Rational Families of Conics and Quadrics

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Abstract

A surface generated by a one parameter family of conics $c(t)$ is called *conic surface*. If $c(t)$ can be described by rational functions, the generated conic surface is rational. An algorithm to construct real rational parametrizations for such surfaces and some examples will be given.
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1 Introduction

In algebraic geometry it is known that a rational one parameter family of (irreducible) conics generates a rational surface. This follows from Tsen's Theorem (see [9], p.73 f) which states that an equation $F(x_1, \ldots, x_n) = 0$ of degree $m < n$ in x_i, whose coefficients are polynomials in one variable t, has a polynomial solution $x_i = p_i(t)$, $i = 1, \ldots, n$.

But since $p_i(t)$ is a solution of a system of polynomial equations, $p_i(t)$ will not define a real curve in general, such that this method will not lead to real rational parametrizations. The aim of this article is to construct real rational parametrizations for real rational conic surfaces.

The article is organized as follows. Section 2 discusses the parametrization problem in the plane. Section 3 provides some geometric properties of conic surfaces. In Section 4 rational parametrizations for rational conic surfaces are constructed. Section 5 discusses the dual surfaces to Section 3, namely envelopes of quadratic cones and finally Section 6 tells about some applications of the obtained results.

2 Rational Family of Conics in the Plane

Let $x = (x_0, x_1, x_2)$ be homogeneous coordinates of points with respect to an arbitrary, but fixed coordinate system in real projective plane $\mathbb{P}^2$.

Let C be a non–zero real symmetric 3×3–matrix with coefficients c_{jk}. A quadratic equation $c : x^T C x = 0$ defines a (not necessarily regular) conic in $\mathbb{P}^2$ as set of points. The conic is called singular, if $\det(C) = 0$. In detail, $\mathrm{rk}(C) = 2$ defines a pair of lines, $\mathrm{rk}(C) = 1$ defines a double counted line. If $\mathrm{rk}(C) = 3$, we call it *regular* or irreducible.

Dualities can be applied in $\mathbb{P}^2$, and we will use $X = (X_0, X_1, X_2)$ as coordinates of lines. Any non trivial quadratic equation $X^T C X = 0$ with a symmetric 3×3–matrix $C \neq 0$ defines a (not necessarily regular) conic in $\mathbb{P}^2$ as set of tangent lines. If $\mathrm{rk}(C) = 2$, the quadratic curve c contains two pencils of lines and if $\mathrm{rk}(C) = 1$ it is a double counted pencil.

If the coefficients c_{jk} depend on a parameter t, the equation

$$c(t) : \sum_{j,k=0}^{2} c_{jk}(t) x_j x_k = x^T C(t) x = 0, \tag{2.1}$$

defines a one parameter family of conics $c(t)$ in the plane $\mathbb{P}^2$. We call $c(t) \subset \mathbb{P}^2$ a *real rational* one parameter family of conics, if there is a representation of $c(t)$, such that the coefficients of the defining equation (2.1) are rational functions and $c(t)$ possesses real points for all real t.

We assume that $c(t)$ is a real rational family of regular conics, that means $\det(C) = 0$ just for finitely many t. For simpler notation, let X, Y and Z be homogeneous coordinates of points or lines in $\mathbb{P}^2$. By rational coordinate transformations we can assume that the family of conics $c(t)$ is given by

$$c(t) : X^2 L(t) + Y^2 M(t) + Z^2 N(t) = 0, \tag{2.2}$$

where L, M and N are considered to be polynomials in $\mathbb{R}[t]$. Our aim is to construct polynomials $x, y, z \in \mathbb{R}[t]$, which satisfy (2.2) identically.

Proposition 2.1 *Let L, M and N be polynomials in $\mathbb{R}[t] \setminus 0$ possessing constant signs for all real t. Further, L, M and N do not have multiple zeros and neither L and M, nor L and N nor M and N possess common zeros.*

If L, M and N define a real rational family of regular conics, there exist polynomials $x(t)$, $y(t)$ and $z(t)$ in $\mathbb{R}[t]$, which satisfy (2.2) identically and $(x, y, z)(t) \neq (0, 0, 0)$.

