SUBDIVISION ALGORITHMS FOR MOTION DESIGN BASED ON HOMOLOGOUS POINTS

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Abstract We present two algorithms for the interpolation of given positions of a moving body by a smooth and fair motion, such that chosen feature points of the moving system run on smooth and fair paths. We outline algorithms which rely on known interpolatory variational subdivision for curves and on registration techniques from Computer Vision. For the numerical solution of the arising optimization problems we propose a geometric method which is based on instantaneous kinematics.

Keywords: motion design, interpolation, variational subdivision, kinematics, registration

1. Introduction

An important topic in motion design are *smooth* motions with interpolation or approximation constraints. Motions of that type, which make use of NURBS techniques, have been investigated in several publications (see e.g. Jüttler and Wagner, 1996; Röschel, 1998). Recently, Hyun et al., 2001 constructed affine rational spline motions with minimal distortion. For motion design based on the quaternion representation of the spherical component and nonlinear extensions of spline constructions in affine spaces to the sphere see Fang et al., 1998.

The present paper provides alternatives for motion design using *subdivision algorithms*. This has several advantages over previous approaches: It is easier to deal with *variational motion design*, it leads in a natural way to a *multiresolution representation* for the designed motion, and it can be extended to motion design in the *presence of obstacles*. *Homol-*

ogous points are the different locations of a single feature point as the moving body takes several positions Our work emphasizes optimality with respect to some fairness criterion based on trajectories of chosen feature points. As a rigid body motion is a curve in the Lie group SE(3), variational motion design could be based on the fairness of that curve (see Park and Ravani, 1997). However, for the applications we have in mind, we believe that it is most important that the body to be moved behaves well.

We present two *subdivision algorithms based on homologous points* for the design of fair and smooth motions. Consider the given positions of the moving body as a coarse approximation of the motion we want to design. Then we refine this motion by iteratively inserting more and more intermediate positions. The resulting dense set of discrete positions of the moving body might be sufficient for the application in mind, or it can easily be interpolated with other motion design techniques without caring about the fairness of the motion anymore. The main difference of the two algorithms is how intermediate positions of the moving body are inserted to refine the motion.

The paper is organized as follows. In Sec. 2, 3, 4 we present the main ingredients for the two motion design algorithms, *interpolatory variational subdivision*, *instantaneous kinematics* and *registration*. In Sec. 5 we present the algorithm *subdivision for motion design I*, which is actually a transfer principle from curve to motion design. A related, but different approach, is the algorithm *subdivision for motion design II*, presented in Sec. 6. We conclude the paper in Sec. 7 with an outlook on possible extensions of the presented algorithms.

2. Interpolatory variational subdivision

Variational subdivision has been introduced by Kobbelt, 1996 and involves the minimization of some energy functional to control the fairness of the curves (or surfaces) that are constructed.

We start with an *open* polygon and in each iteration step we insert new points $\mathbf{q}_i, i = 1, \ldots, N-1$ between the points $\mathbf{p}_i, i = 1, \ldots, N$ of the polygon to be refined such that the second forward differences are minimized (see Fig. 1, left). The position of the new points is thus found by minimizing the objective function

$$F(\mathbf{q}_1, \ldots, \mathbf{q}_{N-1}) = \sum_{i=1}^{N-1} \|\mathbf{m}_i - \mathbf{q}_i\|^2 + \sum_{i=2}^{N-1} \|\mathbf{n}_i - \mathbf{p}_i\|^2 \qquad (1)$$

where $\mathbf{m}_i = (\mathbf{p}_i + \mathbf{p}_{i+1})/2$ and $\mathbf{n}_i = (\mathbf{q}_{i-1} + \mathbf{q}_i)/2$. F is *quadratic* in the unknowns $\mathbf{q}_i \in \mathbb{R}^d$ and thus the minimization of F leads to the following

