Developable Surface Fitting to Point Clouds

Martin Peternell

Vienna University of Technology, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstr. 8–10, A–1040 Vienna, Austria

Abstract

Given a set of data points as measurements from a developable surface, the present paper investigates the recognition and reconstruction of these objects. We investigate the set of estimated tangent planes of the data points and show that classical Laguerre geometry is a useful tool for recognition, classification and reconstruction of developable surfaces. These surfaces can be generated as envelopes of a one-parameter family of tangent planes. Finally we give examples and discuss the problems especially arising from the interpretation of a surface as set of tangent planes.

Key words: developable surface, cylinder of revolution, cone of revolution, reconstruction, recognition, space of planes, Laguerre geometry

1 Introduction

Given a cloud of data points $\mathbf{p}_i$ in $\mathbb{R}^3$, we want to decide whether $\mathbf{p}_i$ are measurements of a *cylinder or cone of revolution*, a *general cylinder* or *cone* or a general *developable surface*. In case where this is true we will approximate the given data points by one of the mentioned shapes. In the following we denote all these shapes by *developable surfaces*. To implement this we use a concept of classical geometry to represent a developable surface not as a two-parameter set of points but as a one-parameter set of tangent planes and show how this interpretation applies to the recognition and reconstruction of developable shapes.

Points and vectors in $\mathbb{R}^3$ or $\mathbb{R}^4$ are denoted by boldface letters, $\mathbf{p}, \mathbf{v}$. Planes and lines are displayed as italic capital letters, T, L. We use Cartesian coordinates in $\mathbb{R}^3$ with axes x, y and z. In $\mathbb{R}^4$ the axes of the Cartesian coordinate system are denoted by $u_1, \ldots, u_4$.

Developable surfaces shall briefly be introduced as special cases of ruled surfaces. A *ruled surface* R carries a one parameter family of straight lines L. These lines are called *generators* or *generating lines*. The general parametrization of a ruled surface R is

$$\mathbf{x}(u,v) = \mathbf{c}(u) + v\mathbf{e}(u), \tag{1}$$

where $\mathbf{c}(u)$ is called *directrix curve* and $\mathbf{e}(u)$ is a vector field along $\mathbf{c}(u)$. For fixed values u, this parametrization represents the straight lines $L(u)$ on R.

The normal vector $\mathbf{n}(u,v)$ of the ruled surface $\mathbf{x}(u,v)$ is computed as cross product of the partial derivative vectors $\mathbf{x}_u$ and $\mathbf{x}_v$, and we obtain

$$\mathbf{n}(u,v) = \dot{\mathbf{c}}(u) \times \mathbf{e}(u) + v\dot{\mathbf{e}}(u) \times \mathbf{e}(u). \tag{2}$$

For fixed $u = u_0$, the normal vectors $\mathbf{n}(u_0, v)$ along $L(u_0)$ are linear combinations of the vectors $\dot{\mathbf{c}}(u_0) \times \mathbf{e}(u_0)$ and $\dot{\mathbf{e}}(u_0) \times \mathbf{e}(u_0)$. The parametrization $\mathbf{x}(u,v)$ represents a *developable surface* D if for each generator L all points $\mathbf{x} \in L$ have the same tangent plane (with exception of the singular point on L). This implies that the vectors $\dot{\mathbf{c}} \times \mathbf{e}$ and $\dot{\mathbf{e}} \times \mathbf{e}$ are linearly dependent which is expressed equivalently by the following condition

$$\det(\dot{\mathbf{c}}, \mathbf{e}, \dot{\mathbf{e}}) = 0. \tag{3}$$

Any regular generator $L(u)$ of a developable surface D carries a unique *singular point* $\mathbf{s}(u)$ which does not possess a tangent plane in the above defined sense, and $\mathbf{s}(u) = \mathbf{x}(u, v_s)$ is determined by the parameter value

$$v_s = -\frac{(\dot{\mathbf{c}} \times \mathbf{e}) \cdot (\dot{\mathbf{e}} \times \mathbf{e})}{(\dot{\mathbf{e}} \times \mathbf{e})^2}. \tag{4}$$

If $\mathbf{e}$ and $\dot{\mathbf{e}}$ are linearly dependent, the singular point $\mathbf{s}$ is at infinity, otherwise it is a proper point. In Euclidean space $\mathbb{R}^3$ there exist three different basic classes of developable surfaces:

(1) Cylinder: the singular curve degenerates to a single point at infinity.
(2) Cone: the singular curve degenerates to a single proper point, which is called *vertex*.
(3) Surface consisting of the *tangent lines of a regular space curve* $\mathbf{s}(u)$, which is the *singular curve* of the surface.

In all three cases the surface D can be generated as *envelope* of its *one parameter family of tangent planes*. This is called the *dual representation* of D. A cylinder of revolution is obtained by rotating a plane around an axis which is parallel to this plane. A cone of revolution is obtained by rotating a plane around a general axis, but which is not perpendicular to this plane. Further, it is known that smooth developable surfaces can be characterized by vanishing Gaussian curvature. In applications surfaces appear which are composed of these three basic types.

