Envelopes – Computational Theory and Applications

Category: survey

Abstract

Based on classical geometric concepts we discuss the computational geometry of envelopes. The main focus is on envelopes of planes and natural quadrics. There, it turns out that projective duality and sphere geometry are powerful tools for developing efficient algorithms. The general concepts are illustrated at hand of applications in geometric modeling. Those include the design and NURBS representation of developable surfaces, canal surfaces and offsets. Moreover, we present applications in NC machining, geometrical optics, geometric tolerancing and error analysis in CAD constructions.

Keywords: envelope, duality, sphere geometry, NURBS surface, developable surface, canal surface, geometric tolerancing.

1 Introduction

2 Definition and computation of envelopes

2.1 First order analysis

We will define and discuss envelopes at hand of a spatial interpretation. Let us first illustrate this approach with a simple special case. Consider a surface in Euclidean 3-space $\mathbb{R}^3$, which has the regular parameterization

$$X(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v)).$$

Here and in the sequel, we always assume sufficient differentiability. The curves $u = const$ and $v = const$ form two one-parameter families of curves ('parameter lines') on the surface. Under a projection, e.g. the normal projection onto the plane $x_3 = 0$, each family of parameter lines is mapped onto a one-parameter set of planar curves. The envelope of each of these two curve families is the *silhouette* s of the surface X (see Fig.1). It is defined as follows: On the surface X, we search for points, whose tangent plane maps to a line under the projection. These points form the so-called *contour* of the surface with respect to the given projection; the image of the contour is the silhouette. Since the tangent t at a point p of the silhouette is the projection of the tangent plane at a surface point, both the u-curve and the v-curve passing through p have t as tangent there. This shows the envelope property. Points on the envelope are also called *characteristic points*.

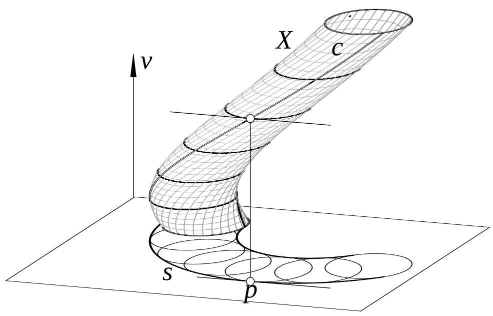

Figure 1: The silhouette of a surface as envelope of the projections of the parameter lines

For an analytic representation we note that the tangent plane of the surface is spanned by the tangent vectors to the parameter lines, which are the partial derivatives with respect to u and v,

$$\frac{\partial X}{\partial u} =: X_u, \ \frac{\partial X}{\partial v} =: X_v.$$

Under the normal projection onto the plane $x_3 = 0$, the tangent plane has a line as image iff it is parallel to the x_3-axis. In other words, the projections x_u, x_v of the two vectors X_u, X_v must be linearly dependent, i.e.,

$$\det(x_u, x_v) = 0 \iff \begin{vmatrix} x_{1,u} & x_{2,u} \\ x_{1,v} & x_{2,v} \end{vmatrix} = 0. \quad (1)$$

The computation of the silhouette (envelope) s is therefore equivalent to finding the zero set of a bivariate function. This is done with a standard contouring algorithm, a special case of a surface/surface intersection algorithm (see [21]). Let us recall that (1) characterizes the envelope of two curve families, described by $x(u,v) = (x_1(u,v), x_2(u,v))$: the set of u-lines (where u is the curve parameter and v is the set parameter) and the set of v-lines (where v is the curve parameter and u is the set parameter).

Given any one-parameter set of curves in the plane with a parametric representation $x(u,v)$, we can interpret it as projection of a surface onto that plane; it is sufficient to set $x_3(u,v) = v$. If the curve family is given in *implicit form*,

$$f(x_1, x_2, v) = 0,$$

we interpret it again as projection of a surface with the level curves $f(x_1,x_2,v) = 0$ in the planes $x_3 = v$. The tangent planes of this surface $f(x_1,x_2,x_3) = 0$ are normal to the gradient vectors

$$\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}).$$

Hence, a tangent plane is projected onto a line iff the gradient vector is normal to the x_3-axis, i.e.

$$\frac{\partial f}{\partial x_3} = 0.$$

We see that the silhouette (= envelope) is the solution of the system (we set again $x_3 = v$),

$$f(x_1,x_2,v) = 0, \ \ \frac{\partial f}{\partial v}(x_1,x_2,v) = 0. \qquad (2)$$

This is a well-known result. Its computational treatment can follow our derivation: one has to compute a projection of the intersection of two implicit surfaces (see [21]). An implicit representation of the envelope follows by elimination of v from (1). A parametric representation $(x_1(v), x_2(v))$ is obtained by solving for x_1 and x_2. By the spatial interpretation, various algorithms for computing silhouettes may also be applied [1, 21].

With the understanding of this special case, it is straightforward to extend the results to the general case. It concerns *envelopes of k-parameter sets of n-dimensional surfaces in $\mathbb{R}^d$* ($n < d$). We set $k + n \geq d$, since otherwise there is in general no envelope.

Let us start with the parametric representation in the form

$$(u_1, \ldots, u_n, \lambda_1, \ldots, \lambda_k) \mapsto$$
$$f(u_1, \ldots, u_n, \lambda_1, \ldots, \lambda_k) \in \mathbb{R}^d. \qquad (3)$$

The u_i are the parameters of the n-dimensional surfaces and the λ_k are the set parameters. We interpret this set of surfaces as normal projection of a hypersurface Φ in $\mathbb{R}^{k+n+1}$ onto $\mathbb{R}^d$ ($x_{d+1} = \ldots = x_{k+n+1} = 0$). At any point of Φ, the tangent hyperplane is spanned by the partial derivatives with respect to $u_1, \ldots, u_n, \lambda_1, \ldots, \lambda_k$. Their projections are the partial derivatives of f. At a point of the silhouette (= envelope), these derivative vectors must span at most a hyperplane in $\mathbb{R}^d$. This yields the *envelope condition*

$$\text{rank}(\frac{\partial f}{\partial u_1}, \ldots, \frac{\partial f}{\partial u_n}, \frac{\partial f}{\partial \lambda_1}, \ldots, \frac{\partial f}{\partial \lambda_k}) < d. \qquad (4)$$

It characterizes points on the envelope by a rank deficiency of the Jacobian matrix of f.

Analogously, we find the following known result. The envelope of a k-parameter set of *hypersurfaces* in $\mathbb{R}^d$, analytically represented by

$$f(x_1, \ldots, x_d, \lambda_1, \ldots, \lambda_k) = 0, \qquad (5)$$

is the solution of the system

$$f(x_1, \ldots, \lambda_k) = 0, \qquad (6)$$
$$\frac{\partial f}{\partial \lambda_1}(x_1, \ldots, \lambda_k) = 0, \ldots, \frac{\partial f}{\partial \lambda_k}(x_1, \ldots, \lambda_k) = 0.$$

The envelope conditions elucidate the computational difficulties: we have to solve a nonlinear system of equations.

In special cases, a variety of methods are available. Let us give an important example. For a one-parameter set of surfaces in $\mathbb{R}^3$ ($n = 2, k = 1$), equation (4) can be formulated as

$$\det(\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}, \frac{\partial f}{\partial \lambda_1}) = 0.$$

Hence, we have to compute the zero set of a trivariate function, which can be done, for example, with marching cubes [31]. The solution is the preimage of the envelope in the parameter domain, and application of f to it results in the envelope itself.

For systems of polynomial or rational equations, one can use algorithms which employ the subdivision property of Bézier and NURBS representations [10, 25, 56]. In more general situations one realizes the lack of algorithms for geometric processing in higher dimensions.

2.2 Singularities, curvature computation and approximation of envelopes

Let us continue with the special case of a one-parameter family of curves in the plane and its envelope, viewed as silhouette of a surface in $\mathbb{R}^3$. It is well known in classical constructive geometry [6] that the silhouette of a surface possesses a cusp if the projection ray is an osculating tangent at the corresponding point of the contour. Therefore, the condition for cusps follows by expressing that the x_3-parallel projection ray has contact of order two with the surface.

