APPROXIMATION IN LINE SPACE – APPLICATIONS IN ROBOT KINEMATICS AND SURFACE RECONSTRUCTION

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Abstract. Combining classical line geometry with techniques from numerical approximation, we develop algorithms for approximation in line space. In particular, linear complexes, linear congruences and reguli are fitted to given sets of lines or line segments. The results are applied to computationally robust detection of special robot configurations and to reconstruction of fundamental surface shapes from scattered points.

1. Introduction and fundamentals

The close relation between spatial kinematics and line geometry is one of the classical fundamentals of kinematic geometry (Bottema, 1990; Hunt, 1978; Husty *et al.*, 1997). It forms the basis of screw theory (Hunt, 1978) and has found a variety of applications including the characterization of special robot configurations (Hunt, 1986; Husty *et al.*, 1997) and the study of singular positions of parallel manipulators (Merlet, 1992). However, the computational treatment of this beautiful theory seems to have found little attention so far (Ge *et al.*, 1994; Ravani and Wang, 1991). In practical applications several sources for errors (manufacturing, material properties, computing, ...) are hardly avoidable. How do we then check whether a set of lines lies in a special configuration like a linear complex? In fact, the question is whether the lines near their realization on the object, i.e. line segments, are close – within some tolerance – to lines of a linear complex.

This is an approximation or regression problem in line space. Moreover, proximity to special positions may be undesirable anyway. For example, a parallel manipulator may snap into a neighboring configuration near a singular position. Therefore, we will present here initial results on approximation in line space. This study actually arose from a reverse engineering problem, namely the reconstruction of cylinders and surfaces of revolution from scattered points (Pottmann and Randrup, 1997). We will briefly describe this application, along with directions for future research.

Let us introduce our notation and point to a few essential fundamentals. In 3–dimensional real Euclidean space E^3 we use a Cartesian coordinate system and represent a straight line L by a normalized direction vector $\mathbf{l}$, $\|\mathbf{l}\| = 1$, and its moment vector $\bar{\mathbf{l}} := \mathbf{x} \times \mathbf{l}$ with respect to the origin. Here, $\mathbf{x}$ denotes the coordinate vector of an arbitrary point on L. The 6 coordinates of $\mathbf{l}, \bar{\mathbf{l}}$ are the *normalized Plücker coordinates* of L. They satisfy the Plücker relation $\mathbf{l} \cdot \bar{\mathbf{l}} = 0$. Any 6–tupel $(\mathbf{l}, \bar{\mathbf{l}}) \in \mathbb{R}^6$ with $\|\mathbf{l}\| = 1, \mathbf{l} \cdot \bar{\mathbf{l}} = 0$ represents a line in E^3, where $(\mathbf{l}, \bar{\mathbf{l}})$ and $(-\mathbf{l}, -\bar{\mathbf{l}})$ describe the same line.

Dropping the normalization, we can work in the projective extension P^3 of E^3, and represent lines L at infinity by $\mathbf{l} = 0, \bar{\mathbf{l}} \neq 0$. Here, $\bar{\mathbf{l}}$ is a normal vector of planes through L. These homogeneous Plücker coordinates define a mapping from the set $\mathcal{L}$ of lines in P^3 to ordered, homogeneous 6–tupels $\mathsf{L} = (\mathbf{l}, \bar{\mathbf{l}}) \in \mathbb{R}^6 \setminus \{0\}$, which may be interpreted as points in real projective 5–space P^5. We write $\mathsf{L}\mathbb{R}$ to indicate homogeneity and also denote points in P^5 in this way. By the Plücker relation, exactly those points in P^5 are 'Klein images' of lines in P^3, which lie on the *Klein quadric*

$$M_2^4 : \ \mathbf{l} \cdot \bar{\mathbf{l}} = l_1 l_4 + l_2 l_5 + l_3 l_6 = 0. \tag{1}$$