There is quite a lot of literature on modeling with developable surfaces, see [1–6] and their references. B-spline representations and the dual representation are well known. The dual representation has been used for interpolation and approximation of tangent planes and generating lines. Pottmann and Wallner [5] study approximation of tangent planes, generating lines and points. The treatment of the singular points of the surface is included in the approximation with relatively little costs. To implement all these tasks, a local coordinate system is used for the representation of developable surfaces such that their tangent planes $T(t)$ are given by $T(t) : e_4(t) + e_1(t)x + ty - z = 0$. This concept can be used for surface fitting too, but the representation is a bit restrictive.

We note a few problems occurring in surface fitting with developable B-spline surfaces. In general, for fitting a B-spline surface

$$\mathbf{b}(u, v) = \sum N_i(u)N_j(v)\mathbf{b}_{ij}$$

with control points $\mathbf{b}_{ij}$ to a set of unorganized data points $\mathbf{p}_k$, one estimates parameter values (u_i, v_j) corresponding to $\mathbf{p}_k$. The resulting approximation leads to a linear problem in the unknown control points $\mathbf{b}_{ij}$. For surface fitting with ruled surfaces we might choose the degrees n and 1 for the B-spline functions $N_i(u)$ and $N_j(v)$ over a suitable knot sequence. There occur two main problems in approximating data points by a developable B-spline surface:

- For fitting ruled surfaces to point clouds, we have to estimate in advance the approximate direction of the generating lines of the surface in order to estimate useful parameter values for the given data. To perform this, it is necessary to estimate the asymptotic lines of the surface in a stable way.
- We have to guarantee that the *resulting approximation* $\mathbf{b}(u, v)$ is *developable*, which is expressed by equation (3). Plugging the parametrization $\mathbf{b}(u, v)$ into this condition leads to a highly non-linear side condition in the control points $\mathbf{b}_{ij}$ for the determination of the approximation $\mathbf{b}(u, v)$.

1.1 Contribution of the article

To avoid above mentioned problems, we follow another strategy. The reconstruction of a developable surface from scattered data points is implemented as reconstruction of a one-parameter family of planes which lie close to the estimated tangent planes of the given data points. Carrying out this concept we can automatically guarantee that the approximation is developable. This concept avoids the estimation of parameter values and the estimation of the asymptotic curves. The reconstruction is performed by solving curve approximation techniques in the space of planes.

The proposed algorithm can also be applied to approximate nearly developable surfaces (or better slightly distorted developable surfaces) by developable surfaces. The test implementation has been performed in Matlab and the data has been generated by scanning models of developable surfaces with an optical laser scanner. Some examples use data generated by simulating a scan of mathematical models.

The article is organized as follows: Section 2 presents basic properties concerning the Blaschke image (Blaschke model) of the set of planes in $\mathbb{R}^3$ which is relevant for the implemention of the intended reconstruction. Section 3 tells about a classification, and Section 4 discusses the recognition of developable surfaces in point clouds using the Blaschke image of the set of estimated tangent planes of the point set. Section 5 describes the concept of an algorithm for reconstruction of these surfaces from data points. Finally, we present some examples and discuss problems of this approach and possible solutions.

2 The Blaschke model of oriented planes in $\mathbb{R}^3$

Describing points $\mathbf{x}$ by their Cartesian coordinate vectors $\mathbf{x} = (x, y, z)$, an oriented plane E in Euclidean space $\mathbb{R}^3$ can be written in the Hesse normal form,

$$E : n_1x + n_2y + n_3z + d = 0, \quad n_1^2 + n_2^2 + n_3^2 = 1. \tag{5}$$

We note that $n_1x + n_2y + n_3z + d = \mathrm{dist}(\mathbf{x}, E)$ is the signed distance between the point $\mathbf{x}$ and the plane E. In particular, d is the origin's distance to E. The vector $\mathbf{n} = (n_1, n_2, n_3)$ is the unit normal vector of E. The vector $\mathbf{n}$ and the distance d uniquely define the oriented plane E and we also use the notation $E : \mathbf{n} \cdot \mathbf{x} + d = 0$.

The interpretion of the vector (n_1, n_2, n_3, d) as point coordinates in $\mathbb{R}^4$, defines the *Blaschke mapping*

$$b : E \mapsto b(E) = (n_1, n_2, n_3, d) = (\mathbf{n}, d). \tag{6}$$

In order to carefully distinguish between the original space $\mathbb{R}^3$ and the image space $\mathbb{R}^4$, we denote Cartesian coordinates in the image space $\mathbb{R}^4$ by

(u_1, u_2, u_3, u_4). According to the normalization $\mathbf{n}^2 = 1$ and (6), the set of all oriented planes of $\mathbb{R}^3$ is mapped to the entire point set of the so-called *Blaschke cylinder*,

$$B: \; u_1^2 + u_2^2 + u_3^2 = 1. \tag{7}$$

Thus, the set of planes in $\mathbb{R}^3$ has the structure of a three-dimensional cylinder, whose cross sections with planes $u_4 = const.$ are copies of the unit sphere S^2 (Gaussian sphere). Any point $U \in B$ is image point of an oriented plane in $\mathbb{R}^3$. Obviously, the Blaschke image $b(E) = (\mathbf{n}, d)$ is nothing else than the graph of the support function d (distance to the origin) over the Gaussian image point $\mathbf{n}$.