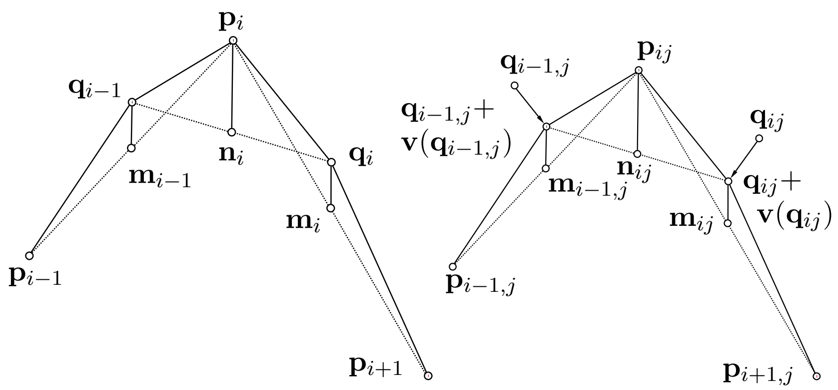

Figure 1. Squared distances that are minimized in Kobbelt's interpolatory variational subdivision (left) and in the simultaneous variational subdivision on the K sequences of homologous points of motion design with subdivision II (right).

tridiagonal linear system of equations, which can be solved efficiently using e.g. sparse matrix techniques,

$$\begin{pmatrix} 5 & 1 & & & \\ 1 & 6 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 6 & 1 \\ & & & 1 & 5 \end{pmatrix} \begin{pmatrix} \mathbf{q_1} \\ \mathbf{q_2} \\ \vdots \\ \vdots \\ \mathbf{q_{N-2}} \\ \mathbf{q_{N-1}} \end{pmatrix} = \begin{pmatrix} 2\mathbf{p}_1 + 4\mathbf{p}_2 \\ 4(\mathbf{p}_2 + \mathbf{p}_3) \\ \vdots \\ \vdots \\ 4(\mathbf{p}_{N-2} + \mathbf{p}_{N-1}) \\ 4\mathbf{p}_{N-1} + 2\mathbf{p}_N \end{pmatrix} \tag{2}$$

This refinement scheme is global, i.e., every new point depends on all points of the polygon to be refined. Interpolation is guaranteed since the old points belong to the newly calculated finer version. Kobbelt, 1996 has shown that this scheme generates at least C^2 curves. The scheme corresponds to an underlying *uniform* parametrization and can be extended to a *non-uniform* parametrization, cf. Kobbelt and Schröder, 1998.

3. Instantaneous kinematics

Consider a smooth one-parameter rigid body motion in Euclidean 3-space. Introducing Cartesian coordinate systems in the moving system Σ and in the fixed system Σ_0, the time dependent position $\mathbf{x}_0(t)$ of a point $\mathbf{x} \in \Sigma$ in the fixed system is given by

$$\mathbf{x}_0(t) = \mathbf{a}(t) + M(t) \cdot \mathbf{x}. \tag{3}$$

Here, the time dependent orthogonal matrix $M(t)$ represents the spherical component of the motion, and $\mathbf{a}(t)$ describes the trajectory of the origin of the moving system. All arising functions shall be C^1. By differentiation we get the velocity vectors. It is well-known that the velocity vector field is linear at any time instant and that it has the form

$$\mathbf{v}(\mathbf{x}) = \bar{\mathbf{c}} + \mathbf{c} \times \mathbf{x}, \tag{4}$$

where $\bar{\mathbf{c}}$ represents the velocity vector of the origin, and $\mathbf{c}$ is the so-called Darboux vector (vector of angular velocity). Only very special one-parameter motions have a time-independent velocity vector field. These motions are a *translation* with constant velocity (if $\mathbf{c} = \mathbf{0}$), a uniform *rotation* about an axis (if $\mathbf{c} \cdot \bar{\mathbf{c}} = 0$) and a uniform *helical motion* (if $\mathbf{c} \cdot \bar{\mathbf{c}} \neq 0$). The most general case is that of a uniform helical motion, which is the superposition of a rotation with constant angular velocity about an axis A and a translation with constant velocity parallel to A. If the moving body rotates about an angle α, the translation distance is $p \cdot \alpha$. We compute axis A, pitch p and angular velocity ω from $\mathbf{c}, \bar{\mathbf{c}}$ by

