LOWNESS FOR EFFECTIVE HAUSDORFF DIMENSION

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ABSTRACT. We examine the sequences A that are *low for dimension*, i.e., those for which the effective (Hausdorff) dimension relative to A is the same as the unrelativized effective dimension. Lowness for dimension is a weakening of *lowness for randomness*, a central notion in effective randomness. By considering analogues of characterizations of lowness for randomness, we show that lowness for dimension can be characterized in several ways. It is equivalent to *lowishness for randomness*, namely, that every Martin-Löf random sequence has effective dimension 1 relative to A, and *lowishness for K*, namely, that the limit of $K^A(n)/K(n)$ is 1.

We show that there is a perfect Π_1^0-class of low for dimension sequences. Since there are only countably many low for random sequences, many more sequences are low for dimension. Finally, we prove that every low for dimension is jump-traceable in order n^ε, for any $\varepsilon > 0$.

1. INTRODUCTION

The *effective (Hausdorff) dimension* of an infinite sequence $A \in 2^\omega$ is

$$\dim(A) = \liminf_{n \to \infty} \frac{K(A \restriction n)}{n}.$$

This notion was introduced by Lutz [14], who effectivized a martingale characterization of Hausdorff dimension and defined $\dim(A)$ to be the effective Hausdorff dimension of $\{A\}$. The characterization above, which reveals effective dimension to be a natural measure of the information density of infinite sequences, was given by Mayordomo [15] (see also Ryabko [19]).

We say that $A \in 2^\omega$ is *low for (effective Hausdorff) dimension* if

$$(\forall X \in 2^\omega)[\dim^A(X) \geq \dim(X)].$$

In other words, A is too weak as an oracle to change the effective Hausdorff dimension of any sequence. It is clear that $\dim^A(X) \leq \dim(X)$, for any $A \in 2^\omega$, so if A is low for dimension we have $\dim^A(X) = \dim(X)$ for all X.

This paper initiates the study of lowness for dimension. Our main result gives several characterizations of the notion. These can be seen as weakenings of well-known characterizations of lowness for randomness. This analogy helped direct our research and will be discussed in detail below.

We need one more definition before we can state the main theorem. Given a set of strings $W \subseteq 2^{<\omega}$ and a real $s \in [0, 1]$, the *s-dimensional weight* of W is $\mathrm{wt}_s(W) = \sum_{\sigma \in W} 2^{-s|\sigma|}$.

Main Theorem. *The following are equivalent for $A \in 2^\omega$:*

(1) *A is* lowish for (Martin-Löf) randomness, *i.e., if $X \in 2^\omega$ is Martin-Löf random, then $\dim^A(X) = 1$.*

(2) *There is an $s \in (0, 1]$ such that $(\forall X \in 2^\omega)[\, \dim(X) = s \Rightarrow \dim^A(X) = s \,]$.*

(3) *A is low for dimension.*

(4) *A has the Σ^0_1-covering property, i.e., if $W \subseteq 2^{<\omega}$ is A-c.e., $0 \le s < t \le 1$, and $\mathrm{wt}_s(W) < 1$, then there is a c.e. set V with $\mathrm{wt}_t(V) < 1$ such that $[W] \subseteq [V]$.*

(5) *A has the* c.e. covering property, *i.e., if $W \subseteq 2^{<\omega}$ is A-c.e., $0 \le s < t \le 1$, and $\mathrm{wt}_s(W) < 1$, then there is a c.e. set $V \supseteq W$ with $\mathrm{wt}_t(V) < \infty$.*

(6) $\displaystyle \liminf_{\sigma \to \infty} \frac{K^A(\sigma) - K(\sigma)}{|\sigma|} \ge 0.$

(7) *A is* lowish for K, *i.e.,* $\displaystyle \liminf_{\sigma \to \infty} \frac{K^A(\sigma)}{K(\sigma)} \ge 1.$

Some of these properties are easily shown to be equivalent to superficially stronger forms. It is clear that $\limsup_{\sigma \to \infty} K^A(\sigma)/K(\sigma) \le 1$, so (7) is equivalent to

(7′) $\displaystyle \lim_{\sigma \to \infty} \frac{K^A(\sigma)}{K(\sigma)} = 1.$

In the same way, (6) is equivalent to

(6′) $\displaystyle \lim_{\sigma \to \infty} \frac{K^A(\sigma) - K(\sigma)}{|\sigma|} = 0.$

Finally, in (5) we can discard finitely many of the strings in $V \smallsetminus W$ to ensure that $\mathrm{wt}_t(V) < 1$.

Analogy with lowness for randomness. Recall that a sequence $A \in 2^\omega$ is *K-trivial* if $K(A \restriction n) \le^+ K(n)$,[1] in other words, if its initial segment prefix-free complexity is as low as possible. The class of K-trivial sequences has been a rich area of study in recent years, and many diverse characterizations have been found. See Nies [18] or Downey and Hirschfeldt [5] for a thorough introduction to K-triviality. The characterizations of lowness for dimension in the main theorem are analogous to some of the characterizations of K-triviality (though not K-triviality itself) stated in the following

Theorem 1.1. *The following are equivalent for $A \in 2^\omega$:*

(a) *A is K-trivial.*

(b) *A is* low for (Martin-Löf) randomness, *i.e., every Martin-Löf random sequence is Martin-Löf random relative to A.*

(c) *A is* low for K, *i.e., $K(\sigma) \le^+ K^A(\sigma)$.*

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[1]We use $f(n) \le^+ g(n)$ to mean $(\exists c)(\forall n)[f(n) \le g(n) + c]$.

(d) *Every $\Sigma_1^0[A]$-class of measure less than one is contained in a Σ_1^0-class of measure less than one.*

It is easy to see that (c) implies (a) and (b). The equivalence of (a)–(c) was proved in Nies [16], where the proof that every K-trivial is low for K is attributed to Hirschfeldt and Nies. Kjos-Hanssen [10] proved that (d) is equivalent to the others.

There is an obvious similarity between the properties in the main theorem and Theorem 1.1. Property (1) is a direct weakening of (b), though (2) and (3) have a similar feel. Properties (6) and (7) are both natural weakenings of (c), while property (4) is a natural analogue of (d). Property (5) is also analogous to a known characterization of K-triviality; see Simpson [20, Lemma 5.11] for a strong version.

It is clear that lowness for randomness implies lowness for dimension. What about the converse? Hirschfeldt and Weber [9] constructed a sequence $A \in 2^\omega$ that has finite self-information (see Section 7) but is not low for randomness. They observed that their sequence is also low for dimension, so lowness for dimension does not imply lowness for randomness. We can also see this as a corollary to Theorem 5.1, where we prove that there are continuum many sequences that are low for dimension (see also Herbert [8]). Chaitin [3] proved that every K-trivial is Δ_2^0, so there are only countably many low for random sequences.

Structure of the paper. The next three sections are dedicated to the proof of the main theorem. The equivalence of the seven properties will be shown as follows:
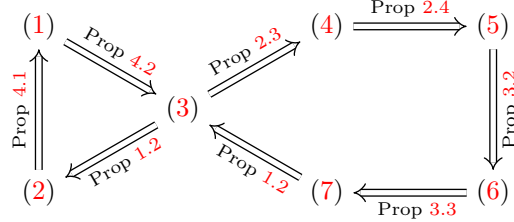
There are several trivial implications between the various properties, two of which appear in the diagram.

Proposition 1.2.
- $(3) \Rightarrow (2)$.
- $(7) \Rightarrow (3)$.

Proof. If A is low for dimension, then for every $X \in 2^\omega$ and $s \in [0,1]$, we have $\dim(X) = s$ implies $\dim^A(X) = s$. So it is clear that (3) implies (2).

Now assume that A is lowish for K, i.e., satisfies (7). Then for any $X \in 2^\omega$,

$$\dim^A(X) = \liminf_{n\to\infty} \frac{K^A(X \restriction n)}{n} = \liminf_{n\to\infty} \frac{K(X \restriction n)}{n} \frac{K^A(X \restriction n)}{K(X \restriction n)}$$

$$\geq \liminf_{n\to\infty} \frac{K(X \restriction n)}{n} = \dim(X).$$

So A is low for dimension, i.e., (3) holds. $\square$

In Section 5, we produce a perfect Π_1^0-class of low for dimension sequences, proving that there are continuum many. In Section 6, we investigate the degrees

of sequences that are low for dimension. The main result in this section is that if $A \in 2^\omega$ is low for dimension and $\varepsilon > 0$, then A is jump-traceable with bound n^ε. No slower-growing bound is known to permit continuum many jump-traceable sequences. The final section presents a few open questions related to our work.

We assume that the reader is familiar with elementary computability theory and algorithmic randomness. Downey and Hirschfeldt [5] and Nies [18] are excellent references for these topics, although Nies does not cover effective dimension. One tool that we use repeatedly is the bounded request set. Let W be a c.e. set of *requests* of the form $\langle \sigma, d \rangle$ where $\sigma \in 2^{<\omega}$ and $d \in \omega$. We call W a *bounded request set* if $\sum_{\langle \sigma, d \rangle \in W} 2^{-d} < \infty$.[2] If W is a bounded request set, then $(\forall \langle \sigma, d \rangle \in W)[K(\sigma) \leq^+ d]$. This follows from the Kraft–Chaitin Theorem.