For a spatial *one–parameter motion*, the velocities $\mathbf{v}$ of points $\mathbf{x}$ at an instant t form a linear vector field,

$$\mathbf{v}(\mathbf{x}) = \bar{\mathbf{c}} + \mathbf{c} \times \mathbf{x}. \tag{2}$$

Provided that $(\mathbf{c}, \bar{\mathbf{c}}) \neq (0,0)$ at t, i.e. that the position at t is not stationary, it agrees with that of a screw motion. Its axis $A = (\mathbf{a}, \bar{\mathbf{a}})$ and pitch p may be computed via

$$\mathbf{a} = \frac{\mathbf{c}}{\|\mathbf{c}\|}, \ \bar{\mathbf{a}} = \frac{\bar{\mathbf{c}} - p\mathbf{c}}{\|\mathbf{c}\|}, \ p = \frac{\mathbf{c} \cdot \bar{\mathbf{c}}}{\mathbf{c}^2}. \tag{3}$$

For $p = 0$ we get an instantaneous rotation, and $\mathbf{c} = 0$ $(p = \infty)$ characterizes an instantaneous translation. A fundamental relation to line geometry is the following. At a fixed instant t, the *path normals* $(\mathbf{l}, \bar{\mathbf{l}})$ (lines through points $\mathbf{x}$ normal to $\mathbf{v}(\mathbf{x})$) satisfy

$$0 = \mathbf{v} \cdot \mathbf{l} = \bar{\mathbf{c}} \cdot \mathbf{l} + (\mathbf{c} \times \mathbf{x}) \cdot \mathbf{l} = \bar{\mathbf{c}} \cdot \mathbf{l} + \mathbf{c} \cdot (\mathbf{x} \times \mathbf{l}) = \bar{\mathbf{c}} \cdot \mathbf{l} + \mathbf{c} \cdot \bar{\mathbf{l}}.$$

Provided that $(\mathbf{c},\bar{\mathbf{c}}) \neq (0,0)$ at t, we see that the path normals form a *linear line complex* $\mathcal{C}$ in E^3. It is called *singular* for $p = 0$ and $p = \infty$. We also speak of pitch p and axis A of a linear complex, which are found via (3). Conversely, in view of (2), any linear complex $\mathcal{C}$, defined as solution set of a linear homogeneous equation in Plücker coordinates,

$$\bar{\mathbf{c}} \cdot \mathbf{l} + \mathbf{c} \cdot \bar{\mathbf{l}} = 0. \tag{4}$$

can be obtained as path normal complex. We also see that the Klein image of the lines of a linear complex is a hyperplanar cut of the Klein quadric M_2^4. Pole of the hyperplane with respect to M_2^4 is the point $\mathsf{C}\mathbb{R} = (\mathbf{c},\bar{\mathbf{c}})\mathbb{R} \in P^5$, which is called *extended Klein image* of the linear complex.

Sets of linear complexes, whose extended Klein images form a k–dimensional subspace of P^5 are called k–dimensional spaces of linear complexes. The intersection of all complexes in such a set is called its *carrier*. For $k = 1$, we obtain a *pencil of linear complexes*. The carrier is in general a *linear congruence* of lines. The case $k = 2$ leads to *bundles of linear congruences*, whose carrier is in general a (not necessarily real) *regulus*. The relation to kinematics is the following: At an instant of a k–parameter motion in E^3, the path normal complexes to one–parameter motions through that instant lie in a $(k-1)$–dimensional space of linear complexes. A detailed investigation, expressed in terms of screw theory, may be found in (Hunt, 1978). For more information on line geometry, see e.g. (Hlavaty, 1953; Hoschek, 1971).

2. Approximating linear complexes

In practice, errors in data are often unavoidable, and thus the question arises how to construct a linear complex $\mathcal{C}$, which – in a sense to be specified – best approximates a given set of lines L_i, $i = 1,\ldots,k$. In other words, we are interested in the construction of a *linear complex of regression* to a given set of data lines.