Proof Since L, M and N possess constant signs, they are of even degrees, say $2l$, $2m$ and $2n$. Note that L, M and N are not divisible by t since they possess no double zeros. The zeros of L, M and N shall be denoted by

$$\rho_1, \ldots, \rho_{2l}, \sigma_1, \ldots, \sigma_{2m} \text{ and } \tau_1 \ldots \tau_{2n} \in \mathbb{C} \setminus \mathbb{R}.$$

We look for polynomials x, y and z which satisfy (2.2) identically. We set

$$\begin{aligned} x(t) &= x_0 + x_1 t + \ldots + x_p t^p, \\ y(t) &= y_0 + y_1 t + \ldots + y_q t^q, \\ z(t) &= z_0 + z_1 t + \ldots + z_r t^r. \end{aligned} \tag{2.3}$$

with $(p+q+r)+3$ unknown real coefficients. We will see soon that the degrees shall be chosen to be

$$p = m + n, q = l + n, r = l + m. \tag{2.4}$$

For simplicity, let $\tau \in \{\tau_1, \ldots, \tau_{2n}\}$ be an arbitrary zero of N. Analogously, let ρ and σ be arbitrary zeros of L and M, respectively. Inserting (2.3) in equation (2.2) and evaluating at these zeros leads to

$$x(\tau)^2 L(\tau) + y(\tau)^2 M(\tau) = 0, \tag{2.5}$$
$$y(\sigma)^2 M(\sigma) + z(\sigma)^2 N(\sigma) = 0, \tag{2.6}$$
$$x(\rho)^2 L(\rho) + z(\rho)^2 N(\rho) = 0. \tag{2.7}$$

If $\mathrm{sgn}(L(\tau)) = \mathrm{sgn}(M(\tau))$ the first equation (2.5) factorizes to

$$\left(x(\tau)\sqrt{L(\tau)} + \mathrm{i}y(\tau)\sqrt{M(\tau)}\right)\left(x(\tau)\sqrt{L(\tau)} - \mathrm{i}y(\tau)\sqrt{M(\tau)}\right) = 0, \tag{2.8}$$

and otherwise we obtain

$$\left(x(\tau)\sqrt{L(\tau)} + y(\tau)\sqrt{M(\tau)}\right)\left(x(\tau)\sqrt{L(\tau)} - y(\tau)\sqrt{M(\tau)}\right) = 0. \tag{2.9}$$

Analogously equations (2.6) and (2.7) can be factorized. Depending on the signs of L and M, equation (2.5) is satisfied if one of the factors in (2.8) or (2.9) is zero. Taking all zeros into account, one obtains $2(l + m + n)$ linear homogeneous equations to compute $2(l + m + n) + 3$ unknowns $x_0, \ldots, x_p$, $y_0, \ldots, y_q$ and $z_0, \ldots, z_r$.

The coefficients of this linear system are complex. But by adding and subtracting equations to a conjugate pair $(\tau, \bar{\tau})$ one obtains a linear system with real coefficients. The zero space of this linear system is at least 3–dimensional. In the case of maximal rank the solutions can be parametrized by

$$x_i = x_{i0}u_0 + x_{i1}u_1 + x_{i2}u_2, \text{ for } i = 0, \ldots, p,$$

with parameters u_0, u_1 and u_2. Analogously for $y_j, j = 0, \ldots, q$ and $z_k, k = 0, \ldots, r$. Now, consider the polynomial

$$P(t) : x(t)^2 L(t) + y(t)^2 M(t) + z(t)^2 N(t).$$

It is of degree $\leq 2(l+m+n)$ in t and possesses $2(l+m+n)$ zeros at ρ_i, σ_j and τ_k. Note that $P(t)$ also depends on the free parameters u_0, u_1, u_2. If $\deg(P(t)) < 2(l+m+n)$ we are already done and $(x, y, z)(t)$ is a solution.