2. THE COVERING PROPERTIES

In this section, we show that if $A \in 2^\omega$ is low for dimension then it satisfies both covering properties. Although the statements of the covering properties are very similar, they are on two sides of a divide between properties involving infinite sequences and properties involving finite strings. This is what separates (1)–(4) from (5)–(7). Bridging this gap is the most delicate part of proving the main theorem.

Let $s \in [0,1]$ and $U \subseteq 2^{<\omega}$. The *prefix-free s-weight* of U is $\mathrm{pfwt}_s(U) = \sup\{\mathrm{wt}_s(V) : V \subseteq U$ is prefix-free$\}$. We say that U is *s-closed* if for every $\tau \in 2^{<\omega}$, if $\mathrm{pfwt}_s(\{\sigma \in U : \tau \preccurlyeq \sigma\}) \geq 2^{-s|\tau|}$, then $\tau \in U$. Under reasonable assumptions, a c.e. set has a c.e. s-closed superset with almost the same prefix-free s-weight:

Lemma 2.1. *Let $r \in [0,1]$ be rational and $R \subseteq 2^{<\omega}$ c.e. For every $\varepsilon > 0$, there is an r-closed c.e. set $U \supseteq R$ such that $\mathrm{pfwt}_r(U) \leq \mathrm{pfwt}_r(R) + \varepsilon$.*

Proof. We may assume that ε is rational. Let $U \subseteq 2^{<\omega}$ be the closure of R under the requirement that for every $\tau \in 2^{<\omega}$, if there is a finite prefix-free set $S \subseteq U$ of extensions of τ such that $\mathrm{pfwt}_r(S) > 2^{-r|\tau|} - 2^{-2|\tau|-1}\varepsilon$, then $\tau \in U$. It is easy to see that U is c.e. and r-closed. Now let $T \subseteq U$ be a finite prefix-free set. By recursively replacing each $\tau \in T \smallsetminus R$ with the finite prefix-free set of extensions that resulted in its inclusion in U, we get a finite prefix-free $T' \subseteq R$ such that

$$\mathrm{wt}_r(T) < \mathrm{wt}_r(T') + \sum_{\tau \in 2^{<\omega}} 2^{-2|\tau|-1}\varepsilon \leq \mathrm{pfwt}_r(R) + \varepsilon.$$

Therefore, $\mathrm{pfwt}_r(U) \leq \mathrm{pfwt}_r(R) + \varepsilon$. $\qquad\square$

If $W \subseteq 2^{<\omega}$, define W^n for all $n \in \omega$ as follows. Let $W^0 = \{\lambda\}$, where λ is the empty sequence. Let $W^{n+1} = \{\sigma\tau : \sigma \in W^n$ and $\tau \in W\}$. The next lemma is completely routine.

Lemma 2.2. *Let $s \in [0,1]$ and let $W \subseteq 2^{<\omega}$ be a c.e. set such that $\mathrm{wt}_s(W) < 1$. If $X \in [W^n]$ for all $n \in \omega$, then $\dim(X) \leq s$.*

Proof. Let $U = \bigcup_{n\in\omega} W^n$. Let $r > s$ be rational. Note that $\mathrm{wt}_s(W^n) = \mathrm{wt}_s(W)^n$ for all $n \in \omega$. Therefore, $\mathrm{wt}_r(U) \leq \mathrm{wt}_s(U) \leq \sum_{n\in\omega} \mathrm{wt}_s(W)^n < \infty$. This means that $\{\langle \sigma, r|\sigma| \rangle : \sigma \in U\}$ is a bounded request set, so $(\forall \sigma \in U)[K(\sigma) \leq^+ r|\sigma|]$. Infinitely many prefixes of X are in U, so $\dim(X) \leq r$. But $r > s$ was any rational number, so $\dim(X) \leq s$. $\qquad\square$

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[2]This definition differs slightly from Nies [18], where the sum is required to be ≤ 1.

Proposition 2.3. (3) $\Rightarrow$ (4).

Proof. We prove the contrapositive, so assume that the Σ_1^0-covering property fails for $A \in 2^\omega$. Fix a counterexample: Let $W \subseteq 2^{<\omega}$ be an A-c.e. set and let $s, t \in [0, 1]$ be real numbers such that $s < t$ and $\mathrm{wt}_s(W) < 1$. Assume that there is no c.e. set V such that $\mathrm{wt}_t(V) < 1$ and $[W] \subseteq [V]$.

Fix a rational r such that $s < r < t$. Consider a c.e. set U such that $\mathrm{pfwt}_r(U) < 1$. We claim that $[W] \not\subseteq [U]$. To see this, note that

$$\mathrm{wt}_t(U) = \sum_{n \in \omega} |U \cap 2^n| 2^{-tn} = \sum_{n \in \omega} |U \cap 2^n| 2^{-rn} 2^{(r-t)n}$$

$$\leq \sum_{n \in \omega} \mathrm{pfwt}_r(U) 2^{(r-t)n} < \sum_{n \in \omega} 2^{(r-t)n} < \infty.$$

Since $\mathrm{wt}_t(U) < \infty$, there is an $m \in \omega$ such that $\mathrm{wt}_t(U \smallsetminus 2^{\leq m}) < 1 - \mathrm{pfwt}_r(U)$. Since $U \cap 2^{\leq m}$ is finite, it is covered by a finite prefix-free subset V_0 of itself. Of course, $\mathrm{wt}_t(V_0) \leq \mathrm{pfwt}_t(U) \leq \mathrm{pfwt}_r(U)$. Let $V = V_0 \cup (U \smallsetminus 2^{\leq m})$. Then V is c.e., $[V] = [U]$, and $\mathrm{wt}_t(V) = \mathrm{wt}_t(V_0) + \mathrm{wt}_t(U \smallsetminus 2^{\leq m}) < \mathrm{pfwt}_r(U) + (1 - \mathrm{pfwt}_r(U)) = 1$. By assumption, $[W] \not\subseteq [V] = [U]$.

Let $R = \{\sigma \in 2^{<\omega} : K(\sigma) \leq r|\sigma|\}$. So R is a c.e. set and

$$\mathrm{pfwt}_r(R) \leq \mathrm{wt}_r(R) = \sum_{\sigma \in R} 2^{-r|\sigma|} \leq \sum_{\sigma \in 2^{<\omega}} 2^{-K(\sigma)} < 1.$$

Note that if $X \notin [R]$, then $\dim(X) \geq r$. By Lemma 2.1, there is an r-closed c.e. set $U \supseteq R$ such that $\mathrm{pfwt}_r(U) < 1$.

We construct a sequence of strings $\{\sigma_k\}_{k \in \omega}$ such that for all $k \in \omega$, $\sigma_k \in W$, and $X = \sigma_0 \sigma_1 \sigma_2 \cdots$ is not in $[U]$. Let $\tau_k = \sigma_0 \cdots \sigma_{k-1}$, so in particular, $\tau_0 = \lambda$. Since $\mathrm{pfwt}_r(U) < 1$, we know that $[\tau_0] \not\subseteq [U]$. Now suppose that τ_k has been defined and $[\tau_k] \not\subseteq [U]$. Consider $U_{\tau_k} = \{\sigma \in 2^{<\omega} : \tau_k \sigma \in U\}$. Since U is r-closed, $\mathrm{pfwt}_r(U_{\tau_k}) \geq 1$ would imply that $\tau_k \in U$. Therefore, $\mathrm{pfwt}_r(U_{\tau_k}) < 1$, so by the discussion above, there is a $\sigma_k \in W$ such that $[\sigma_k] \not\subseteq [U_{\tau_k}]$. This means that $[\tau_{k+1}] = [\tau_k \sigma_k] \not\subseteq [U]$.

The construction ensures that $X \notin [U]$. Since $[R] \subseteq [U]$, we have $X \notin [R]$ and so $\dim(X) \geq r$. On the other hand, $X \in [W^n]$ for all $n \in \omega$, so $\dim^A(X) \leq s$ by Lemma 2.2 relativized to A. Therefore, $\dim^A(X) \leq s < r \leq \dim(X)$. Hence A is not low for dimension. $\square$

We are ready to make the transition from the world of infinite sequences to the world of finite strings.

Proposition 2.4. (4) $\Rightarrow$ (5).

Proof. Assume that $A \in 2^\omega$ has the Σ_1^0-covering property. To prove that A has the c.e. covering property, it is sufficient to consider *rational* numbers s and t with $0 \leq s < t < 1$. Fix these, and let $r = \frac{s}{1-t+s} = t - \frac{(1-t)(t-s)}{1-t+s}$. Note that $s < r < t$. We will apply the Σ_1^0-covering property to the pair s, r, which gives us extra room when proving the c.e. covering property for s, t.