Let us consider a pencil (one-parameter family) of parallel oriented planes $E(t) : \mathbf{n} \cdot \mathbf{x} + t = 0$. The Blaschke mapping (6) implies that the image points $b(E(t)) = (\mathbf{n}, t)$ lie on a generating line of B which is parallel to the u_4-axis.

2.1 Incidence of point and plane

We consider a fixed point $\mathbf{p} = (p_1, p_2, p_3)$ and all planes $E : \mathbf{n} \cdot \mathbf{x} + d = 0$ passing through this point. The incidence between $\mathbf{p}$ and E is expressed by

$$p_1 n_1 + p_2 n_2 + p_3 n_3 + d = \mathbf{p} \cdot \mathbf{n} + d = 0, \tag{8}$$

and therefore the image points $b(E) = (n_1, n_2, n_3, d)$ in $\mathbb{R}^4$ of all planes passing through $\mathbf{p}$ lie in the three-space

$$H : p_1 u_1 + p_2 u_2 + p_3 u_3 + u_4 = 0, \tag{9}$$

passing through the origin of $\mathbb{R}^4$. The intersection $H \cap B$ with the cylinder B is an ellipsoid and any point of $H \cap B$ is image of a plane passing through $\mathbf{p}$. Fig. 1 shows a 2D illustration of this property.

2.2 Tangency of sphere and plane

Let S be the oriented sphere with center $\mathbf{m}$ and signed radius r, $S : (\mathbf{x} - \mathbf{m})^2 - r^2 = 0$. The tangent planes T_S of S are exactly those planes, whose signed distance from $\mathbf{m}$ equals r. Therefore, they satisfy

$$T_S : n_1 m_1 + n_2 m_2 + n_3 m_3 + d = \mathbf{n} \cdot \mathbf{m} + d = r. \tag{10}$$

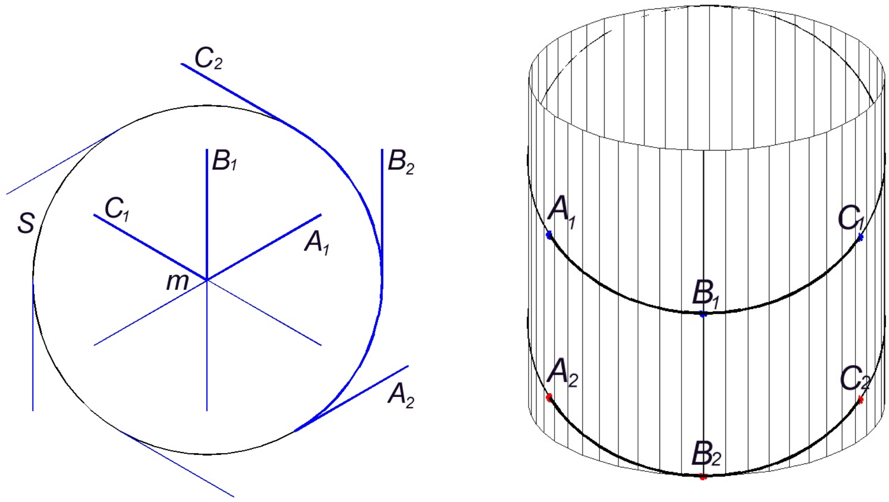

Fig. 1. Blaschke images of a pencil of lines and of lines tangent to an or. circle.

Their Blaschke image points $b(T_S)$ thus lie in the three-space

$$H : m_1u_1 + m_2u_2 + m_3u_3 + u_4 - r = 0, \tag{11}$$

and $b(T_S)$ are the points of the intersection $H \cap B$, which is again an ellipsoid.

This also follows from the fact that S is the offset surface of $\mathbf{m}$ at signed distance r. The offset operation, which maps a surface $F \subset \mathbb{R}^3$ (as set of tangent planes) to its offset F_r at distance r, appears in the Blaschke image B as translation by the vector $(0, 0, 0, r)$, see Fig. 1.

Conversely, if points $\mathbf{q} = (q_1, q_2, q_3, q_4) \in B$ satisfy a linear relation

$$H : a_0 + u_1a_1 + u_2a_2 + u_3a_3 + u_4a_4 = 0,$$

$\mathbf{q} = b(T)$ are Blaschke images of planes T which are tangent to a sphere in case $a_4 \neq 0$. Center and radius are determined by

$$\mathbf{m} = \frac{1}{a_4}(a_1, a_2, a_3), \quad r = \frac{-a_0}{a_4}.$$

If $a_0 = 0$, the planes $b(T)$ pass through the fixed point $\mathbf{m}$. If $a_4 = 0$, the planes T form a constant angle with the direction vector $\mathbf{a} = (a_1, a_2, a_3)$ because of $\mathbf{a} \cdot \mathbf{n} = -a_0$, with $\mathbf{n} = (u_1, u_2, u_3)$.