Let us work out this condition in case that the curve family (surface in $\mathbb{R}^3$) is given in implicit form $f(x_1,x_2,v) = 0$. We pick a point (x_1^0,x_2^0,v^0) on the surface and express that the projection ray (x_1^0,x_2^0,v^0+t), $t \in \mathbb{R}$, has contact of order two at this point. We use the Taylor expansion at (x_1^0,x_2^0,v^0),

$$f(x_1,x_2,v) = \tag{7}$$
$$\frac{\partial f}{\partial x_1}(x_1-x_1^0)+\frac{\partial f}{\partial x_2}(x_2-x_2^0)+\frac{\partial f}{\partial v}(v-v^0)+$$
$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}(x_1-x_1^0)^2+\ldots+\frac{\partial^2 f}{\partial v^2}(v-v^0)^2\right)+(*).$$

Here, all partial derivatives are evaluated at (x_1^0,x_2^0,v^0) and (*) denotes terms of order ≥ 3. Inserting v_0+t for v, we must get a threefold zero at $t=0$ in order to have second order contact. Using the fact that (x_1^0,x_2^0,v^0) is a point on the contour, this requires in addition

$$\frac{\partial^2 f}{\partial v^2}(x_1^0,x_2^0,v^0) = 0. \tag{8}$$

Analogously, we can investigate singular points on the envelope surface of a one-parameter family of surfaces in $\mathbb{R}^3$. These points, if they exist at all, form in general a curve, which is called an *edge of regression*. Planar intersections of the envelope surface possess (in general) a cusp at points of the edge of regression. Points of the edge of regression solve the system

$$f(x_1,x_2,x_3,v) = 0, \tag{9}$$
$$\frac{\partial f}{\partial v}(x_1,x_2,x_3,v) = 0,\ \frac{\partial^2 f}{\partial v^2}(x_1,x_2,x_3,v) = 0.$$

The characteristic curves, along which the surfaces $f(x_1,x_2,x_3,v) = 0$ touch the envelope, are tangent to the regression curve. Thus, the regression curve may be seen as envelope of the family of characteristic curves on the envelope surface. As an example, Fig. 6 shows the envelope surface of a family of planes. The characteristic curves are lines which touch the curve of regression. Note also the cusp in a planar intersection of the surface. The figure also shows cusps at the curve of regression. They are additionally satisfying

$$\frac{\partial^3 f}{\partial v^3}(x_1,x_2,x_3,v) = 0.$$

The transition to higher dimensions is straightforward: A one-parameter family of hypersurfaces in $\mathbb{R}^d$, written as

$$f(x_1,\ldots,x_d,v) = 0,$$

has an envelope hypersurface which also satisfies

$$\frac{\partial f}{\partial v}(x_1,\ldots,x_d,v) = 0.$$

On this envelope, we have a $(d-2)$-dimensional surface of singular points characterized by

$$\frac{\partial^2 f}{\partial v^2}(x_1,\ldots,x_d,v) = 0.$$

The singularities of this surface possess vanishing third derivative of f, and so on. Clearly, the envelope and the singular sets need not be real.

For a reliable approximation of envelope curves/surfaces, the computation of singularities is an important subtask. More generally, we can use results of constructive differential geometry to compute curvatures. These results concern curvature constructions for silhouettes [6]. An example of an application of such a formula to envelope curves has recently been given by Pottmann et al [50]. We will briefly address it in section 7.

There is still a lot of room for future research in this area. Reliable approximation of envelopes is one of our current research topics. It involves a segmentation according to special points of the envelope, curvature computations and approximation schemes based on derivative information up to second order at discrete points.

3 Kinematical applications

Classical kinematical geometry studies rigid body motions in the plane and in 3-space [5, 23, 24]. Because of their appearance in applications such as the construction of gears, simulation and verification for NC machining, collision avoidance in robot motion planning, etc., envelopes have received a lot of attention in this area. We will briefly address some main ideas and point to the literature.

Let us start with a one-parameter motion in the plane $\mathbb{R}^2$ and consider a curve $c(u)$ in the moving plane Σ; its position in the fixed plane Σ_0 at time t shall be $c(u,t)$. Any point $c(u_0)$ of the curve generates a path $c(u_0,t)$ in the fixed plane Σ_0. By equation (1) we get an envelope point, if the tangent vector $\partial c/\partial t$ to the path (velocity vector) is tangent to the curve position (linearly dependent to $\partial c/\partial u$ (see Fig. 2). The construction is simplified by looking at the velocity distribution at a time instant t. We either have the same velocity vector at all points (instantaneous translation) or we have an instantaneous rotation, where the velocity vector of a point x is normal to the connection with the instantaneous rotation center p (pole) and its length is proportional to $||p-x||$ (Fig. 2). With two velocity vectors the pole can be constructed and then the envelope points are the *footpoints of the normals from the pole to the curve position*. For derivations and further studies of this approach we refer to the literature [5, 24]. There, one also finds a variety of applications, in particular the construction of gears.

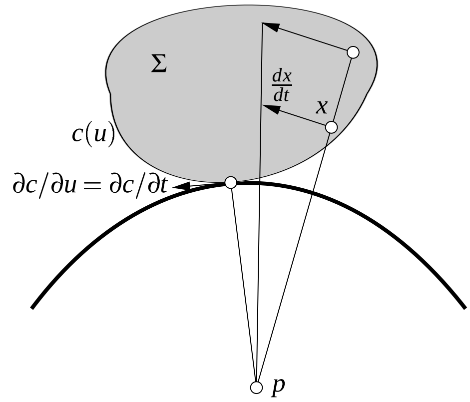

Figure 2: Envelope construction for a motion in the plane

Example: Let us consider the special case of a *translatory motion* in the plane. It is completely defined by prescribing the trajectory of one point, say the path $a(t)$ of the origin of the moving system. A curve $c(u)$ in the moving system then has at time t the position

$$c(u,t) = a(t) + c(u).$$

At any instant t, we have an instantaneous translation parallel to the derivative vector da/dt. Envelope points are characterized by parallelity of the vectors

$$\frac{d}{dt}a(t),\ \frac{d}{du}c(u).$$

Thus, we search on the curves $a(t)$ and $c(u)$ for points $a(t_0), c(u_0)$ with parallel tangents and perform the addition $a(t_0) + c(u_0)$. Since the curvature computation depends just on derivatives up to second order, we can replace the curve a by its osculating circle at $a(t_0)$, and analogously we compute the osculating circle of c at $c(u_0)$. If we translate two circles along each other, the envelope is formed of two circles. Taking orientations into account and respecting signs of the curvature radii ρ_a^0 and ρ_c^0 (defined as the sign of the curvature), we see that the envelope has curvature radius

$$\rho = \rho_a^0 + \rho_c^0. \tag{10}$$

This is illustrated in Fig.3. Recall that the envelope is the silhouette of a surface, which is in the present special case a translational surface. In case that the two given curves are the boundaries of convex domains A and C, the outer part of the envelope is convex. It is the boundary of the *Minkowski sum* $A+B$ of the two domains A,B. The Minkowski sum is defined as locus of all points $x+y$ with $x \in A$ and $y \in B$. For the computational treatment of Minkowski sums, we refer to Kaul and Farouki [27]. In the non convex case, the extraction of those parts of the envelope, which lie on the boundary of the sum $A+B$, is a subtle task (see Lee et al. [30]). Farouki et al. [12] transformed another envelope problem occurring at so-called Minkowski products to the present one.

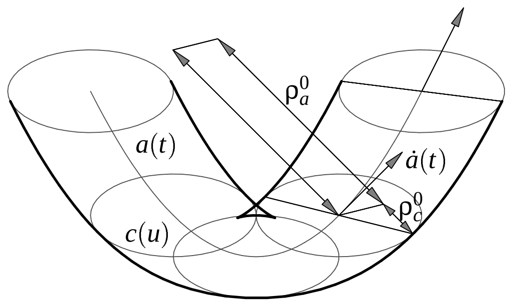

Figure 3: Envelope construction for a translatory motion in the plane

For an instant of a one-parameter motion in 3-space, the velocity distribution is also quite simple [5, 24]. The velocity vector $v(x)$ of a point x can be expressed with two vectors c, $\bar{c}$ in the form

$$v(x) = \bar{c} + c \times x. \tag{11}$$

This is in general the velocity field of a helical motion, whose axis is parallel to the vector c. In special cases we have an instantaneous rotation or translation. In order to construct the envelope surface of a surface in the moving system, one has to find on the positions $\Phi(t)$ of the moving surface those points, where the velocity vector touches $\Phi(t)$.

The computation of the envelope of a moving surface occurs for example in NC machining simulation and verification [35]. The cutting tool generates under its fast spinning motion around its axis a rotational surface (surface of revolution), which is the 'cutter' from the geometric viewpoint. Under the milling operation, the cutter removes material from the workpiece. The thereby generated surfaces are (parts of the) envelopes of the moving surface of revolution (see Fig. 4).