Let $\{S_e\}_{e \in \omega}$ be an effective list of all A-c.e. sets of strings (sufficiently universal to apply the Recursion Theorem below). Uniformly in $e \in \omega$, there is an A-c.e. set $\widehat{S}_e \subseteq S_e$ such that $\mathrm{wt}_s(\widehat{S}_e) \leq 2^{-e-1}$ and if $\mathrm{wt}_s(S_e) \leq 2^{-e-1}$, then $\widehat{S}_e = S_e$. Let

$S = \bigcup_{e \in \omega} \widehat{S}_e$. Then

$$\mathrm{wt}_s(S) \leq \sum_{e \in \omega} \mathrm{wt}_s(\widehat{S}_e) < \sum_{e \in \omega} 2^{-e-1} = 1,$$

where the strict inequality follows from the fact that $S_e = \emptyset$ for some $e \in \omega$. By the Σ_1^0-covering property, there is a c.e. set $R \subseteq 2^{<\omega}$ such that $[S] \subseteq [R]$ and $\mathrm{wt}_r(R) < 1$. This is our only use of the Σ_1^0-covering property.

Let $W \subseteq 2^{<\omega}$ be an A-c.e. set such that $\mathrm{wt}_s(W) < 1$. Our goal is to find a c.e. set $V \supseteq W$ such that $\mathrm{wt}_t(V) < \infty$. We will build an auxiliary A-c.e. set U. By the Recursion Theorem, we may assume that the construction of U has access to a parameter $e \in \omega$ such that $U = S_e$. As long as we ensure that $\mathrm{wt}_s(U) \leq 2^{-e-1}$, then $[U] = [\widehat{S}_e] \subseteq [R]$.

Before we proceed to the details, it will be helpful to sketch the construction of V. When a string σ is enumerated into W, we will respond by putting a *code string* τ into U and waiting for $[\tau]$ to be covered by $[R]$. Say that this happens because $\rho \preccurlyeq \tau$ goes into R. If ρ and τ are approximately of the same length, then we can use the enumeration of ρ as permission to put σ into V. If ρ is much shorter than τ, then its enumeration gives us permission to try again by putting a new code string into U for the sake of σ. Of course, it is possible that $[\tau]$ is covered by a collection of longer strings in R. To eliminate the need to consider this possibility explicitly, we replace R by the set Q of all strings that are covered by extensions in R. For $n \in \omega$, let $2^{\geq n} = \{\rho \in 2^{<\omega} : |\rho| \geq n\}$. Let

$$Q = \left\{ \tau \in 2^{<\omega} : [\tau] \subseteq \left[R \cap 2^{\geq |\tau|} \right] \right\}.$$

It is clear that $[Q] = [R]$. We want to show that $\mathrm{wt}_r(Q) < \infty$. Consider the contribution that a string $\rho \in R$ makes to $\mathrm{wt}_r(Q)$. For every $\tau \preccurlyeq \rho$ that ends up in Q, we could attribute $2^{|\tau|-|\rho|}$ of its r-weight $2^{-r|\tau|}$ to ρ. In this way, the total cost of $\tau \in Q$ is split up between the strings from R that cover it. So

$$\mathrm{wt}_r(Q) \leq \sum_{\rho \in R} \sum_{m \leq |\rho|} 2^{m-|\rho|} 2^{-rm} \leq \sum_{\rho \in R} O(1) 2^{-r|\rho|} = O(1) \, \mathrm{wt}_r(R) < \infty.$$

We will also use the fact that

$$\mu[Q] = \mu[R] \leq \sum_{\rho \in R} 2^{-|\rho|} \leq \sum_{\rho \in R} 2^{-r|\rho|} = \mathrm{wt}_r(R) < 1.$$

Fix $c \in \omega$ large enough that $\mu[Q] \leq 1 - 2^{-c}$ and $2^{-sc} \leq 2^{-e-3}$. Fix $d \in \omega$ large enough that $2^{-d} \mathrm{wt}_r(Q) \leq 2^{-e-2}$.

Let $\{\rho_k\}_{k \in \omega}$ be a computable enumeration of Q and let $Q_k = \{\rho_i\}_{i<k}$. We dynamically assign code strings. At each stage $k \in \omega$, we have a (computable) injection $\tau_k \colon 2^{<\omega} \to 2^{<\omega}$ such that for every $\sigma \in 2^{<\omega}$

- $|\tau_k(\sigma)| = |\sigma| + c$,
- no prefix of $\tau_k(\sigma)$ is in Q_k, and
- if $\tau_k(\sigma) \neq \tau_{k+1}(\sigma)$, then there is a prefix of $\tau_k(\sigma)$ in Q_{k+1}.

Note that it is always possible to reassign strings: $\mu[Q] < 1 - 2^{-c}$ implies that, for each $n \in \omega$, there are at least $2^{-c} 2^{n+c} = 2^n$ usable strings of length $n + c$.

The purpose of the code strings should be clearer if we describe how Q is decoded to produce a c.e. set $V \subseteq 2^{<\omega}$. If $\rho \in 2^{<\omega}$ enters Q at stage k, $\tau_k(\sigma) \succcurlyeq \rho$, and $|\tau_k(\sigma)| < \frac{r|\rho|+d}{s}$, then put σ into V. Note that several strings may enter V in

response to a single ρ entering Q. Despite this, we can show that $\mathrm{wt}_t(V) < \infty$. Consider $\rho \in Q$ with $n = |\rho|$. The contribution of ρ to $\mathrm{wt}_t(V)$ is at most

$$\sum_{m < \frac{rn+d}{s} - n} 2^m 2^{-t(n+m-c)} \leq O(1) 2^{(1-t)\left[\frac{rn+d}{s} - n\right] - tn} = O(1) 2^{-rn},$$

where the last equality follows from the fact that $(1-t)\left[\frac{r}{s} - 1\right] - t = -r$. (Indeed, this is the reason for our choice of r.) Therefore, $\mathrm{wt}_t(V) \leq \sum_{\rho \in Q} O(1) 2^{-r|\rho|} = O(1) \mathrm{wt}_r(Q) < \infty$.

We are now ready to build U. Recall that $U \subseteq 2^{<\omega}$ is the A-c.e. set that we are constructing, and that if $\mathrm{wt}_s(U) \leq 2^{-e-1}$, then $[U] \subseteq [R] = [Q]$. The goal of the construction is to ensure that $V \supseteq W$. Let $\{\sigma_j\}_{j \in \omega}$ be an A-computable enumeration of W. Let $k_0 = 0$. At stage k_i of the construction, let σ_j be the first element of the enumeration of W that we have not yet successfully coded. Put $\tau = \tau_{k_i}(\sigma_j)$ into U and wait for the first stage k_{i+1} at which some $\rho \preccurlyeq \tau$ enters Q. If $|\tau| < \frac{r|\rho|+d}{s}$, then we have *successfully coded* σ_j, since in that case, $\sigma_j \in V$. At stage k_{i+1}, we either attempt again to code σ_j, or if we were successful, attempt to code σ_{j+1}. This completes the construction.

We must prove that $\mathrm{wt}_s(U) \leq 2^{-e-1}$ without assuming that $[U] \subseteq [Q]$. Note, however, that if $[U] \nsubseteq [Q]$, then there is some final stage k_i at which we enumerate a string into U, and then we wait forever for a stage k_{i+1}. So there is at most one string $\hat\tau \in U$ with $[\hat\tau] \nsubseteq [Q]$. Since $|\hat\tau| \geq c$, $2^{-s|\hat\tau|} \leq 2^{-sc} \leq 2^{-e-3}$.

For all remaining $\tau \in U$, some prefix of τ is necessarily enumerated into Q. The total cost of all of the τ that are put into U that result in a successful coding is $\sum_{\sigma \in W} 2^{-s(|\sigma|+c)} = 2^{-sc} \mathrm{wt}_s(W) < 2^{-sc} \leq 2^{-e-3}$. For every τ other than $\hat\tau$ that is put into U but does not result in a successful coding, there is a corresponding $\rho \in Q$ such that $|\tau| \geq \frac{r|\rho|+d}{s}$. Hence the total cost of these τ is at most

$$\sum_{\rho \in Q} 2^{-s\left(\frac{r|\rho|+d}{s}\right)} = 2^{-d} \mathrm{wt}_r(Q) \leq 2^{-e-2}.$$

Therefore, $\mathrm{wt}_s(U) \leq 2^{-e-3} + 2^{-e-3} + 2^{-e-2} = 2^{-e-1}$. This means that every τ we put into U eventually has a prefix in Q. Furthermore, we cannot put more than $2^{|\sigma_j|+c}$ strings into U for the sake of σ_j, so it must eventually be successfully coded. Hence $W \subseteq V$. $\qquad\square$

3. Lowishness for K

The next goal is to prove that if $A \in 2^\omega$ is low for dimension, then it is lowish for K. An easy lemma shows that the c.e. covering property gives us a connection between K and K^A. From there we show that (6) holds for A, meaning that $K(\sigma) - K^A(\sigma)$ is asymptotically small relative to $|\sigma|$. Finally, we prove that A is lowish for K, in other words, that $K(\sigma) - K^A(\sigma)$ is asymptotically small even relative to $K(\sigma)$.