Here it would lead to far to explain more about *Laguerre geometry*, the geometry of oriented planes and spheres in $\mathbb{R}^3$ and we refer to the literature [7–9] for more details.

Let $T(u)$ be a one-parameter family of planes

$$T(u) : n_4(u) + n_1(u)x + n_2(u)y + n_3(u)z = 0$$

with arbitrary functions $n_i, i = 1, \ldots, 4$. The vector $\mathbf{n}(u) = (n_1, n_2, n_3)(u)$ is a normal vector of $T(u)$. Excluding degenerate cases, the envelope of $T(u)$ is a developable surface D, whose generating lines $L(u)$ are

$$L(u) = T(u) \cap \dot{T}(u),$$

where $\dot{T}(u)$ denotes the derivative with respect to u. The generating lines themselves envelope the singular curve $\mathbf{s}(u)$ which is the intersection

$$\mathbf{s}(u) = T(u) \cap \dot{T}(u) \cap \ddot{T}(u).$$

Taking the normalization $n_1^2 + n_2^2 + n_3^2 = \mathbf{n}(u)^2 = 1$ into account, the Blaschke image $b(T(u)) = b(D)$ of the developable surface D is a curve on the Blaschke cylinder B. This property will be applied later to fitting developable surfaces to point clouds.

3 The classification of developable surfaces according to their image on B

This section will characterize cylinders, cones and other special developable surfaces D by studying their Blaschke images $b(D)$.

Cylinder: D is a *general cylinder* if all its tangent planes $T(u)$ are parallel to a vector $\mathbf{a}$ and thus its normal vectors $\mathbf{n}(u)$ satisfy $\mathbf{n} \cdot \mathbf{a} = 0$. This implies that the image curve $b(T(u)) = b(D)$ is contained in the three-space

$$H : a_1u_1 + a_2u_2 + a_3u_3 = 0. \tag{12}$$

Cone: D is a *general cone* if all its tangent planes $T(u)$ pass through a fixed point $\mathbf{p} = (p_1, p_2, p_3)$. This incidence is expressed by $p_1n_1+p_2n_2+p_3n_3+n_4 = 0$. Thus, the Blaschke image curve $b(T(u)) = b(D)$ is contained in the three-space

$$H : p_1u_1 + p_2u_2 + p_3u_3 + u_4 = 0. \tag{13}$$

There exist other special types of developable surfaces. Two of them will be mentioned here.

The surface D is a *developable of constant slope*, if its normal vectors $\mathbf{n}(u)$ form a constant angle ϕ with a fixed direction vector $\mathbf{a}$. Assuming $\|\mathbf{a}\| = 1$, we get $\cos(\phi) = \mathbf{a} \cdot \mathbf{n}(u) = \gamma = \text{const}$. This implies that the Blaschke images of the tangent planes of D are contained in the three-space

$$H : -\gamma + a_1 u_1 + a_2 u_2 + a_3 u_3 = 0. \tag{14}$$

The developable surface D is *tangent to a sphere* with center $\mathbf{m}$ and radius r, if the tangent planes $T(u)$ of D satisfy $n_4 + n_1 m_1 + n_2 m_2 + n_3 m_3 - r = 0$, according to (11). Thus, the image curve $b(D)$ is contained in the three-space

$$H : -r + u_1 m_1 + u_2 m_2 + u_3 m_3 + u_4 = 0. \tag{15}$$

3.1 Cones and cylinders of revolution

For applications it is of particular interest if a developable surface D is a cone or cylinder of revolution.

Let D be a *cylinder of revolution* with axis A and radius r. The tangent planes T of D are tangent to all spheres of radius r, whose centers vary on A. Let

$$S_1 : (\mathbf{x} - \mathbf{p})^2 - r^2 = 0, \quad S_2 : (\mathbf{x} - \mathbf{q})^2 - r^2 = 0$$

be two such spheres with centers $\mathbf{p}, \mathbf{q}$. According to (11), the images $b(T)$ of the tangent planes T satisfy the relations

$$\begin{aligned} H_1 &: -r + u_1 p_1 + u_2 p_2 + u_3 p_3 + u_4 = 0, \\ H_2 &: -r + u_1 q_1 + u_2 q_2 + u_3 q_3 + u_4 = 0. \end{aligned} \tag{16}$$

Since $\mathbf{p} \neq \mathbf{q}$, the image curve $b(D)$ lies in the plane $P = H_1 \cap H_2$ and $b(D)$ is a conic.

Cones of revolution D can be obtained as envelopes of the common tangent planes of two oriented spheres S_1, S_2 with different radii $r \neq s$. Thus, $b(D)$ is a conic contained in the plane $P = H_1 \cap H_2$ which is defined by

$$\begin{aligned} H_1 &: -r + u_1 p_1 + u_2 p_2 + u_3 p_3 + u_4 = 0, \\ H_2 &: -s + u_1 q_1 + u_2 q_2 + u_3 q_3 + u_4 = 0. \end{aligned} \tag{17}$$

Conversely, if the Blaschke image $b(D)$ of a developable surface is a planar curve $\subset P$, how can we decide whether D is a cone or cylinder of revolution?