$$\mathbf{a} = \frac{\mathbf{c}}{\|\mathbf{c}\|}, \qquad \bar{\mathbf{a}} = \frac{\bar{\mathbf{c}} - p\mathbf{c}}{\|\mathbf{c}\|}, \qquad p = \frac{\mathbf{c} \cdot \bar{\mathbf{c}}}{\mathbf{c}^2}, \qquad \omega = \|\mathbf{c}\|, \tag{5}$$

where the axis A is parallel to $\mathbf{a}$ and $\bar{\mathbf{a}} = \mathbf{p} \times \mathbf{a}$ is the *moment vector* of A, which is independent of the choice of the point $\mathbf{p}$ on A (see e.g. Pottmann and Wallner, 2001).

4. Registration

Consider two clouds X, Y of *corresponding* points $\mathbf{x}_i,\ i = 1, \ldots, N$ and $\mathbf{y}_i,\ i = 1, \ldots, N$, respectively. The problem of applying to one cloud, say X, a Euclidean motion m which brings each $\mathbf{x}_i$ as close as possible to $\mathbf{y_i}$ is well studied. Formulating it in a least squares sense, i.e., minimizing

$$F = \sum_{i=1}^{N} \|m(\mathbf{x}_i) - \mathbf{y}_i\|^2, \tag{6}$$

the solution amounts to an eigenvalue problem (see e.g. Horn, 1987).

Although there is a direct way to solve the minimization of F in Eq. 6, we describe the following *iterative* algorithm which is based on instantaneous kinematics. It has the advantage that it can be applied directly to the simultaneous alignment of more than two clouds of corresponding points, which we use in the algorithm motion design with subdivision II (see Sec. 6). To each point $\mathbf{x}_i$ attach the velocity vector $\mathbf{v}(\mathbf{x}_i) = \bar{\mathbf{c}} + \mathbf{c} \times \mathbf{x}_i$ of the still unknown velocity vector field of an instantaneous helical motion which can be described by a pair $\mathbf{C} = (\mathbf{c}, \bar{\mathbf{c}}) \in \mathbb{R}^6$. The objective function we minimize in each iteration step,

$$F(\mathbf{C}) = \sum_i \|\mathbf{x}_i + \mathbf{v}(\mathbf{x}_i) - \mathbf{y}_i\|^2, \tag{7}$$

is quadratic in $\mathbf{c}, \bar{\mathbf{c}}$ and thus the solution can be computed using a linear system of equations.

Note that the transformation which maps $\mathbf{x}_i$ to $\mathbf{x}_i + \mathbf{v}(\mathbf{x}_i)$ is an affine map and not a Euclidean transformation. Therefore we use the uniform helical motion that is uniquely determined by the velocity vector field represented by the pair $(\mathbf{c}, \bar{\mathbf{c}})$. Applying a rotation about the axis of the helical motion through an angle of $\alpha = \arctan(\|\mathbf{c}\|)$ and a translation parallel to this axis by the distance $p \cdot \alpha$ brings the points close to the tips $\mathbf{x}_i + \mathbf{v}(\mathbf{x}_i)$ of the velocity vectors used in Eq. 7.

The algorithm iteratively minimizes F in Eq. 7 and updates the position of X with the corresponding helical motion.