Lemma 3.1. *Assume that A has the c.e. covering property and that $0 \leq s < t \leq 1$. Then $(\forall \sigma \in 2^{<\omega})[\ K^A(\sigma) \leq s|\sigma| \Rightarrow K(\sigma) \leq^+ t|\sigma|\ ]$, where the constant does not depend on σ.*

Proof. Let $W = \{\sigma : K^A(\sigma) \leq s|\sigma|\}$. So W is A-c.e. and

$$\mathrm{wt}_s(W) = \sum_{\sigma \in W} 2^{-s|\sigma|} \leq \sum_{\sigma \in W} 2^{-K^A(\sigma)} < 1.$$

Therefore, there is a c.e. set $V \supseteq W$ such that $\text{wt}_t(V) < \infty$. The latter condition implies that $U = \{\langle \sigma, t|\sigma| \rangle \colon \sigma \in V\}$ is a bounded request set, so for every $\sigma \in V$, we have $K(\sigma) \leq^+ t|\sigma|$. $\qquad \square$

Proposition 3.2. $(5) \Rightarrow (6)$.

Proof. Assume that (6) fails, so there are a rational $r < 0$ and a sequence of distinct strings $\{\sigma_i\}_{i \in \omega}$ such that $(\forall i)[K^A(\sigma_i) - K(\sigma_i) < r|\sigma_i|]$. Let $\ell = \liminf\limits_{i \to \infty} K^A(\sigma_i)/|\sigma_i|$. Clearly, $\ell \in [0,1]$. Note that

$$\ell = \liminf_{i \to \infty} \frac{K^A(\sigma_i)}{|\sigma_i|} \leq \liminf_{i \to \infty} \frac{K(\sigma_i) + r|\sigma_i|}{|\sigma_i|} \leq 1 + r < 1.$$

Take s and t such that $\ell < s < t < 1$ and $\ell - t > r$. Thin $\{\sigma_i\}_{i \in \omega}$ to a subsequence $\{\tau_i\}_{i \in \omega}$ such that $(\forall i)[K^A(\tau_i) \leq s|\tau_i|]$. By Lemma 3.1, $K(\tau_i) \leq^+ t|\tau_i|$. This means that $\limsup_{i \to \infty} K(\tau_i)/|\tau_i| \leq t$. Since $\{\tau_i\}_{i \in \omega}$ is a subsequence of $\{\sigma_i\}_{i \in \omega}$, we have $\liminf_{i \to \infty} K^A(\tau_i)/|\tau_i| \geq \ell$. Therefore,

$$\liminf_{i \to \infty} \frac{K^A(\tau_i) - K(\tau_i)}{|\tau_i|} \geq \liminf_{i \to \infty} \frac{K^A(\tau_i)}{|\tau_i|} - \limsup_{i \to \infty} \frac{K(\tau_i)}{|\tau_i|} \geq \ell - t > r.$$

This contradicts the fact that $K^A(\tau_i) - K(\tau_i) < r|\tau_i|$ for all $i \in \omega$. $\qquad \square$

Proposition 3.3. $(6) \Rightarrow (7)$.

Proof. We prove the contrapositive. Assume that (7) fails, so there is an $\varepsilon > 0$ such that $K^A(\sigma) < (1 - \varepsilon)K(\sigma)$ for infinitely many σ. This says that there are infinitely many σ that A can compress better by a multiplicative factor of $K(\sigma)$. To prove the failure of (6), we need to show that A compresses infinitely many η better by a multiplicative factor of $|\eta|$. If a witness σ to the failure of (7) has low complexity, then it might not serve as a witness to the failure of (6). The idea is to replace σ with a short η such that $U(\eta) = \sigma$. The choice of η will be made so that it is essentially a compressed witness to the failure of (7), hence also to the failure of (6). In particular, we will find infinitely many η for which $K^A(\eta) - K(\eta) \leq^+ -\varepsilon|\eta|$, which is sufficient.

Let U^X denote the universal prefix-free machine with oracle X. We define a prefix-free machine M^A relative to A as follows. If $U^A(\rho) \downarrow = \sigma$ and $k \in \omega$, let $\tau = \rho 0^k 1$ and search for a string η such that $U(\eta) = \sigma$ and $(1 - \varepsilon)|\eta| \leq |\tau|$. If such an η is found, let $M^A(\tau) = \eta$. The fact that U^A is prefix-free implies that M^A is prefix-free.

Take σ for which $K^A(\sigma) < (1 - \varepsilon)K(\sigma)$. Let ρ be a minimal U^A-program for σ and let $\widehat{\eta}$ be a minimal U-program for σ. By assumption, $|\rho| < (1 - \varepsilon)|\widehat{\eta}|$. Let $k \in \omega$ be minimal such that $(1 - \varepsilon)|\widehat{\eta}| \leq |\rho| + k + 1$ and let $\tau = \rho 0^k 1$. This ensures that $M^A(\tau)$ converges, though possibly not to $\widehat{\eta}$. Let $\eta = M^A(\tau)$. Our choice of k ensures that $|\tau| - 1 < (1 - \varepsilon)|\widehat{\eta}| \leq (1 - \varepsilon)|\eta| \leq |\tau|$, from which it follows that $|\eta| =^+ |\widehat{\eta}|$. Also note that $|\widehat{\eta}| = K(\sigma) = K(U(\eta)) \leq^+ K(\eta)$. Putting everything together, we obtain

$$K^A(\eta) - K(\eta) \leq^+ |\tau| - |\widehat{\eta}| \leq^+ (1 - \varepsilon)|\eta| - |\eta| = -\varepsilon|\eta|. \qquad \square$$

4. Lowishness for randomness and lowness for dimension $s \in (0,1]$

We have almost completed the proof of the main theorem. All that remains it to show that the following properties are also equivalent to lowness for dimension.

(1) A is lowish for randomness, i.e., every Martin-Löf random sequence has dimension 1 relative to A,

(2) A is low for some dimension $s \in (0, 1]$.

It is easy to see that both properties are implied by lowness for dimension. In particular, we have already noted that (3) $\Rightarrow$ (2).

Proposition 4.1. (2) $\Rightarrow$ (1).

Proof. Assume that $A \in 2^\omega$ is low for dimension $s \in (0, 1]$. Given a Martin-Löf random sequence $X \in 2^\omega$, we will spread X out to create a sequence $s[X]$ of dimension s. We will then observe that $\dim^A(s[X]) = s$ implies that $\dim^A(X) = 1$.

For $Z \in 2^\omega$, the spread out version of Z is $s[Z] = \{\lfloor n/s \rfloor : Z(n) = 1\}$. We want to show that $K(s[1^\omega] \restriction n)$ is $O(\log n)$. Fix $n \in \omega$ and let r be the maximum value of $\lfloor m/s \rfloor/m$ for all $m < sn$. Then r is a rational approximation of $1/s$ that is good enough to determine $s[1^\omega] \restriction n$. To see this, consider any $m < sn$. Then $\lfloor rm \rfloor \geq \lfloor \lfloor m/s \rfloor/m \cdot m \rfloor = \lfloor m/s \rfloor$. But it also follows from the definition that $r \leq 1/s$, so $\lfloor rm \rfloor \leq \lfloor m/s \rfloor$. Therefore, $\lfloor rm \rfloor = \lfloor m/s \rfloor$ for all such m. So $s[1^\omega] \restriction n$ can be computed uniformly from r, n and $\max\{m : m < sn\}$. But r is a ratio of natural numbers less than n, so $K(s[1^\omega] \restriction n)$ is $O(\log n)$.

Now note that $K(s[Z] \restriction n)$ differs from $K(Z \restriction \lfloor sn \rfloor)$ by at most $K(s[1^\omega] \restriction n) + O(1)$ because all we need to know to translate between the two strings is the location of the bits of Z in the spread out version. Therefore

$$\dim(s[Z]) = \liminf_{n \to \infty} \frac{K(s[Z] \restriction n)}{n} = \liminf_{n \to \infty} \frac{K(Z \restriction \lfloor sn \rfloor) + O(\log n)}{n}$$
$$= s \liminf_{n \to \infty} \frac{K(Z \restriction \lfloor sn \rfloor)}{sn} + \lim_{n \to \infty} \frac{O(\log n)}{n} = s \dim(Z).$$

Similarly, $\dim^A(s[Z]) = s \dim^A(Z)$ for all $Z \in 2^\omega$.

To finish the proof, let $X \in 2^\omega$ be any Martin-Löf random sequence. Then $\dim(s[X]) = s$, which implies that $\dim^A(s[X]) = s$. So $\dim^A(X) = \frac{1}{s} \dim^A(s[X]) = 1$. This proves that A is lowish for randomness. $\qquad\square$

Proposition 4.2. (1) $\Rightarrow$ (3).

Proof. We prove the contrapositive, so assume that $A \in 2^\omega$ is not low for dimension. Therefore, there is an $X \in 2^\omega$ and a rational $t > 0$ such that $4t \leq \dim(X) - \dim^A(X)$. The idea is to code X compactly into a random sequence $Z \in 2^\omega$ and use the fact that A is better at compressing X to prove that $\dim^A(Z) < 1$.