Figure 4: Envelopes in NC simulation

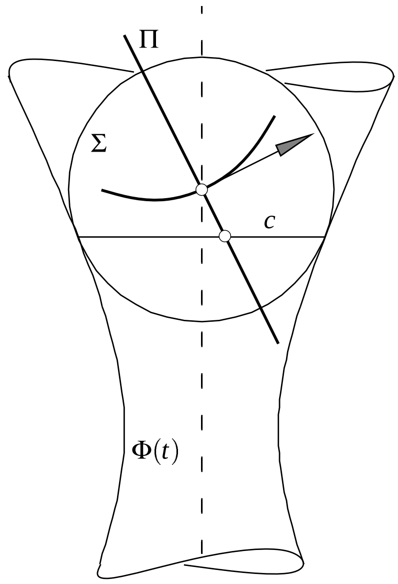

Figure 5: Constructing characteristic points on a moving surface of revolution

A simple geometric method for the computation of points on the envelope is as follows (see Fig. 5): Consider a position $\Phi(t)$ of the moving surface of revolution and pick a circle c on it. We search for the envelope points (characteristic points) on c. Along the circle, the surface is touched by a sphere Σ. Hence, along c, the surfaces Σ and $\Phi(t)$ have the same tangent planes and thus the same characteristic points (points where the velocity vector is tangent to the surface). All characteristic points of the sphere form a great circle in the plane Π normal to the velocity vector of the sphere center (see also section 6). Hence, this normal plane Π intersects c in its characteristic points. Of course, they need not be real, and one has to take those circles of $\Phi(t)$, where one gets real characteristic points. For more details on this approach and on a graphics related method (extension of the z-buffer), we refer to Glaeser and Gröller [16]. A survey on the computation of envelope surfaces has been given by Blackmore et al. [2].

Kinematical geometry has also investigated motions where the positions of the moving system are not congruent to an initial position, but similar or affine to it (see, e.g., the survey [44]). One has again a linear velocity field, and this simplifies the envelope computation. We focussed here on one-parameter motions. However, k-parameter motions, where the positions of the moving system depend on $k > 1$ real parameters, have been studied as well [44].

4 Bézier representations

A wide class of practically interesting cases of envelopes can be discussed if we assume representations in Bézier form. These work in arbitrary dimensions, but we are confining here to envelopes in $\mathbb{R}^3$. Moreover, it is clear that the transition to B-spline representations is straightforward.

We first consider a one-parameter family of surfaces, written in tensor product Bézier form

$$f(u,v,t) = \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} B_i^l(u)B_j^m(v)B_k^n(t)b_{ijk}. \quad (12)$$

We do not necessarily restrict the evaluation to the standard interval $[0,1]$, as is usually done when discussing Bézier solids [21]. Depending on which variables we consider as set parameters and which as curve or surface parameters, equation (12) is a representation of

- three one-parameter families of surfaces, obtained by viewing u or v or t as set parameter ((v,t) or (u,t) or (u,v) as surface parameters, respectively)

- three two-parameter families of curves, obtained by viewing u or v or t as curve parameter ((v,t) or (u,t) or (u,v) as set parameters, respectively)

All these families have the same envelope surface. It is obtained via equation (4),

$$\det\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial t}\right) = 0. \tag{13}$$

Inserting representation (12) and using the linearity of a determinant in each of its variables, we see: *The preimage of the envelope in the* (u,v,t)*-parameter space is an algebraic surface* Ω *of order* $3(l+m+n-1)$. Of course, the algebraic order may reduce in special cases. It might be convenient to write the surface Ω as zero set of a TP Bézier polynomial of degrees $(3l,3m,3n)$ and then apply a subdivision based approach of finding the solution [10, 25].

It has to be pointed out that part of the envelope can appear at the boundary of the corresponding TP Bézier solid (parameterized over $[0,1]^3$), see [25].

There are other trivariate Bézier representations as well [21]. For our purposes, a one-parameter family of surfaces in triangular Bézier form is interesting,

$$f(u,v,w,t) = \sum_{i,j,k,l} B^m_{ijk}(u,v,w) B^n_l(t) b_{ijkl}. \tag{14}$$

Here, u,v,w, $(u+v+w=1)$ are barycentric coordinates for the triangular surface representations, and t is the set parameter. Again, we do not just have a one-parameter family of surfaces (set parameter t), but also a two-parameter family of curves (curve parameter t), which have the same envelope. The envelope computation is performed as above. Restricting the evaluation to (u,v,w) in the domain triangle $(u,v,w \geq 0)$ and to $t \in [0,1]$, equation (14) parameterizes a 'deformed pentahedron', which is called a *pentahedral Bézier solid* of degree (m,n) [21]. Part of the envelope can appear at the boundary of this solid.

In case we are working with implicit representations, a Bézier form may be present in the coefficients of the surface equations. For example, we can discuss a one-parameter family of algebraic surfaces $\Phi(t)$ of order m,

$$f(x,y,z,t) = \sum_{i,j,k} a_{ijk} x^i y^j z^k = 0, \; i+j+k \leq m,$$

$$a_{ijk} = \sum_{l=0}^{n} B^n_l(t) b_{ijkl}. \tag{15}$$

We then have to form the partial derivative with respect to t, which is in general again an algebraic surface $\dot{\Phi}(t)$ of order m, and intersect with the original surface $\Phi(t)$. This results (for each t) in the characteristic curve $c(t)$ along which the surface $\Phi(t)$ touches the envelope. By Bezout's theorem, the characteristic curve is in general an algebraic curve of order $2m$.

In the following sections we will discuss a variety of interesting examples for envelopes computed from Bézier type representations.

Figure 6: Tangent surface of a curve on an ellipsoid

5 Envelopes of planes

The simplest case of a one-parameter family of algebraic surfaces is that of a family of planes $U(t)$, written in the form

$$U(t): \; u_0(t) + u_1(t)x + u_2(t)y + u_3(t)z = 0. \tag{16}$$

Its envelope is found by intersecting planes $U(t)$ with the first derivative planes

$$\dot{U}(t): \; \dot{u}_0(t) + \dot{u}_1(t)x + \dot{u}_2(t)y + \dot{u}_3(t)z = 0. \tag{17}$$

For each t, this yields a straight line $r(t)$. Hence, the envelope surface Φ is a ruled surface which is tangent along each ruling $r(t)$ to a single plane (namely $U(t)$). It is well-known in differential geometry [9] that this characterizes the surface as *developable surface*. It can be mapped isometrically into the Euclidean plane. Because of this property, developable surfaces possess a variety of applications, for example in sheet metal based industries.

Continuing the general concepts from section 2, we compute the *curve of regression*. With the second derivative plane

$$\ddot{U}(t): \; \ddot{u}_0(t) + \ddot{u}_1(t)x + \ddot{u}_2(t)y + \ddot{u}_3(t)z = 0, \tag{18}$$

its points are the intersection points $c(t) = U(t) \cap \dot{U}(t) \cap \ddot{U}(t)$. In special cases $c(t)$ can be fixed (Φ is a general cone) or the system does not have a solution ($c(t)$ is a fixed ideal point and Φ is a general cylinder surface). In the general case, the tangents of $c(t)$ are the rulings $r(t)$ and thus Φ is the *tangent surface* of the regression curve $c(t)$ (cf. the example in Fig. 6).

5.1 Dual representation of curves and surfaces

The coefficients $(u_0,\ldots,u_3)$ in the plane equation (16) are the so-called homogeneous plane coordinates of the plane $U(t)$. Here and in the sequel, we will also denote the vector $(u_0, \ldots, u_3) \in \mathbb{R}^4$ by $U(t)$.

Within projective geometry, we can view planes as points of dual projective space. Thus, in dual space a one-parameter family of planes is seen as a curve and we can apply curve algorithms to compute with them. This point of view has been very fruitful for computing with developable surfaces in NURBS form [3, 4, 22, 48, 52].

A planar curve, represented as envelope of its tangents, is said to be given in *dual representation* (curve in the dual projective plane). Analogously, a surface in 3-space, viewed as envelope of its tangent planes, is said to be in dual representation. If the set of tangent planes is one-dimensional (curve in dual projective space), the surface is a developable surface. Otherwise, we have a two-dimensional set of tangent planes and thus also a surface in dual space.

