1}gh : h \in G\}$.

When is a metric ultraproduct of groups simple?

Suppose $\mathcal{G}_{\text{met}}^* = \prod_{m \in \mathbb{N}}^{\text{met}} G_m$ and assume that $\sup_{m \in \mathbb{N}} \{\| \cdot \|_m\} < \infty$.

Theorem (JG, K. Majcher, M. Ziegler)

The following conditions are equivalent:

- 1 $\mathcal{G}_{\text{met}}^*$ is a simple group,
- 2 for any $r > 0$ and every infinite sequence $(\varepsilon_0, \dots, \varepsilon_n, \dots) \subset \mathbb{R}_{>0}$, there is an $N \in \mathbb{N}$ such that for \mathcal{U} -almost all $m \in \mathbb{N}$, for all $g \in G_m$ with $r < \|g\|$:

$$G_m = C_0(g) \cdot \mathcal{B}(\varepsilon_0) \cup \dots \cup C_N(g) \cdot \mathcal{B}(\varepsilon_N),$$

where $C_k(g) = \left(g^{G_m} \cup g^{-1}G_m \right)^{\leq k}$ and $\mathcal{B}(\varepsilon) = \{g \in G_m : \|g\| < \varepsilon\}$.

Examples of simple metric ultraproducts of groups

The theorem from the previous slide allows us to easily establish the simplicity of the already known metric ultraproducts and to find other simple metric ultraproducts of groups.

Fact

The following groups are simple:

- $\prod_{m \in \mathbb{N}}^{met} (S_m, \frac{1}{m} \|\cdot\|_H)$;
- $\prod_{m \in \mathbb{N}}^{met} (S_\infty, \frac{1}{m} \|\cdot\|_H)$, where $S_\infty = \bigcup_{m \in \mathbb{N}} S_m$;
- $\prod_{m \in \mathbb{N}}^{met} (S_\infty, c_m \|\cdot\|_H)$, where $(c_m)_{m \in \mathbb{N}} \subset \mathbb{R}_{>0}$ is such that $\lim_{m \rightarrow \mathcal{U}} c_m = 0$;
- a metric ultrapower of groups of interval exchange transformations with respect to the $\|g\|_\mu = \mu(\text{supp}(g))$ norm (i.e. the Lebesgue measure of the support).

More examples: metric ultrapower of matrix groups with the rank norm

Suppose K_m is a field and $G_m = \mathrm{PSL}_m(K_m)$ for $m \in \mathbb{N}$ (more generally $G_m(K_m)$ – simple centerless Chevalley group).

If the fields K_m are finite,

$$\prod_{m \in \mathbb{N}}^{\mathrm{met}} (G_m, \|\cdot\|_\ell)$$

is simple, where $\|g\|_\ell := \frac{\log|g^G|}{\log|G|}$ (Stolz - Thom '14, Nikolov '09).

What about infinite fields? Consider another norm, e.g., the rank norm:

$$\|g\|_r := \frac{1}{m} \mathrm{rank}(g - I), \text{ for } g \in G_m.$$

We conjecture that for all fields K_m , the metric ultrapower $\prod_{m \in \mathbb{N}}^{\mathrm{met}} (G_m(K_m), \|\cdot\|_r)$ is simple. We can prove for algebraically closed fields.

Corollary

If each field K_m is algebraically closed, then $\prod_{m \in \mathbb{N}}^{\mathrm{met}} (G_m(K_m), \|\cdot\|_r)$ is simple.

Idea of the proof: $\|\cdot\|_\ell$ and $\|\cdot\|_r$ are asymptotically equivalent. We use the Liebeck-Shalev result for the rank norm, which is first-order expressible.

An open problem

In every ultraproduct of groups $\mathcal{G}^* = \prod_{m \in \mathbb{N}} G_m / \sim$, it is a known fact that if there are elements of arbitrarily large finite order, then there is an element of infinite order:

$$\forall n \in \mathbb{N} \exists g_n \in \mathcal{G}^* \quad n < o(g_n) < \infty \implies \exists g_\infty \in \mathcal{G}^* \quad o(g_\infty) = \infty.$$

What about the metric ultraproduct $\mathcal{G}_{\text{met}}^*$? We have proved this holds if $\mathcal{G}_{\text{met}}^*$ is nilpotent.

However, this does not seem to be true in general for $\mathcal{G}_{\text{met}}^*$, though we have not found a counterexample.

Thank you for your attention!