Otherwise, let L_0, M_0 and N_0 be the trailing coefficients of the polynomials L, M and N, and $(L_0, M_0, N_0) \neq (0, 0, 0)$. The trailing coefficient of P is

$$d = (\sum_{\alpha=0}^{2} x_{0\alpha} u_\alpha)^2 L_0 + (\sum_{\beta=0}^{2} y_{0\beta} u_\beta)^2 M_0 + (\sum_{\gamma=0}^{2} z_{0\gamma} u_\gamma)^2 N_0.$$

If we interpret (u_0, u_1, u_2) as coordinates in a the projective plane, $d = 0$ is a quadratic curve and is generated by transforming the conic $c(t = 0)$ under the map

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} x_{00} & x_{01} & x_{02} \\ y_{00} & y_{01} & y_{02} \\ z_{00} & z_{01} & z_{02} \end{bmatrix} \cdot \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix},$$

which proves that $d = 0$ has real points. Choose (u_0, u_1, u_2) as a real point on $d = 0$. It follows that $P(t)$ possesses a further zero at $t = 0$ such that it is identically zero. This implies that $(x, y, z)(t)$ is a non trivial solution of (2.2). $\square$

If the polynomials possess multiple or/and common zeros, we proceed as follows. Firstly, let $\tau \neq \bar{\tau}$ be a $2k+1$–fold zero of N, that means $N = (t-\tau)^{2k+1}(t-\bar{\tau})^{2k+1}\widetilde{N}$, where $\widetilde{N}$ is a polynomial of degree $2n-4k-2$. Choose

$$x(t) = (t-\tau)^k (t-\bar{\tau})^k \tilde{x}(t), \text{ and } y(t) = (t-\tau)^k (t-\bar{\tau})^k \tilde{y}(t),$$

where $\tilde{x}$, $\tilde{y}$ are of degree $m+n-k$ and $l+n-k$, respectively. We may determine $\tilde{x}(t), \tilde{y}(t)$ and $z(t)$ such that the polynomial $\widetilde{P}(t) : \tilde{x}^2 L + \tilde{y}^2 M + z^2 (t-\tau)(t-\bar{\tau})\widetilde{N}$ is identically zero.

Secondly, let $\tau \neq \bar{\tau}$ be common $2k+1$–fold zero of $L(t)$ and $M(t)$. Let

$$L(t) = (t-\tau)^{2k+1}(t-\bar{\tau})^{2k+1}\widetilde{L}(t) \text{ and } M(t) = (t-\tau)^{2k+1}(t-\bar{\tau})^{2k+1}\widetilde{M}(t),$$

where $\widetilde{L}(t)$ and $\widetilde{M}(t)$ are polynomials of degree $2l-4k-2$ and $2m-4k-2$, respectively. Multiply equation (2.2) by $(t-\tau)(t-\bar{\tau})$, such that $L(t)$ and

$M(t)$ possess τ and $\bar{\tau}$ as zeros of even multiplicity $2k+2$. Choose $z(t) = (t-\tau)^{k+1}(t-\bar{\tau})^{k+1}\tilde{z}(t)$, where $\tilde{z}(t)$ is a polynomial of degree $l+m-k-1$. We determine $x(t)$, $y(t)$ and $\tilde{z}(t)$ such that the polynomial

$$\widetilde{P}(t) : x(t)^2\widetilde{L}(t) + y(t)^2\widetilde{M}(t) + \tilde{z}(t)^2(t-\tau)(t-\bar{\tau})N(t)$$

is identically zero.

Analogously one substitutes in case of $\tau = \bar{\tau}$ τ or if τ is a $2k$–fold zero. It could be necessary to repeat such substitutions discussed above. For instance in the case where $\tau \neq \bar{\tau}$ is of multiplicity $2j$ for L and of multiplicity $2k+1$ for M. But in any case and clearly by a finite number of steps the equation $X^2L(t)+Y^2M(t)+Z^2N(t) = 0$ can be reduced such that it satisfies the conditions of Proposition 2.1.

Theorem 2.1 *Let $c(t)$ be a real rational one parameter family of regular conics in $\mathbb{P}^2$. There exists a rational curve $f(t) = (x, y, z)(t)$, such that $f(t)$ is contained in the conic $c(t)$ for all t.*

This will be the key idea to construct real rational parameterizations of real rational conic surfaces in $\mathbb{P}^3$. This problem was recently also solved by [8].

3 Conic Surfaces

Some important geometric properties of conic surfaces shall be collected. An introduction as well as many, mainly differential geometric details may be found in [1] and [10].