Dual representations in the context of NURBS curves and surfaces, have been first used by J. Hoschek [20]. To illustrate some essential ideas, we briefly discuss the *dual representation of planar rational Bézier curves*.

A rational Bézier curve possesses a polynomial parameterization in homogeneous coordinates $C(t) = (c_0(t), c_1(t), c_2(t))$, which is expressed in terms of the Bernstein polynomials,

$$C(t) = \sum_{i=0}^{n} B_i^n(t) P_i. \tag{19}$$

The coefficients P_i are the homogeneous coordinate vectors of the control points. The tangents $U(t)$ of the curve connect the curve points $C(t)$ with the derivate points $\dot{C}(t)$. This shows that the set of tangents $U(t)$ also possesses a polynomial parameterization in line coordinates, i.e., a dual polynomial parameterization. It can be expressed in the Bernstein basis, which leads to the *dual Bézier representation*,

$$U(t) = \sum_{i=0}^{m} B_i^m(t) U_i. \tag{20}$$

We may interpret this set of lines as a curve in the dual projective plane. Thus, by duality we obtain properties of *dual Bézier curves*, i.e. rational curves in the dual Bézier representation.

When speaking of a Bézier curve we usually mean a curve segment. In the form we have written the Bernstein polynomials, the curve segment is parameterized over the interval [0,1]. For any $t \in [0,1]$, equation (20) yields a line $U(t)$. The

Figure 7: Dual Bézier curve

curve segment we are interested in, is the envelope of the lines $U(t)$, $t \in [0,1]$.

As an example for dualization, let us first discuss the *dual control structure* (see Fig.7). It consists of the *Bézier lines* U_i, $i = 0, \ldots, m$, and the *frame lines* F_i, whose line coordinate vectors are

$$F_i = U_i + U_{i+1}, \;\; i = 0, \ldots, m-1. \tag{21}$$

From (21) we see that the frame line F_i is concurrent with the Bézier lines U_i and U_{i+1}. This is dual to the collinearity of a frame point with the adjacent two Bézier points. We use frame lines – dual to the frame points – instead of weights, since the latter are not projectively invariant. A projective formulation is important for application of projective duality.

The complete geometric input of a dual Bézier curve consists of the $m+1$ Bézier lines and m frame lines. Given these lines, each of it has a one-dimensional subspace of homogeneous coordinate vectors with respect to a given coordinate system. It is possible to choose the Bézier line coordinate vectors U_i such that (21) holds. This choice is unique up to an unimportant common factor of the vectors $U_0, \ldots, U_m$. Now (20) uniquely defines the corresponding curve segment.

For a Bézier curve, the control point P_0 is an end point of the curve segment and the line $P_0 \vee P_1$ is the tangent there. Dual to this property, the end tangents of a dual Bézier curve are U_0 and U_m, and the end points are the intersections $U_0 \cap U_1$ and $U_{m-1} \cap U_m$, respectively.

To evaluate the polynomial (20), one can use the well-known recursive de Casteljau algorithm. It starts with the control lines $U_i = U_i^0$ and constructs recursively lines $U_i^k(t)$ via

$$U_i^k(t) = (1-t) U_i^{k-1}(t) + t U_{i+1}^{k-1}(t). \tag{22}$$

At the end of the resulting triangular array, we get the line $U(t)$. However, usually one would like to get the curve point $C(t)$. It is also delivered by the de Casteljau algorithm, if we note one of its properties in case of the standard representation: in step $m-1$, we get two points $C_0^{m-1}(t), C_1^{m-1}(t)$, which lie on the tangent at $C(t)$. Dual to that, the lines $U_0^{m-1}(t)$ and $U_1^{m-1}(t)$ intersect in the curve point $C(t)$. Hence, we have

$$C(t) = U_0^{m-1}(t) \times U_1^{m-1}(t).$$

Note that the curve point computation based on the dual form has the same efficiency as the computation based on the standard form.

The above procedure leads via

$$U_0^{m-1}(t) = \sum_{i=0}^{m-1} B_i^{m-1}(t) U_i,$$
$$U_1^{m-1}(t) = \sum_{j=0}^{m-1} B_j^{m-1}(t) U_{j+1},$$

to the following *formula for conversion* from the dual Bézier form $U(t)$ of a rational curve segment to its standard point representation $C(t)$,

$$C(t) = \sum_{k=0}^{2m-2} B_k^{2m-2}(t) P_k \tag{23}$$

with

$$P_k = \frac{1}{\binom{2m-2}{k}} \sum_{i+j=k} \binom{m-1}{i}\binom{m-1}{j} U_i \times U_{j+1}.$$

We see that $C(t)$ possesses in general the degree $2m-2$. For more details, degree reductions, further properties and applications, we refer to the literature [20, 43, 46, 45, 53].

5.2 Developable NURBS surfaces as envelopes of planes

The fact that developable surfaces appear as curves in dual space indicates the computational advantages of the dual approach. We will briefly describe a few aspects of computing with the dual form.

A developable NURBS surface can be written as envelope of a family of planes, whose plane coordinate vectors $U(t) \in \mathbb{R}^4$ possess the form

$$U(t) = \sum_{i=0}^{n} N_i^m(t) U_i. \tag{24}$$

The vectors $U_i \in \mathbb{R}^4$ are homogeneous plane coordinate vectors of the *control planes*, which we also call U_i (Fig. 8). To get a geometric input, we additionally use the *frame planes* with coordinate vectors F_i,

$$F_i = U_i + U_{i+1}, \quad i = 0, \ldots, n-1. \tag{25}$$

For several algorithms it is convenient to convert (24) into piecewise Bézier form and then perform geometric processing on these Bézier segments. Therefore, we will in the following restrict our discussion to such segments, i.e., to *developable Bézier surfaces*. In the dual form, they

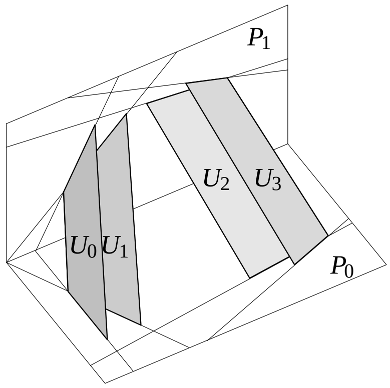

Figure 8: Control planes of a developable Bézier surface

are expressed as

$$U(t) = \sum_{i=0}^{m} B_i^m(t) U_i, \quad t \in [0,1]. \tag{26}$$

Like in the study of planar rational curves in dual Bézier form, we can easily obtain insight into the behaviour of the dual control structure by dualization. Let us start with the behaviour at the interval end points $t=0$ and $t=1$. A rational Bézier curve possesses at its end points the tangents $P_0 \vee P_1$ and $P_{m-1} \vee P_m$. Dual to that, the end rulings of the developable surface (26) are $r(0) = U_0 \cap U_1$ and $r(1) = U_{m-1} \cap U_m$ (Fig. 8). The osculating plane of a Bézier curve at an end point is spanned by this point and the adjacent two control points. Dual to an osculating plane (connections of curve point with the first two derivative points) is a point of regression (intersection of tangent plane with the first two derivative planes). Hence, the end points of the curve of regression $c(t)$ of the developable Bézier surface are

$$c(0) = U_0 \cap U_1 \cap U_2, \quad c(1) = U_{m-2} \cap U_{m-1} \cap U_m.$$

The computation of rulings and points of regression is based on the algorithm of de Casteljau applied to the dual form (26). Stopping this algorithm one step before the last one, there are still two planes in the triangular array. Their intersection is a ruling. One step earlier, we still have three planes, which intersect in the point of regression.

Finally, we note the dual property to the well-known result that any central projection of a rational Bézier curve c yields a rational Bézier curve c' whose control points and frame points are the images of the control points and frame points of c.

Proposition: *The intersection curve of a developable Bézier surface $U(t)$ with a plane P is a* *rational Bézier curve $C(t)$. The control lines and frame lines of $C(t)$ are the intersections of the control planes and frame planes of $U(t)$ with P.*

Figure 9: Control net of the surface patch in Fig. 8

This property is very useful for converting the dual representation into a standard tensor product form (see Fig. 9 and [48]). For other algorithms in connection with developable NURBS surfaces (interpolation, approximation (see Fig. 10), controlling the curve of regression, etc.) we refer the reader to the literature [3, 4, 22, 48, 52].