Let $b(D) = b(T(u))$ be a planar curve $\subset P$ and let P be given as intersection of two independent three-spaces H_1, H_2, with

$$H_i : h_{i0} + h_{i1}u_1 + h_{i2}u_2 + h_{i3}u_3 + h_{i4}u_4 = 0. \tag{18}$$

Using the results of Section 2.2, the incidence relation $b(T(u)) \subset H_1$ implies that $T(u)$ is *tangent to a sphere*, or is *passing through a point* ($h_{10} = 0$), or encloses a *fixed angle with a fixed direction* ($h_{14} = 0$). The same argumentation holds for H_2.

Thus, by excluding the degenerate case $h_{14} = h_{24} = 0$, we can assume that $P = H_1 \cap H_2$ is the intersection by two three-spaces H_1, H_2 of the form (16) or (17).

(1) Let the plane $P = H_1 \cap H_2$ be given by equations (16). Then, the developable surface D is a *cylinder of revolution*. By subtracting the equations (16) it follows that the normal vector $\mathbf{n}(u)$ of $T(u)$ satisfies

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{q}) = 0.$$

Thus, the axis A of D is given by $\mathbf{a} = \mathbf{p} - \mathbf{q}$ and D's radius equals r.

(2) Let the plane $P = H_1 \cap H_2$ be given by equations (17). The pencil of three-spaces $\lambda H_1 + \mu H_2$ contains a unique three-space H, passing through the origin in $\mathbb{R}^4$, whose equation is

$$H : \sum_{i=1}^{3} u_i(sp_i - rq_i) + u_4(s - r) = 0.$$

Thus, the tangent planes of the developable surface D are passing through a fixed point corresponding to H, and D is a *cone of revolution*. Its vertex $\mathbf{v}$ and the inclination angle ϕ between the axis $A : \mathbf{a} = \mathbf{p} - \mathbf{q}$ and the tangent planes $T(u)$ are

$$\mathbf{v} = \frac{1}{s - r}(s\mathbf{p} - r\mathbf{q}), \text{ and } \quad \sin\phi = \frac{|s - r|}{\|\mathbf{q} - \mathbf{p}\|}.$$

4 Recognition of developable surfaces from point clouds

Given a cloud of data points $\mathbf{p}_i$, this section discusses the recognition and classification of developable surfaces according to their Blaschke images. The algorithm contains the following steps:

(1) Estimation of tangent planes T_i at data points $\mathbf{p}_i$ and computation of the image points $b(T_i)$.

(2) Analysis of the structure of the set of image points $b(T_i)$.
(3) If the set $b(T_i)$ is *curve-like*, classification of the developable surface which is close to $\mathbf{p}_i$.

4.1 Estimation of tangent planes

We are given data points $\mathbf{p}_i$, $i = 1, \ldots, N$, with Cartesian coordinates x_i, y_i, z_i in $\mathbb{R}^3$ and a triangulation of the data with triangles t_j. The triangulation gives topological information about the point cloud, and we are able to define adjacent points $\mathbf{q}_k$ for any data point $\mathbf{p}$.

The estimated tangent plane T at $\mathbf{p}$ shall be a plane best fitting the data points $\mathbf{q}_k$. T can be computed as minimizer (in the l_1 or l_2-sense) of the vector of distances $\operatorname{dist}(\mathbf{q}_k, T)$ between the data points $\mathbf{q}_k$ and the plane T. This leads to a set of N estimated tangent planes T_i corresponding to the data points $\mathbf{p}_i$. For more information concerning reverse engineering, see the survey [10].

Assuming that the original surface with measurement points $\mathbf{p}_i$ is a developable surface D, the image points $b(T_i)$ of the estimated tangent planes T_i will form a *curve-like* region on B, see also [10]. To check the property *'curve-like'*, neighborhoods with respect to a metric on B will be defined. Later we will fit a curve $\mathbf{c}(t)$ to the curve-like set of image points $b(T_i)$, and this fitting is implemented according to the chosen metric.

4.2 A Euclidean metric in the set of planes

Now we show that the simplest choice, namely the canonical Euclidean metric in the surrounding space $\mathbb{R}^4$ of the Blaschke cylinder B, is a quite useful metric for data analysis and fitting. This says that the *distance dist*(E, F) *between two planes* E, F

$$E : e_1x_1 + e_2x_2 + e_3x_3 + e_4 = 0, \quad F : f_1x_1 + f_2x_2 + f_3x_3 + f_4 = 0,$$

with normalized normal vectors $\mathbf{e} = (e_1, e_2, e_3)$ and $\mathbf{f} = (f_1, f_2, f_3)$ ($\|\mathbf{e}\| = \|\mathbf{f}\| = 1$) is defined to be the *Euclidean distance of their image points* $b(E)$ and $b(F)$. Thus, the squared distance between E and F is defined by

$$\operatorname{dist}(E, F)^2 = (e_1 - f_1)^2 + (e_2 - f_2)^2 + (e_3 - f_3)^2 + (e_4 - f_4)^2. \tag{19}$$