Let $x = (x_0, \ldots, x_3)$ be homogeneous coordinates of points and $X = (X_0, \ldots, X_3)$ be homogeneous coordinates of planes with respect to an arbitrary, but fixed coordinate system in real projective 3–space $\mathbb{P}^3(\mathbb{R})$. Let C be a symmetric 4×4–matrix with coefficients c_{jk}. A quadratic equation $c : X^TCX = 0$ defines a (not necessarily regular) quadric c in $\mathbb{P}^3$ as set of tangent planes. The quadric is called singular, if $\det(C) = 0$. In detail, if $\mathrm{rk}(C) = 3$, c is a conic as set of tangent planes (see Figure 1). The case $\mathrm{rk}(C) = 2$ characterizes a pair of bundles of planes and finally $\mathrm{rk}(C) = 1$ defines a double counted bundle of planes.

If c_{jk} depend on a parameter t and $\mathrm{rk}(C) = 3$ in I, the equation

$$c(t) : \sum_{j,k=0}^{3} c_{jk}(t)X_jX_k = X^TC(t)X = 0, \tag{3.1}$$

defines a one parameter family of conics in space.

Let $\gamma(t)$ be the plane containing $c(t)$. The homogeneous coordinates $(\gamma_0, \ldots, \gamma_3)(t)$ are solutions of the homogeneous linear system $C(t)X = 0$.

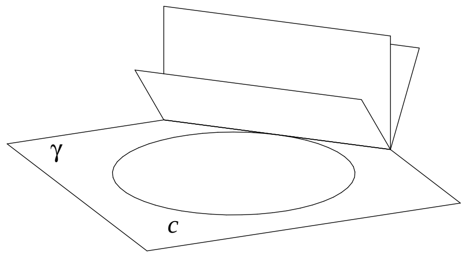

FIGURE 1. Conic as set of tangent planes

If $\gamma(t)$ and $\dot{\gamma}(t)$ are linearly independent, The family $c(t)$ defines a conic surface. Let $g(t) = \gamma(t) \cap \dot{\gamma}(t)$. Depending on the number $0, 1$ or 2 of real intersection points of $c \cap g$, the corresponding conic surface is called *elliptic, parabolic* or *hyperbolic.* This is a local property, such that also the conic c itself shall be called *elliptic, parabolic* or *hyperbolic*, see Figure 2.

Since $\gamma(t)$ satisfies $\gamma(t)^T C(t)\gamma(t) = 0$ identically, the derivative with respect to t leads to the identity

$$\gamma(t)^T \dot{C}(t)\gamma(t) = 0. \tag{3.2}$$

This means that γ is tangent to all regular dual quadrics of the pencil $\lambda c + \mu\dot{c}$. The common tangent planes of quadrics in the pencil $\lambda c + \mu\dot{c}$ are tangent planes of Φ in points of a generating conic c. These planes define a developable surface $D_c(t)$, which is in general rational of class 4. Later we will deal with cases where $D_c(t)$ is a quadratic cone.

Let $p(t)$ be the point of contact between $\gamma(t)$ and any arbitrary regular dual quadric of the pencil $\lambda c(t) + \mu\dot{c}(t)$. Thus we obtain

$$p(t) = \dot{C}(t)\gamma(t) = -C(t)\dot{\gamma}(t). \tag{3.3}$$

Additionally, p is also the pole of g with respect to c, considered as a planar curve in γ, see Figure 2.

4 Rational Conic Surfaces

Let assume that $c(t)$ is a real rational family of conics in space which means that the coefficients $c_{jk}(t)$ of $C(t)$ are polynomials in $\mathbb{R}[t]$ and $c(t)$ contains real points for all real t. For practical calculations it is useful to introduce a local coordinate system. For simplicity we omit the dependency of the parameter t. Let c be of elliptic or hyperbolic type. Let p be the pole of g with respect to c and choose q, r on g, such that q and r are conjugate with respect to c, see Figure 2. Using (p, q, r) as coordinate triangle and an

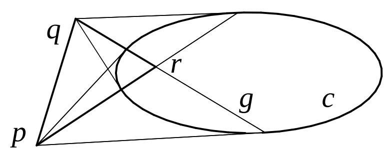

FIGURE 2. Polar triangle of a conic of hyp/ell type

appropriate normalization of the vectors, the conics $c(t)$ can be represented in diagonal form

$$c(t) : c_0(t)X_0^2 + c_1(t)X_1^2 + c_2(t)X_2^2 = 0, \tag{4.1}$$

where c_0, c_1 and c_2 are polynomials in $\mathbb{R}[t]$. Note that with respect to each polar triangle a conic c can be represented in such a diagonal form. Inverting the diagonal matrix $C = \mathrm{diag}(c_0, c_1, c_2)$ leads to a point representation of c.