Figure 10: Approximation of a set of planes by a developable surface

5.3 Developable surfaces via pentahedral Bézier solids

A pentahedral Bézier (PB) solid of degree $(1,n)$ is formed by a one-parameter family of triangles, whose planes envelope a developable surface. We now show how to derive the dual representation of the envelope.

In the special case $m = 1$ of (14), we set

$$b_{100l} =: A_l,\ b_{010l} =: B_l,\ b_{001l} =: C_l,$$

and obtain the representation of the PB solid,

$$F(u,v,w,t) = \sum_{l=0}^{n} B_l^n(t)(uA_l + vB_l + wC_l). \quad (27)$$

We assume a *rational* representation and therefore the involved vectors F, A_l, B_l, C_l are homogeneous coordinate vectors, i.e., vectors in $\mathbb{R}^4$. The solid is generated by a one-parameter family of triangles in planes $U(t)$,

$$U(t) := A(t) \vee B(t) \vee C(t), \quad (28)$$

$$\text{with} \quad A(t) = \sum_{l=0}^{n} B_l^n(t)A_l, \ldots, C(t) = \sum_{l=0}^{n} B_l^n(t)C_l.$$

Note that the rational Bézier curves $A(t), B(t), C(t)$ are edge curves of the pentahedral solid (see Fig. 11).

Given the homogeneous coordinate vectors A, B, C of three points, the plane coordinates of the spanning plane are given by the vector product $A \times B \times C$ (in $\mathbb{R}^4$!). This yields for the plane

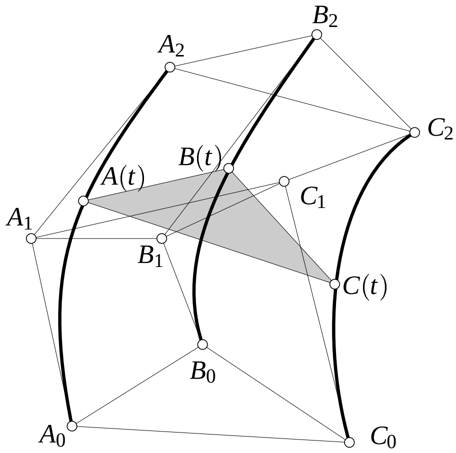

Figure 11: Family of planes generated by a pentahedral Bézier solid of degree (1,2)

family (28),

$$U(t) = A(t) \times B(t) \times C(t) = \sum_{l=0}^{3n} B_l^{3n}(t)U_l, \quad (29)$$

with

$$U_l := \frac{1}{\binom{3n}{l}} \sum_{i+j+k=l} \binom{n}{i}\binom{n}{j}\binom{n}{k}(A_i \times B_j \times C_k).$$

Thus, the generated envelope is in general of dual degree $3n$. Degree reductions are possible. They occur if for some t_0 the vector $U(t_0)$ is zero. Then, all polynomial coordinate functions of $U(t)$ are divisible by $(t - t_0)$, which explains the degree reduction. We omit a detailed study of these reductions and their geometric meaning.

Example: The simplest example belongs to $n = 1$. As envelope Φ we get in general a developable surface of dual degree 3. It is well known that these surfaces are tangent surfaces of rational cubics. In case of degree reductions, Φ may be a quadratic cone, or it degenerates to a line (all planes of the family pass through this line). If we use an integral representation (equation (14) with inhomogeneous coordinates and $n = 1$), Φ is either the tangent surface of a polynomial cubic, a parabolic cylinder or a line (which may be at infinity, i.e., all planes of the family are parallel).

6 Envelopes of spheres and natural quadrics

We will study envelopes of spheres and cones and cylinders of revolution, called natural quadrics. In particular we focus on rational families. The envelope of a one-parameter family of spheres is called *canal surface*. If the family consists of congruent spheres (constant radius), the envelope is called *pipe surface*.

6.1 One-parameter families of spheres

First we start discussing one parameter families of spheres. Let $S(t)$ and $\dot{S}(t)$ denote the spheres and its derivatives,

$$
\begin{aligned}
S(t) \quad &: \quad \sum_{i=1}^{3}(x_i - m_i(t))^2 - r(t)^2 = 0, \qquad (30)\\
\dot{S}(t) \quad &: \quad \sum_{i=1}^{3}(x_i - m_i(t))\dot{m}_i(t) - r(t)\dot{r}(t) = 0,
\end{aligned}
$$

where $M(t) = (m_1, m_2, m_3)(t)$ and $r(t)$ denote centers and radii of $S(t)$. The envelope Φ is tangent to $S(t)$ in points of the characteristic curves $c(t) = S(t) \cap \dot{S}(t)$. Since $\dot{S}$ are planes, $c(t)$ consists of circles. Φ consists of real points if and only if

$$||M(t)||^2 - r(t)^2 \leq 0$$

holds. The envelope possesses singular points if $\ddot{S}(t) \cap c(t)$ consists of real points. $\ddot{S}(t)$ is again a plane, such that each circle $c(t)$ can possess at most two singular points.

If $M(t)$ is a rational center curve and $r(t)$ is rational, $S(t)$ shall be called *rational family of spheres with rational radius function*. It is proved in [38] that the envelope Φ possesses rational parametrizations. Thus it is representable as a NURBS surface. Additionally, all offset surfaces Φ_d at distance d are rational, since they are also canal surfaces with rational center curve and rational radius function.

Let $Q(t)$ be a one-parameter family of spheres represented in the form (15). Equivalently, we have

$$Q(t) : a(x_1^2 + x_2^2 + x_3^2) + bx_1 + cx_2 + dx_3 + e = 0, \qquad (31)$$

where the coefficients $a, \ldots, e$ are rational functions or polynomials in t. $Q(t)$ shall be called *rational family of spheres*. One can verify that the centers $M(t)$ of spheres $Q(t)$ form a rational curve but its radius $r(t)$ is just the square root of a rational function. But also in this case it is proved ([38]) that the envelope Φ of the family $Q(t)$ possesses rational parametrizations. This implies that the *general cyclides* (see [37]), which are algebraic surfaces of order four, are rational. By the way, the offset surfaces are in general not rational since their radius function $r(t)$ will not be a square root of a rational function.

Corollary 6.1 *The envelope Φ of a rational family of spheres is rationally parametrizable. If the family has rational radius function then all offset surfaces Φ_d of Φ are rational, too.*

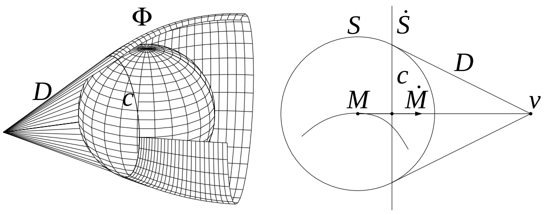

Figure 12: Geometric properties of a canal surface

Figure 12 shows also that the envelope Φ is tangent to a cone of revolution $D(t)$ at the characteristic circle $c(t)$. Thus, Φ is also part of the envelope of the one-parameter family of cones $D(t)$.

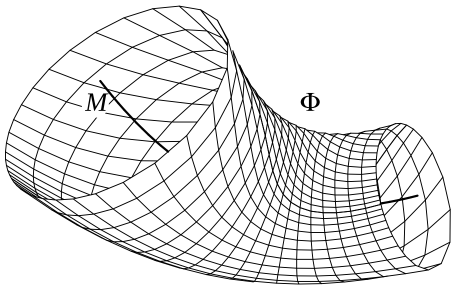

Figure 13: Rational canal surface

6.2 One-parameter families of cylinders and cones of revolution

Next we will study envelopes of a one-parameter family of cylinders or cones of revolution. In the following it is not necessary to distinguish between cones and cylinders and we speak simply of cones of revolution. Let $D(t)$ be such a family. Its derivative $\dot{D}(t)$ defines in general a family of regular quadrics. It can be proved that each surface $\dot{D}(t)$ contains the vertex $v(t)$ of D(t). The characteristic curves $c(t) = D(t) \cap \dot{D}(t)$ are in general rational curves of order four with $v(t)$ as singular point. Let $D(t)$ be given by an implicit quadratic equation in the coordinates x_i,

$$D(t) : \sum_{i,j=1}^{3} a_{ij}(t)x_i x_j + \sum_{i=1}^{3} b_i(t)x_i + c(t) = 0. \qquad (32)$$

The coefficients a_{ij}, b_i, c are functions of the parameter t. Additionally we always can assume that $A = (a_{ij})$ is a symmetric 3×3-matrix. $D(t)$ is a cone of revolution if the matrix a_{ij} possesses a twofold eigenvalue and the extended matrix

$$\begin{bmatrix} c & b_1 & b_2 & b_3 \\ b_1 & a_{11} & a_{12} & a_{13} \\ b_2 & a_{12} & a_{22} & a_{23} \\ b_3 & a_{13} & a_{23} & a_{33} \end{bmatrix}$$

has rank 3. $D(t)$ defines cylinders of revolution if additionally A has rank 2. $D(t)$ is called a *rational family*, if the coefficients in (32) are rational functions of t. In particular, its vertices $v(t)$ form a rational curve.