To illustrate the geometric meaning of $\operatorname{dist}(E, F)^2$ between two planes E and F we choose a fixed plane $M(= F)$ in $\mathbb{R}^3$ as $x - m = 0$. Its Blaschke image is $b(M) = (1, 0, 0, -m)$. All points of the Blaschke cylinder, whose Euclidean distance to $b(M)$ equals r, form the intersection surface S of B with the three-dimensional sphere $(u_1-1)^2+u_2^2+u_3^2+(u_4+m)^2 = r^2$. Thus, S is an algebraic surface of order 4 in general. Its points are Blaschke images $b(E)$ of planes E in $\mathbb{R}^3$ which have constant distance r from M and their coordinates e_i satisfy

$$(e_1-1)^2 + e_2^2 + e_3^2 + (e_4+m)^2 - r^2 = 0. \tag{20}$$

The coefficients e_i satisfy the normalization $e_1^2 + e_2^2 + e_3^2 = 1$. If we consider a general homogeneous equation $E : w_1x_1 + w_2x_2 + w_3x_3 + w_4 = 0$ of E, these coefficients w_i are related to e_i by

$$e_i = \frac{w_i}{\sqrt{w_1^2 + w_2^2 + w_3^2}},\ i = 1, 2, 3, 4.$$

We plug this into (20) and obtain the following homogeneous relation of degree four in plane coordinates w_i,

$$[(2-r^2+m^2)(w_1^2+w_2^2+w_3^2)+w_4^2]^2 = 4(w_1-mw_4)^2(w_1^2+w_2^2+w_3^2). \tag{21}$$

Hence, all planes E, having constant distance $\operatorname{dist}(E, M) = r$ from a fixed plane M, form the tangent planes of an algebraic surface $b^{-1}(S) = U$ of class 4, and U bounds the *tolerance region* of the plane M. If a plane E deviates from a plane M in the sense, that $b(E)$ and $b(M)$ have at most distance r, then the plane E lies in a region of $\mathbb{R}^3$, which is bounded by the surface U (21).

For visualization we choose the 2D-case. Figure 2 shows the boundary curves of tolerance regions of lines $M : x = m$, for values $m = 0, 1.25, 2.5$ and radius $r = 0.25$. The lines M_i are drawn dashed. The largest perpendicular distance of $E(\parallel M)$ and M within the tolerance regions is r. The largest angle of E and M is indicated by the asymptotic lines (dotted style) of the boundary curves. For $m = 0$, the intersection point of the asymptotic lines lies on M_0, but for increasing values of $|m|$ this does not hold in general and the tolerance regions will become asymmetrically. For large values of $|m|$ this intersection point might even be outside the region, and the canonical Euclidean metric in $\mathbb{R}^4$ is then no longer useful for the definition of distances between planes.

The tolerance zone of an oriented plane M is rotationally symmetric with respect to the normal $\mathbf{n}$ of M passing through the origin. In the planes through $\mathbf{n}$ there appears the 2D-case, so that the 2D-case is sufficient for visualization.

The introduced metric is not invariant under all Euclidean motions of the space $\mathbb{R}^3$. The metric is invariant with respect to rotations about the origin, but this does not hold for translations. If the distance $d = m$ of the plane

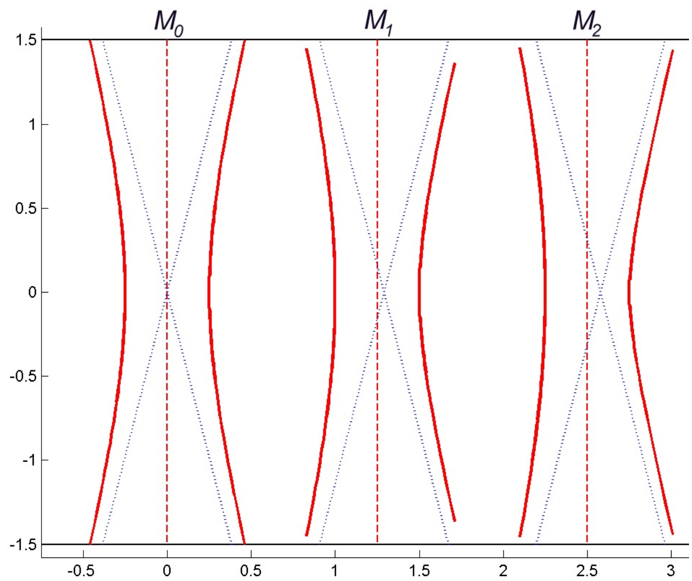

Fig. 2. Boundary curves of the tolerance regions of the center lines M_i.

M to the origin changes, then the shape of the tolerance region changes, too. However, within an area of interest around the origin (e.g. $|m| < 1$), these changes are small and thus the introduced metric is useful.

In practice, we uniformly scale the data in a way that the absolute values of all coordinates x_i, y_i, z_i are smaller than $c = 1/\sqrt{3}$. Then the object is contained in a cube, bounded by the planes $x = \pm c, y = \pm c, z = \pm c$ and the maximum distance of a data point $\mathbf{p}_i$ to the origin is 1. Considering planes passing through the data points $\mathbf{p}_i$, the maximum distance $\mathrm{dist}(O, E)$ of a plane E to the origin is also 1.