5. Motion design with subdivision I

Given are N_0 positions $\Sigma_1, \ldots, \Sigma_{N_0}$ of a moving body Σ which are to be interpolated by a fair and smooth motion. We take a sample of $K \geq 4$ feature points $\mathbf{p}_1, \ldots, \mathbf{p}_K$ from the moving body Σ. In the m-th iteration step we have $N_m = 2^{m-1}(N_0 - 1) + 1$ positions $\Sigma_1, \ldots, \Sigma_{N_m}$ and we insert $N_m - 1$ new positions, one between each two adjacent positions from the previous iteration step. This is done by repeating procedure 1 and 2 outlined below.

Procedure 1. Insert (affinely) distorted copies of the moving body: Apply interpolatory variational subdivision separately to the K sequences of homologous points $\mathbf{p}_{1j}, \ldots, \mathbf{p}_{N_m j}, j = 1, \ldots, K$ (see Fig. 2, left). This gives $N_m - 1$ initial intermediate *affine* copies Σ'_i of the moving body (see Fig. 3, left), which follows from the *linearity* of variational subdivision (Eq. 1). Therefore, it would be sufficient to run the variational subdivision only with four non-coplanar feature points of the moving body. From that one could compute affine maps from an initial position to the inserted affine copies, and use these affine maps to compute the remaining homologous locations of the other $K - 4$ feature points.

Procedure 2. Replace distorted copies of the moving body with Euclidean ones using registration: To each affinely distorted intermediate position Σ'_i we individually register (see Sec. 4) the moving body Σ to find an intermediate position resulting from a rigid body motion. These Euclidean copies of Σ together with those of the previous step are the input to the next iteration step (see Fig. 3, right).

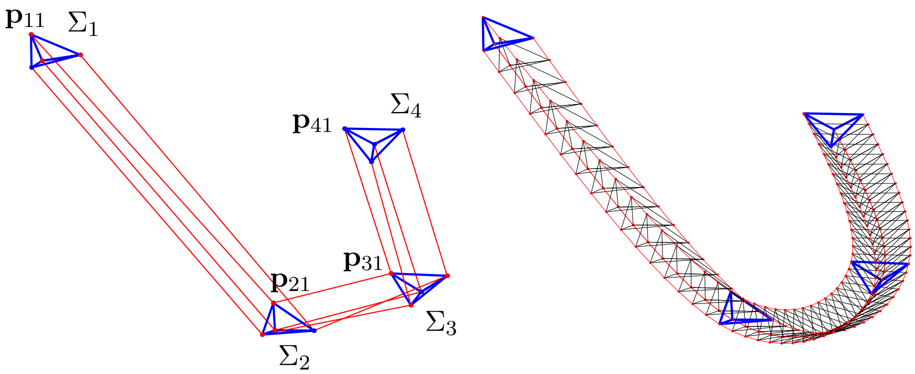

Figure 2. Motion design with subdivision I: The input to the algorithm are four given positions of a moving body and the connecting polygons of the four sequences of homologous points (left). The output after four iterations consists of 49 discrete positions of the moving body (right).

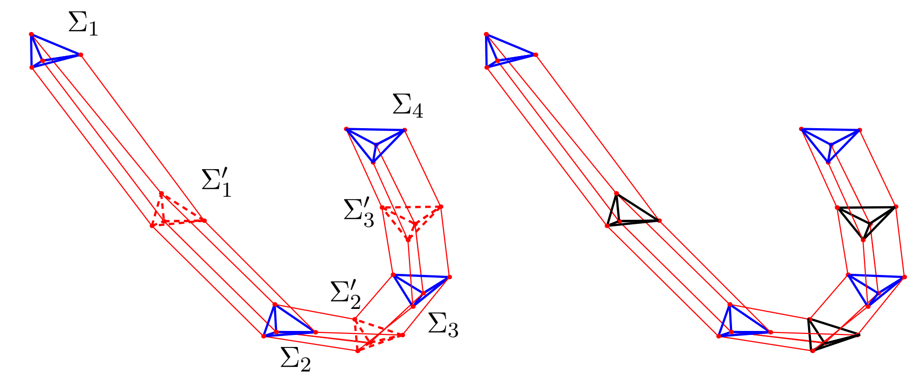

Figure 3. Motion design with subdivision I: To the input we apply procedure 1 of the algorithm (left) and then procedure 2 (right).