Fix a functional Φ and let $W = \{\langle \sigma, d \rangle : d > t|\sigma| - \log \mu(\Phi^{-1}[\sigma])\}$, where $\Phi^{-1}[\sigma] = \{Y : \Phi^Y \restriction |\sigma| = \sigma\}$. Then W is a c.e. set and

$$\sum_{\langle \sigma, d \rangle \in W} 2^{-d} \leq \sum_{\sigma \in 2^{<\omega}} 2^{-t|\sigma|} \mu(\Phi^{-1}[\sigma])$$
$$= \sum_{n \in \omega} 2^{-tn} \sum_{\sigma \in 2^n} \mu(\Phi^{-1}[\sigma]) \leq \sum_{n \in \omega} 2^{-tn} < \infty,$$

where we use the fact that if σ and τ are incomparable strings, then $\Phi^{-1}[\sigma] \cap \Phi^{-1}[\tau] = \emptyset$. Thus W is a bounded request set, so there is a $c \in \omega$ such that for all $\sigma \in 2^{<\omega}$, we have $K(\sigma) \leq t|\sigma| - \log \mu(\Phi^{-1}[\sigma]) + c$. Therefore, $\mu(\Phi^{-1}[\sigma]) \leq 2^{t|\sigma|-K(\sigma)+c}$.

The next step is to code X into a Martin-Löf random sequence. Kučera [12] and Gács [7] both proved that every sequence is computable from a Martin-Löf random. We need the version due to Gács, who proved that for any sequence X, there are a Martin-Löf random Z and a functional Φ such that $\Phi^Z = X$ with use $\varphi(n) = n + o(n)$. We may assume that $\varphi(n) = 2n$, so for any n,

$$Z \restriction 2n \in U_n := \{\rho \in 2^{2n} \colon \Phi^\rho \restriction n = X \restriction n\}.$$

From above, $\mu\left(\Phi^{-1}[X \restriction n]\right) \le 2^{tn - K(X \restriction n) + c}$ for some $c \in \omega$. So U_n has size at most $2^{2n + tn - K(X \restriction n) + c}$ and is uniformly c.e. in $X \restriction n$. We can code $Z \restriction 2n$ using a code for $X \restriction n$ and an index within U_n, therefore

$$K^A(Z \restriction 2n) \le K^A(X \restriction n) + (2n + tn - K(X \restriction n)) + O(\log n).$$

Here we use that if $m \le 2^{2n + tn - K(X \restriction n) + c}$, then $K(m) \le^+ (2n + tn - K(X \restriction n)) + 2\log(2n + tn - K(X \restriction n))$, and $\log(2n + tn - K(X \restriction n)) \le \log(2n + tn) = O(\log n)$. By assumption, there are infinitely many n such that $K(X \restriction n) - K^A(X \restriction n) > 3tn$. For each such n, we have $K^A(Z \restriction 2n) \le 2n - 2tn + O(\log n)$, so $\dim^A(Z) \le 1 - t$. This proves that A is not lowish for randomness. □

5. A perfect class of low for dimension sets

As we mentioned in the introduction, only countably many sets are low for randomness, but continuum many are low for dimension. In this section, we prove the latter fact by constructing a perfect Π^0_1-class containing only lowish for K sequences. This result was also recently proved by Herbert [8]. He built a perfect Π^0_1-class $Q \subseteq 2^\omega$ of sequences with finite self-information (see Section 7) and observed that, as a consequence of his construction, every element of Q is low for dimension. His Π^0_1-class has another interesting property: every sequence is computable from the join of two elements of Q. Compare this to the fact that the join of two low for random sequences is also low for random (as proved for K-trivial sequences by Downey, Hirschfeldt, Nies and Stephan [6]).

Theorem 5.1. *There is a perfect Π^0_1-class $P \subseteq 2^\omega$ such that every $A \in P$ is lowish for K.*

Proof. We construct a computable sequence $T_s \colon 2^{<\omega} \to 2^{<\omega}$ of partial function trees with finite domain such that $T(\sigma) = \lim_{s \to \infty} T_s(\sigma)$ is a total function. Our Π^0_1-class will be $P = [T]$, the set of infinite paths through T. We use σ to denote strings in the domain of T_s, and τ to denote strings in the image of T_s. We let U be the universal prefix-free oracle machine and we write U^τ for both the partial function and its domain. We use η to denote strings in U^τ. If σ has positive length, then σ^- denotes $\sigma \restriction (|\sigma| - 1)$.

Parameters: We say that τ is *on* T_s if $\tau \preccurlyeq T_s(\sigma)$ for some σ in the domain of T_s. For every $s \in \omega$, we keep $T_s(\lambda) = \lambda$ and $T_s(\sigma) \succcurlyeq T_{s-1}(\sigma)$ if both converge. If $T_s(\sigma) \succ T_{s-1}(\sigma)$, we say that σ is *moved* at stage s. At every stage, at most one σ is moved. If σ is moved at stage s, then $T_s(\sigma') \uparrow$ for every $\sigma' \succ \sigma$. We say that such σ' are *initialized*. For convenience, we require that for every σ and s, $T_s(\sigma 0) \downarrow$ iff $T_s(\sigma 1) \downarrow$. To ensure that $[T]$ is a (perfect) Π^0_1-class, we also require that every leaf of T_{s+1} extends a leaf of T_s. That is, if σ is moved at a stage s then we require that the new value $T_s(\sigma)$ extends a leaf of T_{s-1}.

We will make $T(X) = \bigcup_{n \in \omega} T(X \restriction n)$ lowish for K for every $X \in 2^\omega$. We let

$$r_n = (n+2)/(n+1)$$

for $n \in \omega$. We want to ensure that for every $X \in [T]$ and $n \in \omega$, and for almost every η in the domain of U^X, we have that $K(U^X(\eta)) \leq^+ r_n|\eta|$. This would imply that every $X \in [T]$ is lowish for K (see Lemma 5.4).

We enumerate a c.e. set L consisting of sets of triples $\langle \eta, \tau, r_n \rangle$ (called *requests*) such that $\eta, \tau \in 2^{<\omega}$ and $n \in \omega$. The intention is that for each such request, we have $\eta \in U^\tau$ and we want $K(U^\tau(\eta))$ to have complexity $\leq^+ r_n|\eta|$. Such L-requests have to be made for each τ on T, and the entire construction will involve a careful accounting to keep the total weight of L finite. Here we define

$$\mathrm{wt}(L) = \sum_{\langle \eta, \tau, r \rangle \in L} 2^{-r|\eta|}.$$

As long as $\mathrm{wt}(L) < \infty$, we can build a bounded request set from L so that if $\langle \eta, \tau, r \rangle \in L$, then $K(U^\tau(\eta)) \leq^+ r|\eta|$.

Viewed as a domain instead of an operator, we will assume that U^τ is a uniformly computable finite set of strings that increases with respect to τ. As usual, $U^X = \bigcup_{\tau \prec X} U^\tau$ is infinite and has weight less than 1 for each $X \in 2^\omega$. For convenience, we will also assume that $|U^{\tau 0} \smallsetminus U^\tau| = 1$ (which we can ensure by modifying the uses of computations). We order $2^{<\omega}$ first by length, then lexicographically, i.e., $\lambda < 0 < 1 < 00 < \cdots$. Associated with each σ is a c.e. set Bin_σ that we build during the construction. This set will contain certain L-requests. The aim is to ensure that every L-request $\langle \eta, \tau, r_n \rangle$ issued during the construction will fall into Bin_σ for some σ. Intuitively, this means that we will charge the weight of this particular request to the node σ. The entire construction will attempt to keep the total weight in each Bin_σ small so that we can show that $\mathrm{wt}(L) < \infty$. We also maintain counters i_σ for each $\sigma \in 2^{<\omega}$. These record the number of times that σ has been initialized so far (initialization is formally defined in the construction). For every $i \in \omega$, we define the computable sequence $t_{\sigma,i}$ inductively on $|\sigma|$ as follows:

$$t_{\lambda,i} = 0,$$
$$t_{\sigma,i} = \left\langle \frac{t_{\sigma^-,i}}{r_{|\sigma|} - 1}, \frac{\sigma - \log\left(1 - 2^{1 - r_{|\sigma|}}\right) + 5}{r_{|\sigma|} - 1}, i, \sigma \right\rangle, \text{ if } |\sigma| > 0.$$

Here we identify $\sigma \in 2^{<\omega}$ with its code number and all fractions are rounded up, if necessary. This seemingly bizarre choice for $t_{\sigma,i}$ is simply to ensure that certain sums converge and will be made clear during the verification. We prefer to take $\langle \cdot \rangle$ rather than $\max\{\cdot\}$ in the definition for $t_{\sigma,i}$ because we want the quantity $t_{\sigma,i}$ to be strictly increasing in i.

During the construction, when we refer to t_σ at a stage s, we mean the value of t_{σ,i_σ} evaluated at stage s. We sometimes write $t_\sigma[s]$. Each t_σ represents a threshold value for σ.