In view of rational parametrizations of offset surfaces, special rational families of cones of revolution are of interest. Let $v(t)$ be a rational curve and $S(t)$ be a rational family of spheres with rational radius function, as discussed in section 6.1. The unique family of cones of revolution $D(t)$ joining $v(t)$ and $S(t)$ shall be called *rational family with rational radius function*. By the way, there is a one-parameter family of spheres (with rational radius function) being inscribed in each cone. Note that the general rational family (32) has rational vertices $v(t)$ but no further spheres $S(t)$ with rational radii. If a surface Φ is enveloped by a

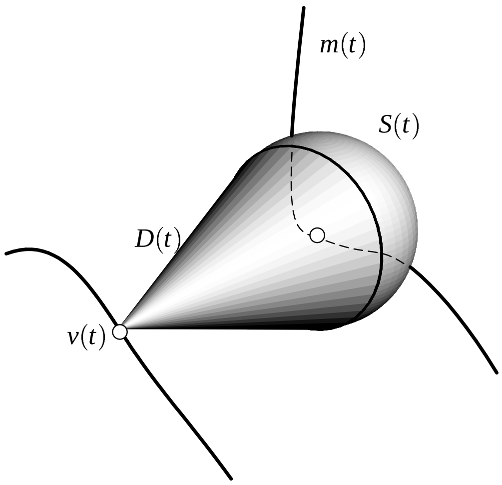

Figure 14: Rational family of cones with rational radius function

one-parameter family of cones (cylinders) of revolution $D(t)$, its offset surface Φ_d at distance d is also enveloped by a one-parameter family of cones(cylinders) of revolution $D_d(t)$, the offset surfaces of $D(t)$. Then, the following theorem holds.

Theorem 6.1 *The envelope Φ of a rational one-parameter family of natural quadrics $D(t)$ possesses rational parametrizations. If additionally, $D(t)$ has rational radius function then all offsets Φ_d of Φ at distance d are also rationally parametrizable.*

A one-parameter family of cones of revolution $D(t)$ enveloping a canal surface is a special case, since the characteristic curve $c(t)$ of order four decomposes into a circle and a not necessarily real pair of lines.

Since the property of possessing rational parametrizations is invariant under projective transformations, first part of theorem 6.1 holds for one-parameter families of general quadratic cones (we drop the double eigenvalue condition). The property of having rational offsets is *not* preserved by projective transformations but by similarities and more generally by sphere-preserving geometric transformations, see [38].

In section 5 we introduced the dual representation of curves in the plane and surfaces in space. Since a natural quadric D is a developable surface, it is enveloped by a one parameter family of planes

$$U(s) : u_0(s) + u_1(s)x_1 + u_2(s)x_2 + u_3(s)x_3 = 0, \tag{33}$$

where we can assume that $u_i(s)$ are quadratic polynomials. Thus, the problem of constructing rational parametrization of a family $D(t)$ is equivalent to the problem of finding a representation of the form

$$U(s,t) : u_0 + u_1x_1 + u_2x_2 + u_3x_3 = 0 \tag{34}$$

with bivariate polynomials $u_i(s,t)$ such that $U(s,t_0)$ represents the tangent planes of the cone $D(t_0)$.

The envelope Φ of the family $D(t)$ is enveloped by the tangent planes $U(s,t)$. The points of contact are obtained by

$$p(s,t) = U(s,t) \cap \frac{\partial U}{\partial s}(s,t) \cap \frac{\partial U}{\partial t}(s,t). \tag{35}$$

6.3 Offset surfaces of regular quadrics

It is known that the offset curves of a conic in the plane are in general not rational algebraic curves of order 8. Besides the trivial case of a circle, only the offset curves of parabolas are rational curves of order 6. Surprisingly it can be proved the following result, see also [32].

Theorem 6.2 *The offset surfaces of all regular quadrics in space can be rationally parametrized.*

The reason is that any regular quadric can be represented as envelope of a rational family of cones of revolution with rational radius function. Details on the parametrization and low degree representations of the offsets can be found in [38].

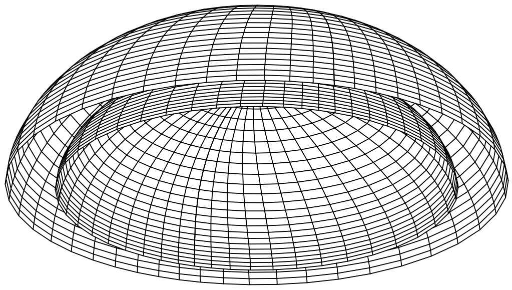

Figure 15: Quadric of revolution and outer offset surface

6.4 Envelope of congruent cylinders of revolution

A special example of a surface enveloped by a one-parameter family of natural quadrics is the envelope of a moving cylinder of revolution. These surfaces appear in applications as results of a (five-axis) milling procedure with a cylindrical cutter. The axes of the one-parameter family of cylinders (cutting tool) generate a ruled surface R, the axes surface. In general, R is a non-developable ruled surface which says that the tangent planes at points of a fixed generating line of R vary. In particular, the relation between contact points and tangent planes of a fixed generator is linearly. A developable ruled surface possesses a fixed tangent plane along a generating line.

A cylinder of revolution D can be considered as envelope of a one-parameter family of congruent spheres, centered at the axis of D. Then it is obvious that the envelope Φ of a moving cylinder of revolution $D(t)$ is an offset surface of R. The distance between R and $\Phi = R_d$ equals the radius of the cylinders $D(t)$. In general, the characteristic curve where $D(t)$ and Φ are in contact, are rational curves of order four, as mentioned in section 6.2.

If the axis surface R is developable such that the axes are tangent lines of the curve of regression c of R, the envelope Φ of the family $D(t)$ decomposes into the offset surface R_d of R and into the pipe surface, interpreted as offset surface of the curve of regression c. The reason is that the characteristic curve $c(t)$ of order four decomposes into a circle and two lines. The circle is located in the normal plane to the curve of regression c. Thus, it generates the pipe surface. The lines generate the offset surface R_d of R which is again a developable surface.

Let $D(t)$ be a rational one-parameter family of congruent cylinders. Theorem 6.1 says that their envelope Φ is rational and possesses rationally parametrizable offsets. More precisely, if the axes surface R is a rational non-developable ruled surface then Φ and its offsets are rational. If R is a rational developable surface, its offsets R_d need not to be rational, but the pipe surface, centered at the line of regression c of R, and its offset pipes are rational. Details on the computation can be found in the survey [55] and in [26]).

Corollary 6.2 *The envelope Φ of a one-parameter family of congruent cylinders of revolution is an offset surface of the axes surface R. If R is rational and non-developable then Φ admits rational parametrizations.*

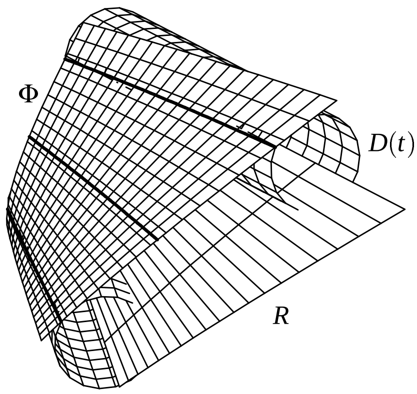

Figure 16: Envelope of a one-parameter family of congruent cylinders of revolution

6.5 Two-parameter families of spheres

One-parameter families of spheres can envelop a canal surface, but also two-parameter families of spheres may have an envelope. Let $S(u,v)$ be such a family and let $S_u(u,v)$ and $S_v(u,v)$ be its partial derivatives with respect to u,v. Thus, we have

$$S(u,v) \quad : \quad \sum_{i=1}^{3}(x_i - m_i(u,v))^2 - r(u,v)^2 = 0,$$

$$S_u(u,v) \quad : \quad \sum_{i=1}^{3}(x_i - m_i)m_{iu} - rr_u = 0,$$

$$S_v(u,v) \quad : \quad \sum_{i=1}^{3}(x_i - m_i)m_{iv} - rr_v = 0. \qquad (36)$$

Points p of the envelope Φ have to satisfy all equations. Since S_u and S_v are planes, p has to lie on the line of intersection $g = S_u \cap S_v$. There might be two, one or no intersection point. If the number of intersection points is two or zero, then Φ consists locally of two or no components. The case of g being tangent to S can hold for isolated points or a two dimensional set of points; usually it is satisfied for a one-parametric family of points.