According to the normalization $e_1^2 + e_2^2 + e_3^2 = 1$, the distance of the Blaschke image $b(E)$ to the origin in $\mathbb{R}^4$ is bounded by 1. This is also important for a discretization of the Blaschke cylinder which we discuss in the following.

4.3 A cell decomposition of the Blaschke cylinder

For practical computations on B we use a cell decomposition of B to define neighborhoods of image points $b(T)$ of (estimated) tangent planes T. We recall that B's equation is $u_1^2+u_2^2+u_3^2 = 1$. Any cross section with a plane $u_4 = \text{const.}$ is a copy of the unit sphere S^2 in $\mathbb{R}^3$. In order to obtain a cell decomposition of B, we start with a triangular decomposition of S^2 and lift it to B.

A tessellation of S^2 can be based on the net of a regular icosahedron. The vertices $v_i, i = 1, \ldots, 12$, with $\|v_i\| = 1$ of a regular icosahedron form twenty triangles t_j and thirty edges. All edges have same arc length. This icosahedral net is subdivided by computing the midpoints of all edges (geodesic circles). Any triangle t_j is subdivided into four new triangles. The inner triangle has equal edge lengths, the outer three have not, but the lengths of the edges to not vary too much. By repeated subdivision one obtains a finer tessellation of the unit sphere.

The cell decomposition of the Blaschke cylinder consists of triangular prismatic cells which are lifted from the triangular tessellation of S^2 in u_4-direction. Since we measure distances according to (19), the height of a prismatic cell has to be approximately equal to the edge length of a triangle. When each triangle of the tessellation is subdivided into four children, each interval in u_4-direction is split into two subintervals.

According to the scaling of the data points $\mathbf{p}_i$, the coordinates of the image points $b(E)$ on B are bounded by ± 1. We start with 20 triangles, 12 vertices and 2 intervals in u_4-direction. The test-implementation uses the resolution after three subdivision steps with 1280 triangles, 642 vertices and 16 intervals in u_4-direction. In addition to the cell structure on B we store adjacency information of these cells.

Remark concerning the visualization: It is easy to visualize the spherical image (first three coordinates) on S^2, but it is hard to visualize the Blaschke image on B. We confine ourself to plot the spherical image on S^2, and if necessary, we add the fourth coordinate (support function) in a separate figure. This seems to be an appropriate visualization of the geometry on the Blaschke cylinder, see Figures 3, 4, 5, 6, 7.

4.4 Analysis and classification of the Blaschke image

Having computed estimates T_i of the tangent planes of the data points and their images $b(T_i)$, we check whether the Blaschke image of the considered surface is *curve-like*. According to Section 4.3, the interesting part of the Blaschke cylinder B is covered by 1280×16 cells C_k. We compute the memberships of image points $b(T_i)$ and cells C_k and obtain a binary image on the cell structure C of B. Let us recall some basic properties of the Blaschke image of a surface.

(1) If the data points $\mathbf{p}_i$ are contained in a single plane P, the image points $b(T_i)$ of estimated tangent planes T_i form a *point-like cluster* around $b(P)$ on B.
(2) If the data points $\mathbf{p}_i$ are contained in a developable surface, the image points $b(T_i)$ form a curve-like region in B, see Figures 3, 4, 5, 6.
(3) If the data points $\mathbf{p}_i$ are contained in a doubly curved surface S, the image points $b(T_i)$ cover a two-dimensional region on B.
(4) If the data points $\mathbf{p}_i$ are contained in a spherical surface S, the image

points $b(T_i)$ cover a two-dimensional region on B which is contained in a three-space.

In the following we assume that the data comes from a smooth developable surface. Since the estimation of tangent planes gives bad results on the boundary of the surface patch and near measurement errors, there will be outliers in the Blaschke image. To find those, we search for cells C_k carrying only a few image points. These cells and image points are not considered for the further computations. The result is referred to as *cleaned Blaschke image*. In addition, a thinning of the Blaschke image can be performed.

After having analyzed and cleaned the Blaschke image from outliers we are able to decide whether the given developable surface D is a general cone or cylinder, a cone or cylinder of revolution, another special developable or a general developable surface.

So, let T_i, $i = 1, \ldots, M$ be the reliable estimated tangent planes of D after the cleaning and let $b(T_i) = \mathbf{b}_i$ be their Blaschke images. As we have worked out in Section 3 we can classify the type of the developable surface D in the following way.