During the construction, if we see some η in U^τ for some τ on T, then we must issue a description for $U^\tau(\eta)$. Fortunately for us, we are allowed to issue a description of length $r|\eta|$ for some $r > 1$, and so we would use up $2^{-r|\eta|}$ much measure to describe $U^\tau(\eta)$ compared to $2^{-|\eta|}$ for the opponent. It might appear that we have much more measure to play with compared to the opponent, but remember that the opponent has used up $2^{-|\eta|}$ much measure in the oracle machine U^τ.

This means that he could repeat this by adding η to $U^{\tau_1}, U^{\tau_2}, \cdots, U^{\tau_k}$ for a large number k (remember that our Π^0_1-class has to be perfect, so we must allow an antichain of arbitrarily large size to be extendible in our Π^0_1-class). So the opponent has really only used up $2^{-|\eta|}$ along each oracle, but we would have to use up $k \cdot 2^{-r|\eta|}$, which could be very large. We can, of course, counter this by picking a large r, but we must eventually use only small values of r to ensure that each path is lowish for K, and so such a naive strategy will not succeed.

The solution to this problem is to block the accumulation of measure by the opponent as described above. We assign a threshold value t_σ to each node σ. When we see new strings added to $U^{T(\sigma')} \smallsetminus U^{T(\sigma)}$ with very small weight, we will issue requests for the new strings and charge the cost of these requests to σ'. If these new strings all have weights smaller than the threshold values $t_{\sigma 0}$ and $t_{\sigma 1}$, then we will never move $T(\sigma)$, $T(\sigma 0)$ or $T(\sigma 1)$, and the cost of describing these new strings are charged to extensions of σ. If, however, we see a new string η added to, say, $U^{T(\sigma 0 \sigma')}$, that increases the cumulative weight above the threshold of $\sigma 0$ (i.e., $\mu[U^{T(\sigma 0 \sigma')} \smallsetminus U^{T(\sigma 0)}] > 2^{-t_{\sigma 0}}$), then we will charge the cost of describing η to σ. We will also move $T(\sigma 0)$ to the position $T(\sigma 0 \sigma')$. Doing this will render (essentially) all of the L-requests previously made for oracles extending $T_s(\sigma 00)$ and $T_s(\sigma 01)$ wasted; however, if we are careful, these will have small weight, and we can reset all of the corresponding thresholds to be much larger.

Now observe that a request issued to describe η is charged to σ only if it was too costly for $\sigma 0$ (or $\sigma 1$), and in this case $\sigma 0$ will be moved so that η is now in $U^{T(\sigma 0)} \smallsetminus U^{T(\sigma)}$. Now the crucial point is that not too much weight can be charged to σ, because otherwise $U^{T(\sigma 0)} \smallsetminus U^{T(\sigma)}$ would exceed the threshold 2^{-t_σ}, and we would then move σ.

Barring the action of higher-priority nodes, σ can only move in this way at most 2^{t_σ} many times, since each time it is moved, we have at least 2^{-t_σ} measure of new strings in $U^{T_{s+1}(\sigma)}$. Hence the total weight of garbage left in L due to the movement of σ in this way can be made small. This allows us to bound the overall weight of L.

Construction: At stage 0, we start with $L = \emptyset$, $\mathrm{Bin}_\sigma = \emptyset$, and $i_\sigma = 0$ for every $\sigma \in 2^{<\omega}$. Let $T_0(\lambda) = \lambda$. At stage $s > 0$, let σ be the least node such that $T_{s-1}(\sigma 0) \uparrow$ (and hence $T_{s-1}(\sigma 1) \uparrow$). We say that we *act for σ*. Let $\tau = T_{s-1}(\sigma)$, and let $\eta_i \in U^{\tau i} \smallsetminus U^\tau$. We first process η_0 (before moving on, if necessary, to η_1, and then end stage s). To process η_0, there are two steps:

> **Step 1:** Let $n \leq |\sigma|$ be largest such that $|\eta_0| > t_{\sigma \upharpoonright n}$. Note that n exists as $t_\lambda = 0$. We enumerate the request $\langle \eta_0, \tau 0, r_n \rangle$ into L.
>
> **Step 2:** We now decide which prefix of σ we will charge this request to. Let $\sigma_0 \preccurlyeq \sigma$ be the minimal node such that $\mu[U^{\tau 0} \smallsetminus U^{T_{s-1}(\sigma_0)}] \geq 2^{-t_{\sigma_0}}$. Note that $\sigma_0 \neq \lambda$. If σ_0 exists, then we enumerate $\langle \eta_0, \tau 0, r_n \rangle$ into $\mathrm{Bin}_{\sigma_0^-}$. We move σ_0 by setting $T_s(\sigma_0) = \tau 0$, initializing every $\sigma' \succ \sigma_0$, and letting T_s copy T_{s-1} everywhere else. Initializing σ' means that we increment $i_{\sigma'}$ and adjust the value of the threshold $t_{\sigma'}$. Note that initialization does not reset $\mathrm{Bin}_{\sigma'}$. Go to the next stage. If σ_0 does not exist, then we set $T_s(\sigma 0) = \tau 0$ and put $\langle \eta_0, \tau 0, r_n \rangle$ into Bin_σ.

If we did not stop the construction while processing η_0, we will move on and process η_1 in the same way.

Verification: It is clear that the sequence T_s meets the conditions stated above (other than the totality of $T = \lim_{s \to \infty} T_s$, which we will verify below). Every L-request is also placed into Bin_σ for some $\sigma \in 2^{<\omega}$, so $\mathrm{wt}(L) \leq \sum_{\sigma \in 2^{<\omega}} \mathrm{wt}(\mathrm{Bin}_\sigma)$. We need to show that T is total, and each $\mathrm{wt}(\mathrm{Bin}_\sigma)$ is small. We first prove a crucial lemma:

Lemma 5.2. *Consider $\sigma \in 2^{<\omega}$ and $s' < s$ such that both $T_{s'}(\sigma) \downarrow$ and $T_s(\sigma) \downarrow$. If σ is not initialized strictly between s' and s, then $\mathrm{wt}[\mathrm{Bin}_{\sigma,s} \setminus \mathrm{Bin}_{\sigma,s'}] < 2^{(1-r_{|\sigma|})t_{\sigma,s}+4}$.*

Proof. Let $s' < s_0 < \cdots < s_k \leq s$ be all the stages at which Bin_σ receives new elements. Let $t = t_{\sigma,s}$, and note that σ^- is not moved at any of these stages. Consequently, it is easy to see that there are at most 2^t many $j \leq k$ for which σ is moved at s_j. For each $i \leq k$, there are three possibilities for s_i:

(i) σ is moved at stage s_i. The only way for σ to be moved and for Bin_σ to increase at stage s_i, is for us to act for σ at stage s_i. Hence Bin_σ receives exactly one new request of the form $\langle \eta, \tau, r_{|\sigma|} \rangle$ when processing η_0 and σ is moved when processing η_1. Since $|\eta| > t$, it follows that $2^{-r_{|\sigma|}|\eta|} < 2^{-r_{|\sigma|}t}$.

(ii) We act for σ at s_i but σ is not moved at stage s_i. In this case, Bin_σ receives two new requests of the form $\langle \eta, \tau, r_{|\sigma|} \rangle$, but note that this item (ii) will not apply again until σ is again moved. As in (i), we have $2^{-r_{|\sigma|}|\eta|} < 2^{-r_{|\sigma|}t}$ for each of the two requests.

(iii) Either $\sigma 0$ or $\sigma 1$ is moved (without loss of generality, assume the former). Then a request of the form $\langle \eta, \tau, r_n \rangle$ is added to Bin_σ, where $|\eta| > t_{\sigma\sigma'}$ and $n = |\sigma\sigma'| \geq |\sigma|$. We claim that $2^{-r_n|\eta|} < 2^{-|\eta|}2^{(1-r_{|\sigma|})t}$. If $|\sigma'| = 0$ then we are done as in case (i). Otherwise suppose that $\sigma \subset \sigma\sigma'$. Now for any string $\alpha \neq \lambda$ and any i, we have that

$$t_{\alpha,i} > \frac{t_{\alpha^-,i}}{r_{|\alpha|}-1} > \frac{r_{|\alpha^-|}-1}{r_{|\alpha|}-1} \cdot t_{\alpha^-,i},$$

where the first inequality follows by the definition of t and the second follows because $1 < r_{|\alpha^-|} < 2$. Applying this, together with the fact that $i_{\sigma\sigma'} \geq i_\sigma$, we obtain $t_{\sigma\sigma'} > \frac{r_{|\sigma|}-1}{r_n-1}t$. It follows that $|\eta| > \frac{r_{|\sigma|}-1}{r_n-1}t$, so our claim follows.