We want to address a special case. If the radius function $r(u,v)$ is constant, then the envelope of $S(u,v)$ is the offset surface of the surface M traced out by the centers (m_1,m_2,m_3) at distance r. If $m_i(u,v)$ are rational functions, then M is a rational surface. In general, its offset surfaces will not be rational. Of course, there are several surfaces possessing rational offsets; see section 6.2. Since a cone or cylinder of revolution can be enveloped by a one-parameter family of spheres, one-parametric families of cones are special cases of two-parametric families of spheres. A sphere geometric approach to rational offsets and envelopes of special two-parameter families of spheres can be found in [38] and [41].

6.6 Two-parameter families of circles in the plane

A one-parameter family of circles in the plane can possess an envelope. In particular, congruent circles will envelope the offset curve of its centers. But also a two-parametric family of circles $c(u,v)$ can have an envelope. Let its centers and radii be $m(u,v) = (m_1,m_2)(u,v)$ and $r(u,v)$, respectively.

To enlighten this geometrically, we will use a mapping ζ which associates a point $\zeta(c) \in \mathbb{R}^3$ to each circle in the following way,

$$c \mapsto \zeta(c) = (m_1,m_2,r) \in \mathbb{R}^3. \qquad (37)$$

We will assume $r(u,v)$ being a signed radius. The transformation ζ is called *cyclographic mapping*. The inverse mapping ζ^{-1} maps points $p = (p_1,p_2,p_3) \in \mathbb{R}^3$ to circles in $\mathbb{R}^2$,

$$\zeta^{-1}(p) : (x_1 - p_1)^2 + (x_2 - p_2)^2 = p_3^2.$$

We identify points in $\mathbb{R}^2$ with circles with zero radii. The cyclographic image $\zeta(c)$ of one or two-parametric families of circles are curves or surfaces in $\mathbb{R}^3$.

Given a point $p \in \mathbb{R}^3$, the circle $\zeta^{-1}(p)$ is the intersection of the cone of revolution $\gamma(p)$ with $\mathbb{R}^2$, where $\gamma(p)$ has p as vertex and its inclination angle with $\mathbb{R}^2$ is $\pi/4 = 45°$. Thus, all tangent planes of $\gamma(p)$ enclose an angle of $\pi/4$ with $\mathbb{R}^2$.

Applying ζ^{-1} to a curve $p(t) \in \mathbb{R}^3$, one obtains a one-parameter family of circles $\zeta^{-1}(p(t))$. The envelope of the one-parameter family of cones $\gamma(p(t))$ is a developable surface $\Gamma(p(t))$. Thus, the envelope of the family of circles $\zeta^{-1}(p(t))$ is the intersection of $\Gamma(p(t))$ with $\mathbb{R}^2$.

Given a two-parametric family of circles $c(u,v)$, it may have an envelope. Its cyclographic image $\zeta(c(u,v))$ is a surface in $\mathbb{R}^3$. If this surface possesses tangent planes τ enclosing an angle of $\pi/4$ with $\mathbb{R}^2$, then $c(u,v)$ will possess an envelope.

Let us assume that $c(u,v)$ has an envelope. Then there exists a curve $p(t) \subset \zeta(c(u,v))$ whose tangent planes $\tau(p(t))$ have slope $\pi/4$. These planes $\tau(p(t))$ envelope a developable surface $\Gamma(c)$. Thus, we finally obtain the envelope of $c(u,v)$ by intersecting $\Gamma(c)$ with $\mathbb{R}^2$.

We will discuss some examples. At first, let $p(u,v) \in \mathbb{R}^3$ be the parametrization of a plane g, enclosing an angle of $\pi/4$ with $\mathbb{R}^2$. The envelope of the family of circles $\zeta^{-1}(p(u,v))$ is simply the intersection $g \cap \mathbb{R}^2$.

At second, let $p(u,v) \in \mathbb{R}^3$ be a sphere. The set of points $q(t)$ possessing tangent planes with slope $\pi/4$ are two circles. These tangent planes envelope two cones of revolution, being tangent to the sphere in points of these circles. Finally, the envelope of $\zeta^{-1}(p(u,v))$ consists of two circles if $p(u,v)$ is not centered in $\mathbb{R}^2$. Otherwise it is one circle.

6.7 Families of spheres in space

Analogously to two-parametric families of circles in the plane are three-parametric families of spheres in space. Since a sphere is determined by its center $M = (m_1,m_2,m_3)$ and its radius r, we will define a cyclographic mapping ζ,

$$S \mapsto \zeta(S) = (m_1,m_2,m_3,r) \in \mathbb{R}^4, \qquad (38)$$

which maps the spheres in $\mathbb{R}^3$ onto points of $\mathbb{R}^4$. We will again assume that r is a signed radius. A three-parametric family of spheres in $\mathbb{R}^3$ is mapped to a hypersurface in $\mathbb{R}^4$. If we apply an analogous construction as done in section 6.6 we arrive at a geometric interpretation of the computation of the envelope of a three-parametric family of spheres. Let their image hypersurface in $\mathbb{R}^4$ be denoted by $p(u,v,w)$. If this hypersurface possesses tangent hyperplanes (3-spaces) enclosing an angle of $\pi/4$ with $\mathbb{R}^3$, then there exists an envelope of the family of spheres $\zeta^{-1}(p)$. We will not go into details here.

The cyclographic mapping is also helpful for constructing envelopes of two-parameter families of spheres. Details about these constructions with an emphasis on rational families of spheres and surfaces possessing rational offset surfaces can be found in [41, 51]. In connection with tolerance analysis, the cyclographic mapping was fruitfully applied in [50].

6.8 Two-parameter families of lines and cylinders

We have discussed in section 6.4 that the axes of a one-parameter family of congruent cylinders of revolution form a ruled surface and the envelope of the cylinders itself is an offset surface of this ruled axis-surface. In case that we have a two-parameter family of congruent cylinders of revolution, the situation is much more involved. First of all we will study two-parameter families of lines (axes), since this will partially answer the question how to compute the envelope of a two-parameter family of cylinders $D(u,v)$.

A two-parameter family of straight lines $\mathcal{C}$ is called *line congruence*. It shall be defined by the following parametrization. Let $M = (m_1,m_2,m_3)(u,v)$ a smooth surface in 3-space and $E = (e_1,e_2,e_3)(u,v)$ a vector field. We may assume that $\|E\| = 1$ for all $(u,v) \subset U$. Then,

$$M(u,v) + wE(u,v),\ w \in \mathbb{R} \tag{39}$$

parametrizes a two-parameter family of lines. The points of a fixed line are parametrized by $w \in \mathbb{R}$. Since the direction vectors $E(u,v)$ are normalized, $E(u,v)$ parametrizes a domain in the unit sphere. We will assume that this domain is two-dimensional and does not degenerate to a single curve or a single point. Mathematically this is guaranteed by linearly independence of the partial derivatives $\partial E/\partial u$ and $\partial E/\partial v$.

Let $L = M(u_0,v_0) + wE(u_0,v_0)$ be a fixed line in the congruence and let $R_u : M(u,v_0) + wE(u,v_0)$ be a ruled u-surface passing through the fixed chosen line L. Analogously, let $R_v : M(u_0,v) + wE(u_0,v)$ be the ruled v-surface passing through L. We want to study the distribution of the tangent planes of R_u and R_v along L. It is an elementary computation that the surface normals N_u and N_v of the ruled surfaces R_u, R_v are

$$N_u = (\frac{\partial M}{\partial u} \times E + w\frac{\partial E}{\partial u} \times E),$$

$$N_v = (\frac{\partial M}{\partial v} \times E + w\frac{\partial E}{\partial v} \times E).$$

The tangent planes of the surfaces R_u and R_v coincide if and only if the normals N_u, N_v (which are assumed to be not zero) are linearly dependent,

$$N_u \times N_v = (0,0,0).$$

Elaborating this one obtains the following quadratic equation

$$w^2 \det(\frac{\partial E}{\partial u}, E, \frac{\partial E}{\partial v}) + w[\det(\frac{\partial E}{\partial u}, E, \frac{\partial M}{\partial v}) + \det(\frac{\partial M}{\partial u}, E, \frac{\partial E}{\partial v})] + (\frac{\partial M}{\partial u}, E, \frac{\partial M}{\partial v}) = 0. \tag{40}$$

Depending on the number $2,1$ or 0 of real solutions, the line L is called hyperbolic, parabolic or elliptic. If L is hyperbolic or elliptic this property holds for lines in the congruence $\mathcal{C}$ being close to L. So we might say that the congruence $\mathcal{C}$ itself is *hyperbolic* or *elliptic*. There might be isolated parabolic lines as well as one or two-dimensional domains of parabolic lines of the congruence. In the following we will focus at the hyperbolic case. We want to note that if the congruence $\mathcal{C}$ consists of normals of a smooth surface (different from a plane or a sphere), then $\mathcal{C}$ is always hyperbolic.