An important input to the solution of the problem is an appropriate measure of the deviation of a given line L from a linear complex $\mathcal{C}$. Let us represent L with *normalized* Plücker coordinates $\mathsf{L} = (\mathbf{l},\bar{\mathbf{l}})$, $\|\mathbf{l}\| = 1$. The linear complex $\mathcal{C}$ shall have equation $\bar{\mathbf{c}} \cdot \mathbf{x} + \mathbf{c} \cdot \bar{\mathbf{x}} = 0$. For the moment, $\mathcal{C}$ shall satisfy $\mathbf{c} \neq 0$ and thus be different from the path normal complex of a translation. According to F. Klein (Klein, 1921) we use the *moment of L with respect to $\mathcal{C}$*,

$$m(L,\mathcal{C}) := \frac{|\bar{\mathbf{c}} \cdot \mathbf{l} + \mathbf{c} \cdot \bar{\mathbf{l}}|}{\|\mathbf{c}\|}. \tag{5}$$

Usually one defines the moment between an oriented line and an oriented linear complex (linear complex with oriented axis) and thus omits the absolute value in the definition. Then, the moment is also interpretable as virtual work between two normalized oriented screws, namely $\mathsf{L}\mathbb{R}^+$ (with pitch 0) and $\mathsf{C}\mathbb{R}^+$. We do not need orientations and thus define the moment as a nonnegative real number.

The moment m has the following geometric interpretation. Pick an arbitrary point $\mathbf{x}$ on L and let r be its distance from the axis A of $\mathcal{C}$. Let $\alpha \in [0, \pi/2]$ be the smallest angle between L and a line of $\mathcal{C}$ through $\mathbf{x}$, which is the angle between L and the path normal plane of $\mathbf{x}$. Then one obtains

$$m(L, \mathcal{C}) = \sqrt{r^2 + p^2} \sin \alpha. \tag{6}$$

Lines with vanishing moment, $m(L, \mathcal{C}) = 0$, are exactly the lines of $\mathcal{C}$. All lines forming the same moment $m \neq 0$ with $\mathcal{C}$ lie in a so–called *cyclic quadratic complex.* This line complex $\mathcal{K}$ possesses cones of revolution as complex cones and circles as planar complex curves (Wunderlich, 1964).

We now want to compute a linear complex $\mathcal{C}$ which is as close as possible to the given lines L_i, $i = 1, \ldots, k$ with normalized Plücker coordinates $\mathsf{L}_i = (\mathbf{l}_i, \bar{\mathbf{l}}_i) = (l_{i1}, \ldots, l_{i6})$. For that we compute $\mathcal{C}$ as minimizer of

$$\sum_{i=1}^{k} m(L_i, \mathcal{X})^2 \tag{7}$$

among all linear complexes $\mathcal{X}$, represented by $\mathsf{X} = (\mathbf{x}, \bar{\mathbf{x}})$. With (5) this is equivalent to minimizing the positive semidefinite quadratic form

$$F(\mathsf{X}) := \sum_{i=1}^{k} (\bar{\mathbf{x}} \cdot \mathbf{l}_i + \mathbf{x} \cdot \bar{\mathbf{l}}_i)^2 =: \mathsf{X}^T \cdot M \cdot \mathsf{X} \tag{8}$$

subject to the normalization condition

$$1 = \|\mathbf{x}\|^2 =: \mathsf{X}^T \cdot D \cdot \mathsf{X}. \tag{9}$$

D is the 6×6 diagonal matrix $(1, 1, 1, 0, 0, 0)$. This is a familiar general eigenvalue problem. Using a Lagrangian multiplier λ, we have to solve the system

$$(M - \lambda D) \cdot \mathsf{X} = 0, \ \mathsf{X}^T \cdot D \cdot \mathsf{X} = 1. \tag{10}$$

Hence, λ must be a root of the equation

$$\det(M - \lambda D) = 0. \tag{11}$$

Because three diagonal elements in D are zero, this is just a cubic equation in λ. For any root λ and corresponding general eigenvector $\mathsf{X} = (\mathbf{x}, \bar{\mathbf{x}})$ (i.e. a solution of $(M - \lambda D) \cdot \mathsf{X} = 0$) with $\|\mathbf{x}\| = 1$, we have

$$F(\mathsf{X}) = \mathsf{X}^T \cdot M \cdot \mathsf{X} = \lambda \mathsf{X}^T \cdot D \cdot \mathsf{X} = \lambda.$$

Therefore, all roots λ are nonnegative and the *solution* C *is a general eigenvector to the smallest general eigenvalue* $\lambda \geq 0$.