Now suppose that $s_i < s_j$ are two consecutive stages where σ is moved (the same analysis below applies if either $s_i = s'$ or $s_j = s$). We now account for how much weight is added to Bin_σ at those stages s_{i_0}, where $s_i < s_{i_0} < s_j$. (ii) contributes on at most one stage, and the sum of the contributions from (iii) is at most $\sum_\eta 2^{-|\eta|}2^{(1-r_{|\sigma|})t} = 2^{(1-r_{|\sigma|})t} \cdot \sum_\eta 2^{-|\eta|}$, where the sum is taken over all strings η in $U^{T(\sigma 0)} \setminus U^{T(\sigma)}$ or $U^{T(\sigma 1)} \setminus U^{T(\sigma)}$. Now clearly this sum $\sum_\eta 2^{-|\eta|}$ is less than twice the threshold value $2 \cdot 2^{-t}$, because otherwise one of the two $U^{T(\sigma 0)} \setminus U^{T(\sigma)}$ or $U^{T(\sigma 1)} \setminus U^{T(\sigma)}$ would get too heavy and we would be forced to move σ before s_j. Thus we conclude that the sum of contributions from (iii) is at most $2 \cdot 2^{-t}2^{(1-r_{|\sigma|})t} = 2^{-r_{|\sigma|}t+1}$.

Totaling the contribution of all the stages s_i, we have $\mathrm{wt}(\mathrm{Bin}_{\sigma,s} \setminus \mathrm{Bin}_{\sigma,s'}) \leq (2^t + 1)((\mathrm{i})+(\mathrm{ii})+(\mathrm{iii})) \leq (2^t + 1)\left(3 \cdot 2^{-r_{|\sigma|}t} + 2^{-r_{|\sigma|}t+1}\right) < 2^{(1-r_{|\sigma|})t+4}$. $\square$

Lemma 5.3. *For each $\sigma \in 2^{<\omega}$, we have $\mathrm{wt}(\mathrm{Bin}_\sigma) < 2^{-\sigma}$. (Here we identify σ with its code number).*

Proof. Fix $\sigma \in 2^{<\omega}$. We do not add anything to Bin_σ at a stage $s \in \omega$ unless $T_{s-1}(\sigma) \downarrow$. Each time σ is initialized, t_σ is increased. To calculate an upper bound on $\text{wt}(\text{Bin}_\sigma)$, we consider the stages $s_0 < s_1 < \cdots$ where σ is initialized. The total weight added to Bin_σ for stages $s_i < s < s_{i+1}$, for every i, can be bounded by applying Lemma 5.2. This weight is at most

$$\sum_{t \geq t_{\sigma,0}} 2^{\left(1-r_{|\sigma|}\right)t+4} = \frac{2^{\left(1-r_{|\sigma|}\right)t_{\sigma,0}+4}}{1 - 2^{\left(1-r_{|\sigma|}\right)}} \leq 2^{-\sigma-1}.$$

The first equality follows by applying the sum of a geometric series, while the second inequality follows from the fact that $t_{\sigma,0} < \frac{\sigma - \log\left(1 - 2^{1-r_{|\sigma|}}\right) + 5}{r_{|\sigma|-1}}$.

On the other hand, if anything is added to Bin_σ at stage s_i, then we must have acted for σ at stage s_i, and the request was added to Bin_σ during the processing of η_0. This is at most $2^{-r_{|\sigma|}t_\sigma[s_i]} < 2^{-t_\sigma[s_i]}$. So the sum of the weights of stuff added at all s_i is bounded above by

$$\sum_i 2^{-t_{\sigma,i}} < 2^{-t_{\sigma,0}+1} < 2^{-\sigma-1}.$$

Therefore adding both parts together we have $\text{wt}(\text{Bin}_\sigma) < 2^{-\sigma}$. $\qquad\square$

The previous lemma tells us that $\text{wt}(L) \leq 1$. It follows by a straightforward induction on $|\sigma|$ that each σ is moved finitely often and each σ is initialized finitely often. Hence T is total. It is routine to verify (by induction on s) that for every s, every τ on T_s, and every $\eta \in U^\tau \smallsetminus U^{\tau^-}$, we have $\langle \eta, \tau, r \rangle$ in L for some r.

Lemma 5.4. *For each $X \in 2^\omega$, $T(X)$ is lowish for K.*

Proof. Let $X \in 2^\omega$, and fix an $n \in \omega$. We want to show that for almost every $\eta \in U^{T(X)}$, we enumerate $\langle \eta, \tau, r_m \rangle$ into L for the sake of $U^{T(X)}$, for some $m \geq n$. Let s be a stage where both $t_{X \restriction n}$ and $T_s(X \restriction n)$ are stable. Clearly, for any $\eta \in U^{T(X)}$, $\langle \eta, \tau, r_m \rangle \in L$ for some $\tau \prec T(X)$ and m. For any $\eta \in U^{T(X)}$ where $|\eta| > t_{X \restriction n}$ and $\langle \eta, \tau, r_m \rangle$ enters L after stage s, we must have $m \geq n$.

By the fact that L is a bounded request set, we have some constant c such that for almost every σ, $K(\sigma) \leq r_m K^{T(X)}(\sigma) + c$ for some $m \geq n$, and so

$$K(\sigma) \leq r_n K^{T(X)}(\sigma) + c.$$

That is, we have for almost every σ,

$$\frac{K^{T(X)}(\sigma)}{K(\sigma)} \geq \frac{n+1}{n+2} - \frac{c}{K(\sigma)r_n}.$$

Hence $T(X)$ is lowish for K. $\qquad\square$

This completes the proof of Theorem 5.1. $\qquad\square$

6. Relationship to other degree classes

In this section we make a few observations about the degrees of sequences that are low for dimension. The first is that if $A \in 2^\omega$ is low for dimension, then it is jump-traceable with a reasonably slow-growing bound. We recall the necessary definitions. A function $h \colon \omega \to \omega$ is an *order function* if it is computable, nondecreasing and unbounded. If h is an order function, then $A \in 2^\omega$ is *jump-traceable*

with bound h if for every partial A-computable function f, there is a c.e. "trace" (sequence of c.e. sets) $\{T_e\}_{e \in \omega}$ such that

$$(\forall e)[\ |T_e| \leq h(e) \text{ and if } f(e) \downarrow, \text{ then } f(e) \in T_e \ ].$$

We could relax the above statement to only hold for sufficiently large e, and the class of reals so defined would remain unchanged. We say that A is *jump-traceable* if A is jump-traceable with some bound.

Theorem 6.1. *Suppose that $A \in 2^\omega$ is low for dimension. Then for every $\varepsilon > 0$, A is jump-traceable with bound n^ε.*

This is an adaptation of a proof found in Nies [17] for the jump-traceability of K-trivials.

Proof. Let $A \in 2^\omega$ be low for dimension, hence lowish for K. Fix f partial A-computable. Then for sufficiently long σ, $K(\sigma) < (1 + \varepsilon/4)K^A(\sigma)$. In particular, $K(n, f(n)) < (1+\varepsilon/4)K^A(n, f(n))$ for sufficiently large n in the domain of f, where $K(n, m)$ is the prefix-free complexity of the pair $\langle n, m \rangle$. Since $K^A(n, f(n)) <^+ K^A(n) <^+ K(n)$, we have $K(n, f(n)) < (1 + \varepsilon/3)K(n)$ for sufficiently large n in the domain of f.

Fix a universal prefix-free machine U. We enumerate a c.e. set W by enumerating $\langle n, d \rangle$ into W when we see $\sum_{U(\sigma)=\langle n,c \rangle} 2^{-|\sigma|} > 2^{-d}$. For each n, let d_n be least such that we enumerate $\langle n, d_n \rangle$ into W.

Then

$$\sum_{\langle n,d \rangle \in W} 2^{-d} \leq \sum_n \sum_{d \geq d_n} 2^{-d} = 2 \sum_n 2^{-d_n} \leq 2 \sum_n \sum_{U(\sigma)=\langle n,c \rangle} 2^{-|\sigma|} < 2,$$

and so W is a bounded request set.

Hence there is a b such that $K(n) \leq d_n + b$. Since $\langle n, K(n) - b - 1 \rangle$ was not enumerated into W, it follows that $\sum_{U(\sigma)=\langle n,c \rangle} 2^{-|\sigma|} \leq 2^{b+1-K(n)}$. So if we count the number of distinct c with $K(n, c) \leq (1 + \varepsilon/3)K(n)$, since each contributes at least $2^{-(1+\varepsilon/3)K(n)}$ to this sum, there can be at most $2^{b+1+(\varepsilon/3)K(n)}$ such c. Since $K(n) \leq 2 \log n$, we see that there are at most $Bn^{2\varepsilon/3}$ such c for some constant B, and if n is in the domain of f, then $f(n)$ is one such c.

We now build our trace as follows: We approximate $K(n)$ from above, beginning with $K_0(n) = 2 \log n$. At stage s, if we see a c such that $K_s(n, c) \leq (1 + \varepsilon/3)K_s(n)$, and we have not yet enumerated $Bn^{2\varepsilon/3}$ many such c for this current guess of $K(n)$, then we enumerate c into T_n.