4.1 Stereographic Projection

Let c be a conic in $\mathbb{P}^2$ and f a real point on c. Let a frame by chosen such that c as point set is represented by the equation

$$c_0x_0^2 + c_1x_1^2 + c_2x_2^2 = 0,$$

where (x_0, x_1, x_2) are homogeneous coordinates of points in the plane and $f = (f_0, f_1, f_2)$. Projecting c from one of its points, for instance f, to a line g with $f \notin g$, is a birational map. The inverse map $\sigma : g \to c$ is called *stereographic projection*. With help of the map σ one can derive rational parametrizations of conics.

Let $g : x_0 = 0$ be parametrized by $q(u) = (0, 1, u)$, where u is an inhomogeneous projective parameter, thus $u \in \mathbb{R} \cup \infty$. Let $a(u) = \lambda f + \mu q(u)$ be the pencil of lines passing through f, where (λ, μ) is a homogeneous projective parameter on the line $a(u)$. The intersection points of $a(u) \cap c$ are obtained for parameter values $\mu = 0$ and

$$\lambda_0 = c_1 + c_2u^2, \mu_0 = -2(c_1f_1 + c_2f_2u).$$

The point f corresponds to $\mu = 0$. This leads to the quadratic parametrization

$$c : z(u) = \lambda_0 f + \mu_0 q(u). \tag{4.2}$$

Since the principle of duality applies in $\mathbb{P}^2$, a conic as set of tangents can be represented by an equation

$$c : c_0X_0^2 + c_1X_1^2 + c_2X_2^2 = 0,$$

where X_i are homogeneous line coordinates. If y is a real tangent line, one finds completely dual to the above construction a quadratic parametrization of c as set of tangent lines.

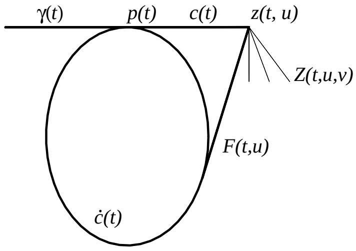

FIGURE 3. Construction of a rational tangent plane

4.2 Rational Parametrizations

Let $c(t)$ be a real rational one parameter family of regular conics in $\mathbb{P}^3$. Let $c(t)$ be of elliptic or hyperbolic type. Representing $c(t)$ with respect to an appropriate coordinate system we can assume that $c(t)$ is given by (4.1).

In the first step one applies Theorem 2.1 to compute polynomials y_0, y_1 and y_2 which satisfy (4.1) identically. Interpreting y_i as homogeneous line coordinates in $\gamma(t)$, the vector $y(t) = (y_0, y_1, y_2)(t)$ defines a real rational tangent line of $c(t)$. Applying the dual stereographic projection to $c(t)$ leads to a dual rational parametrization of $c(t)$ by functions $z(t, u)$.

The tangent line $z(t, u)$ is carrier of a pencil of tangent planes $Z(t, u, v)$ of the conic $c(t)$. The tangent plane $F(t, u) \subset Z(t, u, v)$ of the conic surface Φ is also tangent to the quadric $\dot{c}$. There are two tangent planes of $\dot{c}$, passing through $z(t, u)$. But since one is $\gamma(t)$, the remaining $F(t, u)$ can be computed linearly, see Figure 3. These tangent planes $F(t, u)$ are a dual parametrization of Φ. For fixed t_0 the planes $F(t_0, u)$ are tangent along the conic $c(t_0)$ and therefore parametrize the developable $D_c(t_0)$. For a fixed u_0 the planes $F(t, u_0)$ are tangent to $D_c(t)$ for all t.

It remains to discuss the case where $c(t)$ is parabolic for (almost) all t in $\mathbb{R}$. Since $g(t)$ is already a real rational tangent line one immediately applies a stereographic projection to obtain a dual parametrization $z(t, u)$ of the conic $c(t)$. A real rational dual parametrization $F(t, u)$ of Φ is constructed analogously to the elliptic and hyperbolic case.