If (40) has two real solutions w_1, w_2, they correspond to points with coinciding tangent planes. These points generate the so called *focal surfaces*

$$F_i = M(u,v) + w_i E(u,v),\ i = 1,2.$$

We return to the previous problem of computing the envelope of a two-parameter family of congruent cylinders of revolution $D(u,v)$. These cylinders have axes $A(u,v)$ which form a line congruence. If the congruence is hyperbolic, the axes envelop the focal surfaces F_1, F_2. Thus, the envelope of the cylinders itself contains the offset surfaces G_1, G_2 of F_1, F_2 at distance d, which equals the radii of the cylinders $D(u,v)$. The offset surfaces admit the following representation

$$G_i = F_i(u,v) \pm dN_i^0(u,v)$$

with $N_i^0(u,v) = N_u^0(u,v,w_i) = N_v^0(u,v,w_i)$ as unit normal of the corresponding focal surface F_i. If the congruence is parabolic, the envelope contains just one surface G being the offset of the single focal surface F. In case of an elliptic congruence there exists no real envelope.

Theorem 6.3 *The envelope of a two-parameter family of congruent cylinders of revolution, whose*

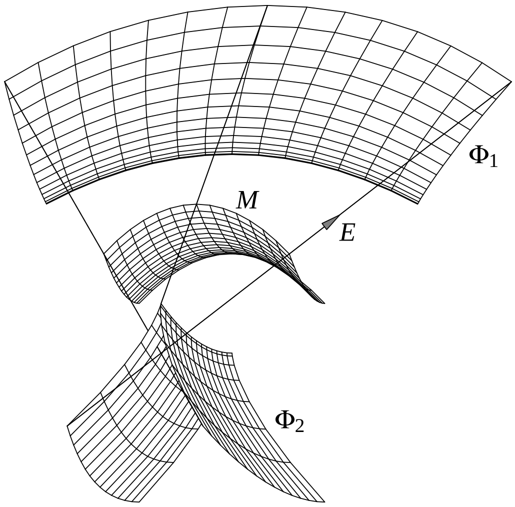

Figure 17: Focal surfaces of the line congruence formed by the normals of a surface

axes lie in a hyperbolic (parabolic?) congruence of lines, consists of the offset surfaces of the focal surfaces of the congruence.

A Bézier representation of a rational line congruence is obtained by taking a trivariate tensor product Bézier representation

$$P(u,v,w) = \sum_{k=0}^{1}\sum_{i=0}^{m}\sum_{j=0}^{n} B_k^1(w)B_i^m(u)B_j^n(v)b_{ijk},$$

which is linear in the parameter w. Evaluating P at $w=0$ gives the surface M, evaluation at $w=1$ gives $M+E$, which are both (m,n)-tensor product surfaces. Bézier representations of a two-parametric family of congruent cylinders can be found in [61].

Remark: Since parameter values w_i are computed as solutions of a quadratic equation, the focal surfaces are in general not rational. It seems to be quite difficult to study rational line congruences with rationally parametrizeable focal surfaces, whose offset surfaces are also rationally parametrizeable.

6.9 Geometrical optics

Geometrical optics has a close relation to line congruences and sphere geometry. The geometric properties of light rays are found by studying geometric properties of two-parametric families of lines (line congruences). In particular, those families are important which intersect a given surface Φ_1 orthogonally. In this context, the theorem of Malus-Dupin is of great interest which states the following.

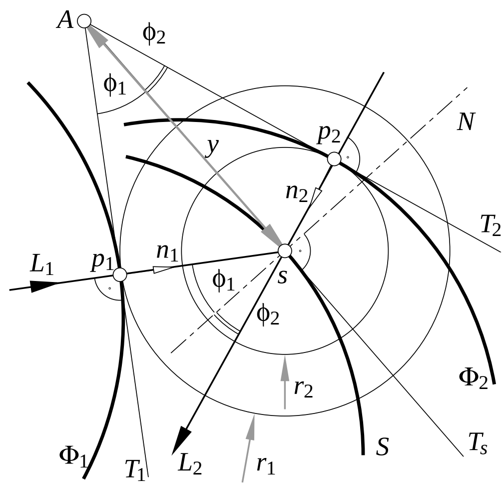

Figure 18: Theorem of Malus-Dupin

Theorem 6.4 *(Malus-Dupin) Given a two-parameter family of lines $\mathcal{L}_1$ being perpendicular to a surface Φ_1. If this family of lines is refracted at an arbitrary smooth surface S in such a way that this refraction satisfies Snellius law, then the refracted two-parameter family of lines $\mathcal{L}_2$ is again perpendicular to a surface Φ_2.*

There are several methods to prove this theorem. We will give a sphere-geometric proof, since it is closely related to envelopes. Actually, the proof is done by a careful study of figure 18. Let L_1 be a line which intersects Φ_1 perpendicularly in a point p_1. This line shall be refracted at $s \in S$ according to Snellius law. It says that the sin-values of the angles ϕ_1, ϕ_2 of incoming and refracted rays L_1, L_2 formed with the normal N at $s \in S$ are proportional,

$$\sin\phi_1 = k\sin\phi_2,\ k = \text{const.}$$

Let S_1 be a sphere, centered at s and tangent to Φ_1 at p_1. Its radius is r_1. The tangent planes T_1, T_s of Φ_1 and S intersect in a line A. The angle $\phi_1 = \angle(L_1, N)$ equals the angle $\angle(T_1, T_s)$. We centre a second sphere at s whose radius is $r_2 = r_1/k$. Further, let T_2 be a plane which is tangent to S_2 and passes through A. The angles ϕ_i between L_i and N also occur between the tangent planes T_i and T_s. Thus it follows that

$$\frac{\sin\phi_1}{\sin\phi_2} = \frac{r_1}{r_2} = k.$$

Since there are two planes which are tangent to S_2 and pass through A, we pre-arrange that positive k shall define equally oriented angles phi_1, ϕ_2. Figure 18 shows an example for $k < 0$. So, choosing this plane T_2 determines the line L_2 carrying the refracted ray. It intersects perpendicularly S_2 at the point p_2.

The surface Φ_1 is enveloped by the two-parameter family of spheres S_1 and the family S_2 envelopes a surface Φ_2, being perpendicular to the lines L_2. Φ_2 is called *anticaustic of refraction* with respect to the illumination perpendicular to Φ_1 (and vice versa). $\square$

Additionally we note that all offset surfaces of Φ_2 are also perpendicular to lines $L_2 \in \mathcal{L}_2$. From the previous section we know that both families of lines $\mathcal{L}_1$ and $\mathcal{L}_2$ may envelope focal surfaces. Actually it can be proved that for line congruences which are perpendicular to a surface, equation (40) always possesses real zeros which imply reality of the focal surfaces.

For some more information and relation of geometric optics and sphere geometry we like to refer to [51].

7 Envelopes in geometric tolerancing and error analysis in CAD constructions

Basic concepts without analytic details; just brief description of linear constructions (including Bezier).

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Envelopes – Computational Theory and Applications

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Based on classical geometric concepts we discuss the computational geometry of envelopes. The main focus is on envelopes of planes and natural quadrics. There, it turns out that projective duality and sphere geometry are powerful tools for developing efficient algorithms. The general concepts are illustrated at hand of applications in geometric modeling. Those include the design and NURBS representation of developable surfaces, canal surfaces and offsets. Moreover, we present applications in NC machining, geometrical optics, geometric tolerancing and error analysis in CAD constructions.
Keywords: envelope, duality, sphere geometry, NURBS surface, developable surface, canal surface, geometric tolerancing.