The standard deviation of the approximating linear complex $\mathcal{C}$ at the given normals is $\sigma = \sqrt{\lambda/(k-5)}$.

In case of a sufficiently small deviation σ, we use (3) to compute axis A and pitch p of the complex $\mathcal{C}$. Note that the moment m, the deviation σ as well as the pitch p are distances in Euclidean geometry. Their magnitude has to be seen in comparison to the error tolerance. Let us now focus on some important special cases.

Complexes with pitch $p = 0$*:* If p turns out to be very small, one might be interested in a fit of the data lines by a singular linear complex $\mathcal{C}_0$ with pitch $p = 0$. Recall that $\mathcal{C}_0$ consists of all lines which intersect or are parallel to an axis $A_0 \subset E^3$. The simplest way is to take A_0 as axis A of the linear complex $\mathcal{C}$ computed as outlined above.

In a refined algorithm, we can mimimize the sum of squared moments (7) in the set of all linear complexes with $p = 0$. This amounts to the minimization of (8) subject to the condition (9) and

$$0 = \Omega(\mathsf{X}) = \mathbf{x} \cdot \bar{\mathbf{x}} =: \mathsf{X}^T \cdot K \cdot \mathsf{X}. \tag{12}$$

K is the coefficient matrix in the equation of the Klein quadric. With two Lagrangian multipliers λ, μ, we have to solve the system

$$(M - \lambda D - \mu K) \cdot \mathsf{X} = 0,\ \mathsf{X}^T \cdot D \cdot \mathsf{X} = 1,\ \mathsf{X}^T \cdot K \cdot \mathsf{X} = 0. \tag{13}$$

Hence, λ, μ are restricted to the algebraic curve s of order 6 in the (λ, μ)–plane given by $\det(M - \lambda D - \mu K) = 0$. Corresponding to the points of s, we get solutions C of the linear homogeneous system $(M - \lambda D - \mu K) \cdot \mathsf{X} = 0$, which define points $\mathsf{C}\mathbb{R}$ in P^5. They lie on an algebraic curve P, whose order turns out to be ≤ 30. Note that the solutions C of our minimization problem also need to satisfy (12), which expresses the Klein quadric M_2^4 in P^5. In the algebraic sense, there are ≤ 60 intersections of P and M_2^4. Representing an intersection point by a coordinate vector $\mathsf{C} \in \mathbb{R}^6$ normalized by (9), we obtain $F(\mathsf{C}) = \lambda$. Hence, *the minimizer of* F *belongs to the smallest value of* λ *among those pairs* (λ, μ) *that characterize a point of* $P \cap M_2^4$. Because of the high degree of the problem, a further algebraic investigation and a study of degree reductions is not performed. We compute the solution numerically by an iterative algorithm. A good starting point is C_0 as outlined above.

Complexes with pitch $p = \infty$*:* Linear complexes $\mathcal{C}$ with $p = \infty$ have so far been excluded. There, $\mathbf{c} = 0$, and the complex consists of lines in E^3 which are orthogonal to the vector $\bar{\mathbf{c}}$. The deviation of a line L from such a complex can simply be taken as cosine of the angle between L and $\bar{\mathbf{c}}$. Thus, we now minimize

$$\sum_{i=1}^{k} (\bar{\mathbf{x}} \cdot \mathbf{l}_i)^2 \tag{14}$$

over all unit vectors $\bar{\mathbf{x}} \in \mathbb{R}^3$. This is an eigenvalue problem in $\mathbb{R}^3$. Note that one might not know in advance whether an approximation of the given data lines L_i with this special type of a singular linear complex (axis at infinity) would be possible. If it is possible, the general algorithm as outlined above will cause numerical problems since all coefficients in (11) will be close to zero. There are two simple ways to overcome this problem. One can either check for a complex with $p = \infty$ first or run at first the general algorithm with another normalization condition, namely $\mathsf{X}^2 = 1$. The latter case is equivalent to setting D as unit matrix. Note, however, that this leads to a characteristic equation (11) of degree 6.