Procedure 1 is a global operation that works separately on each sequence of homologous points. It controls the fairness and smoothness of the path of each feature point. Using only procedure 1 we would get an *affine* motion. Therefore, in procedure 2 we use registration to find the best fit, in the least squares sense, of the rigid moving body to each affinely distorted intermediate position. Procedure 2 is a local operation that works on every intermediate position separately.

We know that the variational subdivision scheme produces at least C^2 curves (for a proof we refer to Kobbelt, 1996). Numerical tests give

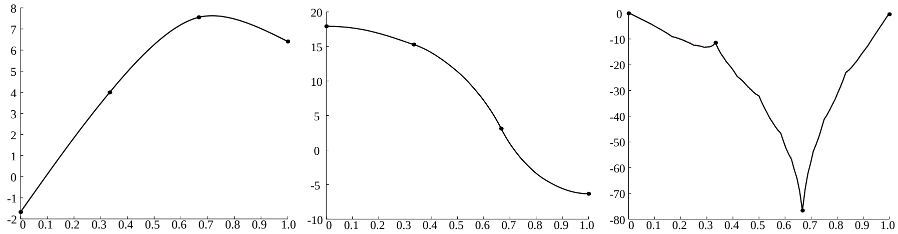

Figure 4. Motion design with subdivision I: Plot of the functions $x(t)$, $x'(t)$, $x''(t)$ where $x(t)$ is the function that assumes over *uniform* parameter values from the interval $[0, 1]$ the x-values of the path of one feature point of the motion.

evidence that using Kobbelt's variational subdivision together with the registration step results in C^2 paths of the feature points and in a C^2 motion (Fig. 4).

The computational cost of the algorithm in the m-th iteration step is the following. First we have to solve K tridiagonal systems of size $N_m - 1$ for the refinement of the paths of the feature points. Then, if we use the iterative registration described by Eq. 7, we need to solve $M(N_m - 1)$ linear systems for 6 unknowns. With a good starting position (centroid of distorted body and moving body coincide), the number of iterations M needed in the registration step is usually 3.

To reduce the computational cost one may stop the algorithm described above after 4-5 iterations, since then a variational scheme is not necessary anymore. One may switch to a local subdivision scheme (see e.g. Warren and Weimer, 2001) and insert as many further positions as needed.

Remarks. Any linear subdivision scheme applied to sequences of homologous points will result in affine intermediate positions. Insertion of intermediate points may also be done with splines, i.e. cubic C^2 splines which are evaluated at intermediate positions. This is one way to address the time distribution of the motion. Another way would be the use of the subdivision scheme corresponding to a *non-uniform* parametrization, see Kobbelt and Schröder, 1998. We may also use nonlinear schemes, but then we do no longer get affine copies and thus have to apply subdivision to *all* sequences of homologous points. In the subsequent registration step, it does not matter whether we have an affine copy or not. But in any case, one has to use all K feature points for the registration.

6. Motion design with subdivision II

Similar to the first method we use K chosen feature points $\mathbf{p}_1, ..., \mathbf{p}_K$ from the moving body Σ. In the m-th iteration step we consider the homologous points $\mathbf{p}_{ij}, i = 1, \ldots, N_m, j = 1, \ldots, K$. Our calculations also use the *centroid* (center of mass) $\mathbf{s}$ of the moving body Σ. In this second algorithm we first insert initial Euclidean copies Σ_i' between each pair of adjacent positions Σ_i and Σ_{i+1}. Then we reposition $\Sigma_1', \ldots, \Sigma_{N_m-1}'$ *simultaneously*. Thereby, the fairness of the result is envoked by using a simultaneous variational subdivision on the K sequences of homologous points. The iterative algorithm repeats the following two procedures.