Proposition 4.1 *A conic surface Φ defined by a real rational one parameter family of regular conics $c(t)$ possesses a real rational dual parametrization.*

A real rational parametrization of Φ as point set can be obtained in the following way. We assume that $c(t)$ is given by (4.1) but convert this to a point representation of $c(t)$, considered as a planar curve in $\gamma(t)$. Applying Theorem 2.1 leads to a real rational curve $f(t)$ which satisfies $f(t) \in c(t)$ for all t. By a stereographic projection one computes a real rational parametrization $z(t,u)$ of Φ. Each fixed conic $c(t_0)$ is parametrized by $z(t_0,u)$. On the other hand, for fixed u_0 one obtains a curve with $z(t,u_0) \in c(t)$ for all t.

If $c(t)$ defines a parabolic conic surface Φ the construction simplifies since a rational curve $f(t) = p(t)$ is already given.

Proposition 4.2 *A conic surface Φ defined by a real rational one parameter family of regular conics $c(t)$ possesses a real rational parametrization.*

5 Envelopes of Quadratic Cones

Dual to one parameter families of conics in $\mathbb{P}^3$ are one parameter families of quadratic cones. Therefore, dual to conic surfaces are envelopes of quadratic cones. We use the same notation as before and will outline some basic properties. Let $(x_0,\dots,x_3)$ be coordinate vectors of points x in $\mathbb{P}^3$. A quadratic equation $c: x^TCx = 0$ defines a quadratic cone, if the matrix C possesses rank 3. Rank 2 defines a pair of planes, rank 1 a double plane.

If the coefficients c_{jk} of C are functions of a parameter t, a one parameter family of quadratic cones is defined by

$$c(t): \sum_{j,k=0}^{3} c_{jk}(t)x_jx_k = x^TC(t)x = 0. \tag{5.1}$$

Let $v(t)$ be the vertex of $c(t)$. It is the solution of the homogeneous linear system $Cx = 0$. If v and $\dot{v}$ are linearly independent, the cones $c(t)$ envelope a surface Φ. The vertices v of the cones c form a curve with tangent lines $g = v \vee \dot{v}$. Depending on the number $0,1$ or 2 of real tangent planes of c, which pass through g, the corresponding envelopes Φ and also the quadratic cones itself are called *elliptic, parabolic* or *hyperbolic*. Note that this is a local property of a cone c and its derivative quadric $\dot{c}$.

We obtain the analogous identity

$$v(t)^T\dot{C}(t)v(t) = 0 \tag{5.2}$$

as in the case of conic surfaces, which says that $v(t)$ is a common point of all quadrics of the pencil $\lambda c(t) + \mu\dot{c}(t)$. Further, $\phi(t)$ denotes the common tangent plane of all regular quadrics of the pencil $\lambda c(t) + \mu\dot{c}(t)$ in the point v. A vector representation of $\phi(t)$ is obtained by $\phi = \dot{C}v = -C\dot{v}$. Additionally, ϕ is the polar plane to g with respect to c.

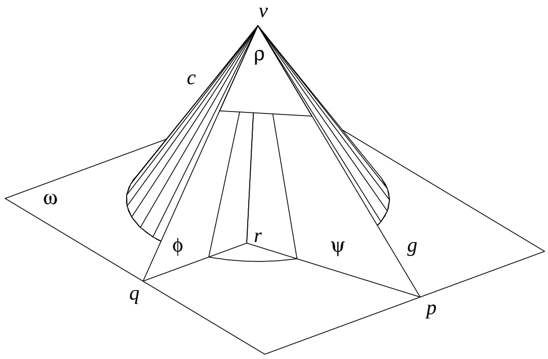

FIGURE 4. Local frame of a one parameter family of ell./hyp. quadratic cones

5.1 Rational Envelopes of Quadratic Cones

We call $c(t) \subset \mathbb{P}^3$ a *real rational* one parameter family of quadratic cones if a representation exists such that the defining equation possesses rational coefficients and $c(t) - v(t)$ contains real points.