Families of solution complexes: For two small eigenvalues λ_1, λ_2, we obtain two nearly equally good solution complexes $\mathcal{C}_1, \mathcal{C}_2$. Since the given lines are close to both complexes, they are in fact close to all complexes of a pencil, spanned by $\mathcal{C}_1, \mathcal{C}_2$. This means that the given lines may be well approximated by the carrier of a pencil, which is in general a linear congruence. Analogously, three small eigenvalues $\lambda_1, \lambda_2, \lambda_3$ define three linear complexes which span a bundle. The given data lines are close to the carrier of the bundle, which is a regulus, in general.

So far, we did not deal with an important aspect: In applications, the deviation of a line L from some set is often just essential in a certain domain of interest. In other words, one is actually interested in the *deviation of a line segment* $\bar{L} \subset L$. We will now show how to modify the general algorithm above to take care of this requirement.

At first, we need an appropriate definition for the distance of a line segment $\bar{L}$ to a linear complex $\mathcal{C}$. Let $\mathbf{a}$ and $\mathbf{b}$ be the boundary points of $\bar{L}$. All lines of $\mathcal{C} : \bar{\mathbf{c}} \cdot \mathbf{l} + \mathbf{c} \cdot \bar{\mathbf{l}} = 0$ passing trough $\mathbf{a}$ lie in a plane with normal vector $\mathbf{n}_a = \bar{\mathbf{c}} + \mathbf{c} \times \mathbf{a}$. Using (non normalized) Plücker coordinates $(\mathbf{l}, \bar{\mathbf{l}})$ of L with $\mathbf{l} = \mathbf{b} - \mathbf{a}$ and inserting $\mathbf{n}_a$, the distance d_b of $\mathbf{b}$ to this plane is

$$d_b = \frac{|\mathbf{n}_a \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}_a\|} = \frac{\bar{\mathbf{c}} \cdot \mathbf{l} + \mathbf{c} \cdot \bar{\mathbf{l}}}{\|\bar{\mathbf{c}} + \mathbf{c} \times \mathbf{a}\|}.$$

An analogous expression we obtain for the distance d_a of $\mathbf{a}$ to the plane of complex lines through $\mathbf{b}$. A useful deviation measure between $\bar{L}$ and $\mathcal{C}$ is now defined via

$$d^2(\bar{L}, \mathcal{C}) = d_a^2 + d_b^2 = (\bar{\mathbf{c}} \cdot \mathbf{l} + \mathbf{c} \cdot \bar{\mathbf{l}})^2 (\frac{1}{v_a^2} + \frac{1}{v_b^2}), \tag{15}$$

with

$$v_a = \|\bar{\mathbf{c}} + \mathbf{c} \times \mathbf{a}\|, \ v_b = \|\bar{\mathbf{c}} + \mathbf{c} \times \mathbf{b}\|. \tag{16}$$

Note that v_a is the norm of the velocity vector $\mathbf{n}_a$ of point $\mathbf{a}$, i.e. the velocity of $\mathbf{a}$ under the helical motion whose path normals form $\mathcal{C}$. With p as pitch and r_a as distance of $\mathbf{a}$ from the axis it is also expressed by $v_a = \sqrt{r_a^2 + p^2}$.