Procedure 1. Insert intermediate Euclidean copies of the moving body: First, we apply to the homologous positions $\mathbf{s}_1, \ldots, \mathbf{s}_{N_m}$ of the *centroid* $\mathbf{s}$ one iteration step of the interpolatory variational subdivision to obtain intermediate positions $\mathbf{s}_1', \ldots, \mathbf{s}_{N_m-1}'$. Second, we compute the angle ϕ_i of the rotational part A_i of the unique helical motion $\mathbf{x} \mapsto A_i\mathbf{x} + a_i$ that transforms Σ_i to the adjacent system Σ_{i+1}, for $i = 1, \ldots, N_m - 1$. The initial intermediate position Σ_i' between Σ_i and Σ_{i+1} is chosen such that its centroid is $\mathbf{s}_i'$ and a rotation through $\frac{\phi_i}{2}$ is applied to Σ_i (see Fig. 5, left).

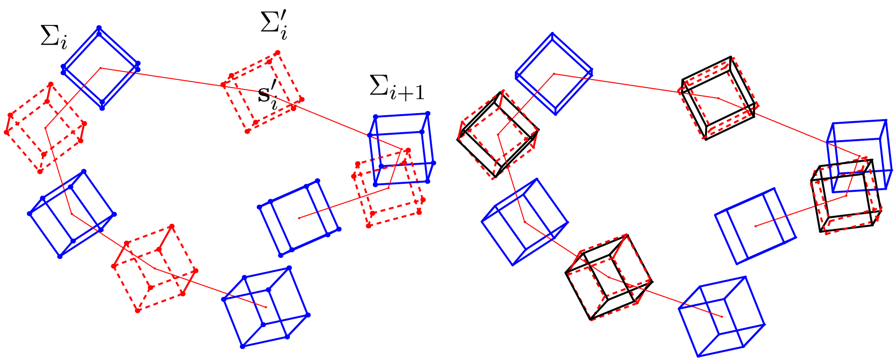

Figure 5. Motion design with subdivision II. The result after procedure 1 (left) and then procedure 2 (right) have been applied to the input data once.

Procedure 2. Simultaneously reposition the inserted Euclidean copies with respect to a fairness criterion: For the simultaneous optimization (with respect to fairness) of the intermediate positions we use linearized motions (velocity fields). To the feature points in each single position Σ_i' we attach vectors belonging to the linear velocity vector field of an instantaneous helical motion (see Eq. 4) described by the pair $(\mathbf{c}_i, \bar{\mathbf{c}}_i)$. The velocity vectors are used for first order estimates of the new positions. Simultaneous variational subdivision on the K sequences of homologous points amounts to the minimization of the following objective function

$$F(\mathbf{C}) = \sum_{j=1}^{K} \left(\sum_{i=1}^{N_m-1} \|\mathbf{m}_{ij} - \mathbf{q}_{ij} - \mathbf{v}(\mathbf{q}_{ij})\|^2 + \sum_{i=2}^{N_m-1} \|\mathbf{n}_{ij} - \mathbf{p}_{ij}\|^2 \right) \quad (8)$$

where $\mathbf{C}$ is the $(2N_m - 2)$-tupel $(\mathbf{c}_1, \bar{\mathbf{c}}_1, \ldots, \mathbf{c}_{N_m-1}, \bar{\mathbf{c}}_{N_m-1})$,

$$\mathbf{m}_{ij} = (\mathbf{p}_{ij} + \mathbf{p}_{i+1,j})/2, \quad \mathbf{v}(\mathbf{q}_{ij}) = \bar{\mathbf{c}}_i + \mathbf{c}_i \times \mathbf{q}_{ij} \quad (9)$$