As in case of conic surfaces it is again useful to represent $c(t)$ in a local coordinate system. Let $c(t)$ be of elliptic or hyperbolic type, that means $g(t)$ is not a generator line of $c(t)$. Let $v(t)$ be the origin of the local frame. Further let $\phi(t)$, $\psi(t)$ and $\rho(t)$ be pairwise conjugate planes with respect to $c(t)$. If the coordinate system shall be connected in an invariant way with $c(t)$, one will choose $\phi(t)$ to be the polar plane to $g(t)$ with respect to c. Then, $c(t)$ is represented by the diagonal form

$$c(t) : c_1(t)x_1^2 + c_2(t)x_2^2 + c_3(t)x_3^2 = 0. \tag{5.3}$$

Note that with respect to each three pairwise conjugate planes and $v(t)$ as origin, we obtain a diagonal form of $c(t)$. One can interpret (x_1, x_2, x_3) as homogeneous projective coordinate vectors of lines in the bundle $v(t)$, or of points in plane $\omega : \bar{x}_0 = 0$.

5.2 Rational Parametrizations

Completely dual to rational conic surfaces one computes real rational parametrizations of envelopes of real rational one parameter families of quadratic cones.

In case of elliptic or hyperbolic type one applies Theorem 2.1 to compute polynomials $(y_1, y_2, y_3)(t)$ which satisfy (5.3) identically. On the one hand these polynomials define a *real rational* generating line $y(t)$ of each quadratic cone $c(t)$. On the other hand $y(t)$ defines a *real rational* point on the conic $c(t) \cap \omega$. In the parabolic case a real rational generating line is $g(t)$.

Interpreting $y(t)$ as real rational generator, the polarity with respect to $c(t)$ maps $y(t)$ to a real rational tangent plane $Y(t)$.

A real rational parametrization $Z(t,u)$ of Φ is then computed by applying a dual stereographic projection to the cones $c(t)$. The parametrization $Z(t,u)$ represents Φ as envelope of its tangent planes.

To compute a point representation of Φ one can proceed as follows. Let $y(t)$ be a rational generator line of $c(t)$ as above. Each cone $c(t)$ is tangent to Φ in points of the characteristic curve $d_c(t)$. Since the generator $y(t)$ intersects $\dot{c}(t)$ in the known vertex $v(t)$, the further intersection point $f(t)$ is rational in t. So, $f(t)$ is a rational curve on Φ and for each t the curve point $f(t)$ lies on $d_c(t)$. To obtain the entire parametrization one will first apply a stereographic projection to the cone $c(t)$, based on the rational generator $y(t)$. The resulting parametrization shall be denoted by $z(t,u)$ and represents for fixed t_0 the generators of a cone $c(t_0)$. Next, one intersects these generators $z(t,u)$ with the quadrics $\dot{c}(t)$ for each t. Then, $Z(t,u) = z(t,u) \cap \dot{c}(t)$ is already a real rational parametrization of Φ as point set. For a fixed t_0, $Z(t_0,u)$ parametrizes the characteristic curve $d_c(t_0)$, and for a fixed u_0 we obtain a rational curve $Z(t,u_0) = f(t)$, such that for all t the point $f(t) \in d_c(t)$.

Proposition 5.1 *The envelope Φ of a real rational one parameter family of quadratic cones $c(t)$ possesses rational parametrizations.*

6 Applications

The families of conic surfaces and envelopes of quadratic cones is quite large. Certain classes, as quadrics and Dupin cyclides are also used in surface modeling. We give some more general examples in Euclidean 3–space.

6.1 Rational Canal Surfaces in Euclidean 3–Space

Points y in Euclidean 3–space are represented by their coordinate vectors (y_1, y_2, y_3) with respect to a fixed, but arbitrary coordinate system. Let a canal surface Φ be defined as envelope of a one parameter family of spheres $S(t)$. A family of spheres shall be called *rational*, if the defining equation of $S(t)$ possesses only rational coefficients.

The envelope Φ of a rational one parameter set of spheres is defined

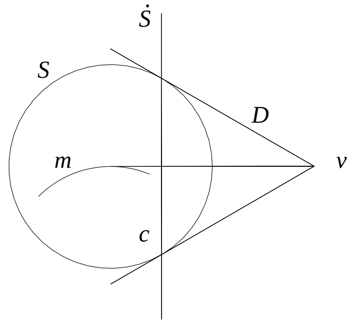

FIGURE 5. Local properties of a canal surface

by the equations

$$S(t) : \sum_1^3 (y_j - m_j(t))^2 - r(t)^2 = 0,\ \dot{S}(t) : \sum_1^3 (y_j - m_j(t))\dot{m}_j(t) + r(t)\dot{r}(t) = 0, \tag{6.1}$$

where $m_j(t)$ are coordinates of the rational center curve $m(t)$ of $S(t)$. The radius function $r(t)$ of the spheres is not necessarily rational, but a square root of a rational function.