Assume, we are given k lines $L_1, \dots, L_k$. On each of these lines we prescribe points $\mathbf{a}_i, \mathbf{b}_i$ bounding the line segment $\bar{L}_i \subset L_i$ of interest. An approximating complex $\mathcal{C}$ to the line segments can now be defined as minimizer of

$$\sum_{i=1}^{k} d^2(\bar{L}_i, \mathcal{X}) = \sum_{i=1}^{k} (\frac{1}{v_{ai}^2} + \frac{1}{v_{bi}^2})(\bar{\mathbf{x}} \cdot \mathbf{l}_i + \mathbf{x} \cdot \bar{\mathbf{l}}_i)^2 = \sum_{i=1}^{k} w_i (\bar{\mathbf{x}} \cdot \mathbf{l}_i + \mathbf{x} \cdot \bar{\mathbf{l}}_i)^2. \tag{17}$$

Here, v_{ai}, v_{bi} are the velocities of $\mathbf{a}_i, \mathbf{b}_i$ and $\mathbf{l}_i = \mathbf{b}_i - \mathbf{a}_i$. The solution can be computed with a *weight iteration.* In each step, the weighted sum (17) is minimized under the normalization condition $\|\mathbf{x}\| = 1$ (or $\|\bar{\mathbf{x}}\| = 1$, if a solution complex with a very large pitch is expected). This is a general eigenvalue problem as discussed above. The weights are taken as $w_i = 1/v_{ai}^2 + 1/v_{bi}^2$, where v_{ai}, v_{bi} are the velocities of points $\mathbf{a}_i$ and $\mathbf{b}_i$ for the previous solution. For the initial solution, all weights are set to 1. The iteration is stopped after the change of weights falls below a given threshold. Problems are caused by points $\mathbf{a}_i$ or $\mathbf{b}_i$ with small velocities. We neglect them, since the computation of their path normal planes is not robust.

3. Approximation of surfaces or scattered points by rotational or helical surfaces

Applying the concepts discussed so far, we look at a problem that arises in *reverse engineering.* Whereas engineering uses CAD/CAM systems to create real parts, reverse engineering transforms a real part into a computer model. The surface of a part may consist of different surface types like planes, spheres, cones and cylinders of revolution and tori, or also more general surfaces of revolution, general cylinders, helical surfaces or general freeform surfaces. Both a CAD representation and a manufacturing of the part requires the recognition of simple surface types.

The approach is based on the following well–known result. *The normals of a surface of differentiability class C^1 lie in a linear complex if and only if the surface is (part of) a cylinder, a surface of revolution or a helical surface. A C^2 surface all whose normals belong to two different linear complexes, is (part of) a plane, a sphere or a cylinder of revolution.* To approximate a set of scattered data points (or a given surface Φ^0) with a helical surface Φ or one of its limit forms (surface of revolution, cylinder), we first compute the *generating motion* of the approximant as follows.

Let $N_i,\ i = 1, \dots, k$ be estimated or exact surface normals at data points $\mathbf{d}_i,\ i = 1, \dots, k$ (or points of Φ^0, respectively).

We have to find an approximating linear complex $\mathcal{C}$ to the normals N_i as outlined in section 2. In case of a small deviation σ, we use (3) to compute axis and pitch p of the generating motion and get important information for *type recognition.*

For a very small value of p, we may approximate with a surface of revolution and compute a fit with a linear complex with pitch $p = 0$. We also have to take care of the special case of a complex with $p = \infty$ that belongs to a cylinder surface and use the ideas presented for this case.

The surface reconstruction itself can be computed by projecting data points $\mathbf{d}_i,\ i = 1, \dots, k$ with help of their trajectories into an appropriate plane π, depending on the type of the generating motion. This generates a point set in π which will be approximated by a curve c. Finally, moving c with help of the determined motion generates the approximating surface. A detailed discussion can be found in (Pottmann and Randrup, 1997).

Future research includes the segmentation of range data into regions that can be fitted well with fundamental shapes as well as region growing techniques (Leonardis *et al.*, 1995; Sapidis and Besl, 1995).