Since $K_s(n)$ is non-increasing and positive, it takes on at most $2 \log n$ distinct values. For each such value, we enumerate at most $Bn^{2\varepsilon/3}$ elements into T_n, and so $|T_n| \leq Bn^{2\varepsilon/3} \cdot 2 \log n < n^\varepsilon$ for sufficiently large n. If n is in the domain of f, then once $K_s(n)$ takes its final value, $f(n)$ will be one of the c we enumerate into T_n. Thus $\{T_n\}_{n \in \omega}$ is a trace for f with bound n^ε. $\square$

Nies [17] constructed a perfect Π^0_1-class such that every element is jump-traceable with bound $2 \cdot 4^e$. He remarked that his argument could be "massaged a bit to obtain a bound close to 2^e." Together with Theorem 5.1, we get a much better bound:

Corollary 6.2. *For any $\varepsilon > 0$, there is a perfect Π^0_1-class $P \subseteq 2^\omega$ such that every $A \in P$ is jump-traceable with bound n^ε.*

Hence there are continuum many n^ε jump-traceable sequences.[3] Downey and Greenberg [4] proved that for a sufficiently slow-growing order function $h\colon \omega \to \omega$, every sequence that is jump-traceable with bound h is Δ_2^0 (and even K-trivial). But it is not clear where the transition is between having countably many and continuum many jump-traceable sequences (even allowing for the dependence on the numbering of partial computable functions).

Every jump-traceable sequence $A \in 2^\omega$ is GL_1, i.e., $A' \leq_T A \oplus \emptyset'$ (see Nies [17]), hence this is true if A is low for dimension. We can also draw some degree-theoretic conclusions from the perfect Π_1^0-class $P \subseteq 2^\omega$ of low for dimension sequences constructed in Theorem 5.1. Let $T\colon 2^{<\omega} \to 2^{<\omega}$ be the function tree such that $P = [T]$. Consider the low for dimension path $A = T(\emptyset'')$, Then

$$\emptyset'' \leq_T A \oplus T \leq_T A \oplus \emptyset' \leq_T T \oplus \emptyset'' \leq_T \emptyset'',$$

so $A' \equiv_T A \oplus \emptyset' \equiv_T \emptyset''$. In other words, A is high. More generally, for any $C \geq_T \emptyset'$, the low for dimension sequence $A = T(C)$ satisfies $A' \equiv_T A \oplus \emptyset' \equiv_T C$. Finally, it is easy to see that any perfect Π_1^0-class must contain a sequence $A \leq_T \emptyset'$ that is not ω-c.e.

7. Open questions

We highlight three questions related to our work. Two ask about the relationship between lowness for dimension and other notions: lowness for packing dimension and having finite self-information. We also ask to what extent our various characterizations remain equivalent when we pass to partially relativized "two oracle" versions.

Lowness for packing dimension. The *effective packing dimension* of $X \in 2^\omega$ is

$$\mathrm{Dim}(X) = \limsup_{n \in \omega} \frac{K(X \restriction n)}{n}.$$

Though it is not clear from this definition, Athreya, Hitchcock, Lutz and Mayordomo [1] proved that effective packing dimension is the effective analogue of packing dimension in the same way that effective dimension is the analogue of Hausdorff dimension. We relativize effective packing definition in the obvious way and consider the corresponding lowness notion:

Definition 7.1. We call $A \in 2^\omega$ *low for packing dimension* if $(\forall X)[\mathrm{Dim}^A(X) \geq \mathrm{Dim}(X)]$.

How does this notion relate to lowness for (Hausdorff) dimension? Are they equivalent? One direction follows easily from the main theorem:

Corollary 7.2. *Lowness for dimension implies lowness for packing dimension.*

Proof. Assume that $A \in 2^\omega$ is low for dimension, hence lowish for K. Then for any $X \in 2^\omega$,

$$\mathrm{Dim}^A(X) = \limsup_{n \to \infty} \frac{K^A(X \restriction n)}{n} = \limsup_{n \to \infty} \frac{K(X \restriction n)}{n} \frac{K^A(X \restriction n)}{K(X \restriction n)}$$

$$\geq \limsup_{n \to \infty} \frac{K(X \restriction n)}{n} = \mathrm{Dim}(X).$$

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[3]Independently, Yu and Hirschfeldt showed that for any $\varepsilon > 0$, there are continuum many $n^{1+\varepsilon}$ jump-traceable sequences.

So A is low for packing dimension.   □

We suspect that lowness for packing dimension is a weaker property than lowness for dimension, but it remains open.

Question 7.3. Does lowness for packing dimension imply lowness for dimension?

Finite self-information. Levin [13] defined the *mutual information* of $A, B \in 2^\omega$ to be
$$I(A\colon B) = \log \sum_{\sigma,\tau \in 2^{<\omega}} 2^{-K(\sigma,\tau)-K^A(\sigma)-K^B(\tau)+K(\sigma)+K(\tau)}.$$
This is one of several definitions that Levin has put forward to capture the notion of mutual information. Hirschfeldt and Weber [9] investigated the sequences $A \in 2^\omega$ that have *finite self-information* for this definition, i.e., $I(A\colon A) < \infty$. It is easy to see that if $A \in 2^\omega$ is low for randomness, or more specifically, low for K, then it has finite self-information. Hirschfeldt and Weber constructed an $A \in 2^\omega$ that has finite self-information but is not low for randomness. Not surprisingly, the key to their construction is to ensure that $K^A(\sigma) - K(\sigma)$ grows very slowly. They observe that the sequence A they build is both low for (Hausdorff) dimension and low for packing dimension. This led them to ask:

Question 7.4 (Hirschfeldt and Weber [9, Question 20])**.** What is the relationship between having finite self-information and being low for dimension?

It is possible that neither property implies the other.

Partial relativization. All of the properties in the main theorem have natural "two oracle" versions. For example, Given $A, B \in 2^\omega$, we could consider:
  (1) If $X \in 2^\omega$ is Martin-Löf random relative to B, then $\dim^A(X) = 1$.
  (3) $(\forall X \in 2^\omega)[\dim^A(X) \geq \dim^B(X)]$.
  (4) If $W \subseteq 2^{<\omega}$ is A-c.e., $0 \leq s < t \leq 1$, and $\mathrm{wt}_s(W) < 1$, then there is a B-c.e. set V with $\mathrm{wt}_t(V) < 1$ such that $[W] \subseteq [V]$.
  (7) $\liminf_{\sigma \to \infty} \dfrac{K^A(\sigma)}{K^B(\sigma)} \geq 1$.

If $A \geq_T B$, then these and the analogues of the other properties can be shown to be equivalent by relativizing the proofs we have given. What about the general case? Consider Proposition 2.4, where we show that (4) implies (5). The proof is dynamic and requires that the A-computable enumeration of U can respond to the enumeration of elements into Q. If Q is only assumed to be B-c.e., so not necessarily enumerable from A, then it is not clear how to modify the proof to make it work. A similar problem arises in Proposition 3.3, where we prove that (6) implies (7).

Question 7.5. Are the two oracle versions of the properties in the main theorem equivalent?

Consider the two oracle versions of properties (b)–(d) in Theorem 1.1. Nies [16] defined $A \leq_{LR} B$ to mean that every sequence that is Martin-Löf random relative to B is Martin-Löf random relative to A. Similarly, he defined $A \leq_{LK} B$ to mean that $K^B(\sigma) \leq^+ K^A(\sigma)$. These are clearly the two oracle versions of (b) and (c) from the theorem. Kjos-Hanssen [10] proved that $A \leq_{LR} B$ is equivalent to the two

oracle version of (d): every $\Sigma_1^0[A]$-class of measure less than one is contained in a $\Sigma_1^0[B]$-class of measure less than one. Kjos-Hanssen, Miller and Solomon [11] built on this result to prove that $A \leq_{LR} B$ if and only if $A \leq_{LK} B$. So the analogy with lowness for randomness might suggest a positive answer to Question 7.5. However, not all results relativize in this way. Write $A \leq_{W2R} B$ to mean that every sequence that is weak 2-random relative to B is weak 2-random relative to A. Lowness for randomness is equivalent to lowness for weak 2-randomness (by Kjos-Hanssen, Miller and Solomon [11] and Nies [18]), but $A \leq_{LR} B$ is *not* equivalent to $A \leq_{W2R} B$ (by Barmpalias, Miller and Nies [2]).

In partial answer to the question, we can show that the two oracle versions of lowness for dimension and the Σ_1^0-covering property are equivalent.

Proposition 7.6. *In the two oracle case,* (3) $\Leftrightarrow$ (4).

Proof. $\Rightarrow$: Note that the proof of (3) $\Rightarrow$ (4) in Proposition 2.3 works for the two oracle versions.

$\Leftarrow$: Assume that $A, B \in 2^\omega$ satisfy the two oracle version of property (4). Fix $X \in 2^\omega$ and let s and t be rational numbers such that $\dim^A(X) < s < t$. Let $W = \{\sigma \in 2^{<\omega} \colon K^A(\sigma) \leq s|\sigma|\}$. So W is an A-c.e. set, $\mathrm{wt}_s(W) < 1$, and every tail of X is in $[W]$. By assumption, there is a B-c.e. set V such that $\mathrm{wt}_t(V) < 1$ and $[V] \supseteq [W]$. We have $X \in [W^n] \subseteq [V^n]$ for every $n \in \omega$, so $\dim^B(X) \leq t$ by Lemma 2.2 relativized to B. But t was any rational greater than $\dim^A(X)$, so $\dim^B(X) \leq \dim^A(X)$. $\square$

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