The generating conics of Φ are the characteristic circles $c = S \cap \dot{S}$. The circles contain real points, if and only if $\dot{m}^2 - \dot{r}^2 \geq 0$. Equality holds, if the plane $\dot{S}$ is tangent to S and c degenerates to a point. Further, Φ is enveloped by a one parameter family of cones of revolution $D(t)$, which are tangent to Φ in points of c. The axis of D is the tangent line of the center curve $m(t)$.

Let $\lambda S(t) + \mu \dot{S}(t)$ be a pencil of quadrics, where $\dot{S}(t)$ is considered to be a double plane. The cone of revolution $D(t)$ is a further singular quadric, contained in this pencil. Since $\det(\lambda S + \mu \dot{S})$ possesses a 3–fold zero at λ, it follows that $D(t)$ is given by an equation with rational coefficients.

Proposition 6.1 *The real envelope Φ of a rational one parameter family of spheres $S(t)$ can be generated as envelope of a real rational one parameter family of cones of revolution $D(t)$, in the sense of Proposition 5.1.*

We mention that the envelope of the one parameter family of cones of revolution $D(t)$ also contains two, not necessarily real developable surfaces, which are not considered in the above Theorem. From Proposition 5.1 it is clear that Φ is a rational surface.

Special rational canal surfaces are those, whose radius function is rational and not just a square root of a rational function. Those surfaces possess additionally rational unit normals. A detailed description of algorithms and low degree representations is given in [5].

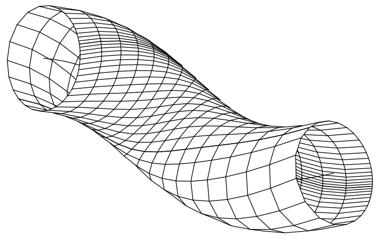

FIGURE 6. Pipe surface with cubic center curve

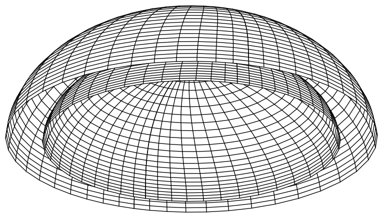

FIGURE 7. Ellipsoid of revolution and outside offset

6.2 Rational Surfaces with Rational Offsets

The results about conic surfaces apply to parametrizing certain rational surfaces with rational offsets. It is proofed in [6] that the envelope of a rational one parameter family of cones of revolution $D(t)$ with rational radius function is a rational surface with rational offset surfaces. Such a cone of revolution $D(t)$ is given by an equation with rational coefficients, but additionally $D(t)$ possesses an inscribed sphere $S(t)$ with rational center and rational radius function. Further it is proofed there that all those envelopes possess rational unit normals. Specializing the family $D(t)$ one obtains the following results, see [4] and [6].

Proposition 6.2 *All regular quadrics in Euclidean* 3*–space possess rational offsets.*

One Proof is based on the fact that a quadric is the envelope of a rational one parameter family of cones of revolution. The offset surfaces can be represented by explicit formulae. An ellipsoid of revolution can be parametrized as rational tensor product surface of degrees 4 and 2; its offset surface possesses degrees 8 and 2. An example is shown in Figure 7. Another proof of Proposition 6.2 can be found in [3].

A further example are ruled surfaces. Those are envelopes of a one parameter family of lines. Viewing the generating lines as cylinders of revolution with zero radius, an offset of a ruled surface at distance d is enveloped by cylinders of revolution of radius d. For rational ruled surfaces, the family of cylinders is rational and possesses a rational radius function. Another proof of the following result can be found in [7].

Proposition 6.3 *All offset surfaces of rational non developable ruled surfaces are rational.*

Remark: As outlined in [4] most of the results described here can be generalized to projective or Euclidean n–space.

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