4. On the stability of a parallel manipulator's position

Let us consider a *parallel manipulator*, where the moving system Σ^0 and the fixed base system Σ are connected by $k \geq 6$ legs, represented by lines $L_i,\ i = 1, \dots, k$. Note that we include redundant legs for $k > 6$. We will now introduce a *stability concept based on approximation in line space.*

Keeping the leg lengths fixed at a position $\Sigma(t)$ of the moving system, we should obtain a rigid system. However, it may be infinitesimally movable and even admit a finite motion. Such positions are referred to as *singular.* Since the legs L_i of fixed length r_i are fixed at points $\mathsf{m}_i \in \Sigma$, they can only move on spheres with centers m_i and radii r_i. Hence, the legs appear as path normals and we immediately arrive at the following well–known result (Husty *et al.*, 1997; Merlet, 1992). *A position of a parallel robot with $k \geq 6$ legs is singular if and only if the positions of the legs lie in a linear complex.* In practice, such positions are avoided. Moreover, if a robot has two different, but in space nearby configurations for the same leg lengths, the robot may snap into the neighboring position, which is clearly undesirable. Such positions are close to a singular position (Husty *et al.*, 1997). Therefore snapping into a neighboring position can be avoided by avoiding *positions where the legs can be fitted well by a linear complex.* Note that we have given the computational tools for checking this. Moreover, various

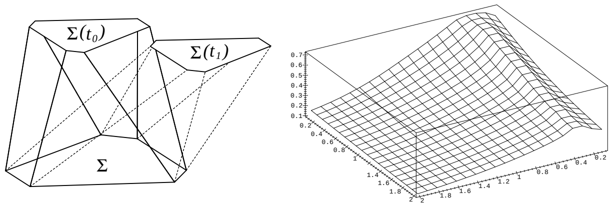

Figure 1. Platform and stability function

sources for errors occur in practice such that the test for a singular position itself should be formulated as a regression problem.

In case that the leg positions $L_i(t)$ lie in the carrier of a pencil of linear complexes, we even have an instantaneous movability of a two–parameter motion. Finally, if the legs lie in the carrier of a bundle of linear complexes, the indeterminacy of the points of the moving system is in three independent directions in general (in two directions for the spherical links) (Merlet, 1992). Again, our results are suitable to detect such positions.

An example is a *Stewart–Gough platform*, where the spherical links are arranged in two planes $\pi \subset \Sigma, \pi^0 \subset \Sigma^0$. We consider a special two–parameter motion of the moving plane π^0 by moving it translatory inside some plane $\alpha \subset E^3$. The position of a reference point $\mathbf{a}^0$ in π^0 then completely defines the current position of the moving system. On a grid of positions we have computed the standard deviation of the leg positions from the best approximating linear complex (computed with the line segment method (17)) and derived a spline fit of these data. The resulting bivariate *stability function* is visualized in figure 1.

An analogous application are *serial robots*. If the robot has six revolute joints, a necessary (but not sufficient) condition for a singular position (defined by vanishing Jacobian of the mapping from the 6–dimensional configuration space to the motion group $SE(3)$) is that the positions of joint axes lie in a linear complex (Hunt, 1986; Husty *et al.*, 1997).

5. Conclusion and future research

We have shown how to construct approximating linear complexes, linear congruences and reguli to given data lines or line segments and briefly outlined applications in robot kinematics and surface reconstruction. There is a variety of open problems in this area of *computational line geometry.* Other concepts for approximation in line space need to be studied. One possibility, based on local mappings of the Klein quadric into Euclidean 4–space via stereographic projection, has been introduced in (Chen and Pottmann, 1997) in connection with approximation by ruled surfaces. This has applications in NC milling (with cylindrical cutters under peripheral milling) and wire–EDM (Ravani and Wang, 1991) of sculptured surfaces. Future research should also take care of algorithmic efficiency for basic line geometric tasks (Chazelle *et al.*, 1996).

6. Acknowledgements

This work has been supported by grant No. P11357-MAT of the Austrian Science Foundation and through a grant from the California Department of Transportation.

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