$$\mathbf{n}_{ij} = [\mathbf{q}_{i-1,j} + \mathbf{v}(\mathbf{q}_{i-1,j}) + \mathbf{q}_{ij} + \mathbf{v}(\mathbf{q}_{ij})]/2, \quad (10)$$

and $\mathbf{q}_{ij}, i = 1, \ldots, N_m - 1, j = 1, \ldots, K$ are the initial intermediate positions of the chosen feature points (see e.g. Fig. 1, right). F is quadratic in the $2N_m - 2$ unknowns $\mathbf{c}_1, \bar{\mathbf{c}}_1, \ldots, \mathbf{c}_{N_m-1}, \bar{\mathbf{c}}_{N_m-1}$, the minimization leads to the solution of a banded linear system of equations which can be solved efficiently.

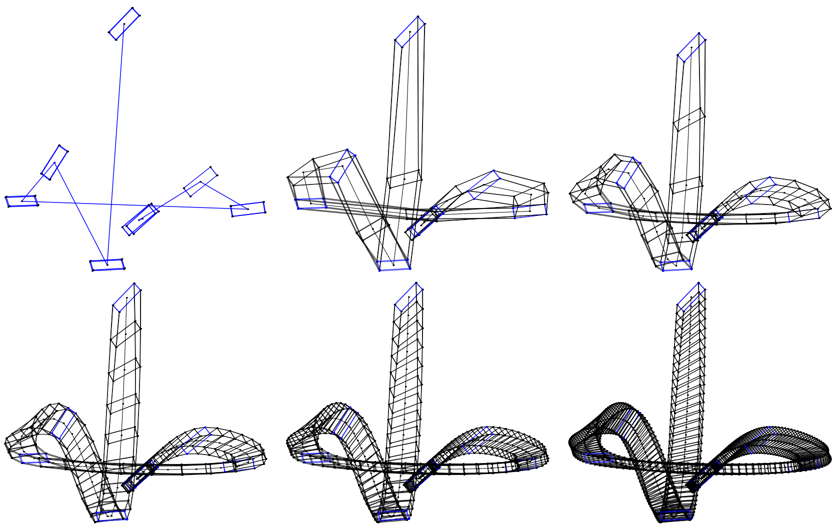

Figure 6. Example of motion design with subdivision II. Shown are the input to the algorithm and the first to fifth iteration step.

Now the initial intermediate positions Σ'_i are corrected using the helical motion that is uniquely determined by the velocity vector field represented by the pair $(\mathbf{c}_i, \bar{\mathbf{c}}_i)$. This means that we apply to Σ'_i a rotation through an angle of $\alpha_i = \arctan(\|\mathbf{c}_i\|)$ and a translation parallel to the axis $(\mathbf{a}_i, \bar{\mathbf{a}}_i)$ by the distance $p_i \cdot \alpha_i$. It brings the points $\mathbf{q}_{ij}$ close to the tips $\mathbf{q}_{ij} + \mathbf{v}(\mathbf{q}_{ij})$ of the velocity vectors used in Eq. 8.

Remarks. The smoothness analysis of the resulting motion is analogous to the one performed in Sec. 5. Numerical tests again suggest that the resulting motion is C^2. The computational cost of the algorithm in the m-th iteration step is the following. We have $N_m = 2^{m-1}(N_0 - 1) + 1$ positions and insert $N_m - 1$. Thus, first we have to solve one linear system of equations of size $N_m - 1$ to refine the sequence of homologous positions of the centroid $\mathbf{s}$ of the moving body. Then, for the simultaneous repositioning of the inserted Euclidean copies of the moving body we have to solve a banded linear system of $6(N_m - 1)$ equations.

7. Conclusions

We have presented two subdivision algorithms for motion design that generate C^2 motions with respect to some optimality criterion. Our approach deals with variational motion design. Both algorithms can be modified for the design of motions approximating given input positions. Algorithm II is extendable to motion design in the presence of obstacles.

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