To check if the point cloud $\mathbf{b}_i$ on B can be fitted well by a hyperplane H,

$$H: \; h_0 + h_1 u_1 + \ldots + h_4 u_4 = 0, \quad h_1^2 + \ldots + h_4^2 = 1. \tag{22}$$

we perform a *principal component analysis* on the points $\mathbf{b}_i$. This is equivalent to computing the ellipsoid of inertia of the points $\mathbf{b}_i$. It is known that the best fitting hyperplane passes through the barycenter $\mathbf{c} = (\sum \mathbf{b}_i)/M$ of the M data points $\mathbf{b}_i$. Using $\mathbf{c}$ as new origin, the coordinate vectors of the data points are $\mathbf{q}_i = \mathbf{b}_i - \mathbf{c}$ and the unknown three-space H has vanishing coefficient, $h_0 = 0$. The signed Euclidean distance $d(\mathbf{b}_i, H)$ of a point $\mathbf{q}_i$ and the unknown three-space H is

$$d(\mathbf{q}_i, H) = h_1 q_{i,1} + \ldots + h_4 q_{i,4} = \mathbf{h} \cdot \mathbf{q}_i, \tag{23}$$

where $\mathbf{h} = (h_1, \ldots, h_4)$ denotes the unit normal vector of H. The minimization of the sum of squared distances,

$$F(h_1, h_2, h_3, h_4) = \frac{1}{M} \sum_{i=1}^{M} d^2(\mathbf{q}_i, H) = \frac{1}{M} \sum_{i=1}^{M} (\mathbf{q}_i \cdot \mathbf{h}_i)^2. \tag{24}$$

with respect to $\mathbf{h}^2 = 1$ is an ordinary eigenvalue problem. Using a matrix notation with vectors as columns, it is written as

$$F(\mathbf{h}) = \mathbf{h}^T \cdot C \cdot \mathbf{h}, \quad \text{with } C := \frac{1}{M} \sum_{i=1}^{M} \mathbf{q}_i \cdot \mathbf{q}_i^T. \tag{25}$$

The symmetric matrix C is known as covariance matrix in statistics and as inertia tensor in mechanics. Let λ_i be an eigenvalue of C and let $\mathbf{v}_i$ be the corresponding normalized eigenvector ($\mathbf{v}_i^2 = 1$). Then, $\lambda_i = F_2(\mathbf{v}_i)$ holds and thus the best fitting three-space V_1 belongs to the smallest eigenvalue λ_1. The statistical standard deviation of the fit with V_1 is

$$\sigma_1 = \sqrt{\lambda_1/(n-4)}. \tag{26}$$

The *distribution of the eigenvalues* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_4$ of the covariance matrix C (and the corresponding standard deviations $\sigma_1 \leq \cdots \leq \sigma_4$) gives important information on the shape of the surface D:

(1) Two small eigenvalues λ_1, λ_2 and different coefficients h_{10}, h_{20}, ($|h_{10} - h_{20}| > \varepsilon$): The surface D can be well approximated by a *cone of revolution*, compare (17). The vertex and the inclination angle are computed according to Section 3.

(2) Two small eigenvalues λ_1, λ_2 but nearly equal coefficients h_{10}, h_{20}, ($|h_{10} - h_{20}| \leq \varepsilon$): The surface D can be well approximated by a *cylinder of revolution*, compare with (16). The axis and the radius are computed according to Section 3.

(3) One small eigenvalue λ_1 and small coefficient h_{10} (compare with (13)): The surface D is a *general cone* and its vertex is

$$\mathbf{v} = \frac{1}{h_{14}}(h_{11}, h_{12}, h_{13}).$$

(4) One small eigenvalue λ_1 and small coefficients h_{10} and h_{14} (compare with (12)): The surface D is a *general cylinder* and its axis is parallel to the vector

$$\mathbf{a} = (h_{11}, h_{12}, h_{13}).$$

(5) One small eigenvalue λ_1 and small coefficient h_{14} (compare with (14)): The surface D is a *developable of constant slope*. The tangent planes of D form a constant angle with respect to an axis. The angle and the axis are found according to formula (14). An example is displayed in Figure 4.

(6) One small eigenvalue λ_1 characterizes a developable surface D whose tangent planes T_i are *tangent to a sphere* (compare with (15)). Its center

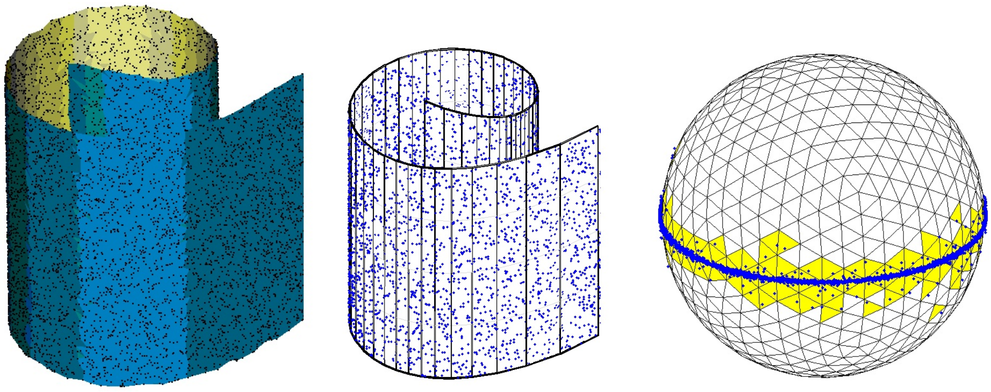

Fig. 3. Left: General cylinder. Middle: Triangulated data points and approximation. Right: Original Blaschke image (projected onto S^2).

and radius are

$$\mathbf{m} = \frac{1}{h_{14}}(h_{11}, h_{12}, h_{13}), \quad r = \frac{-h_{10}}{h_{14}}.$$

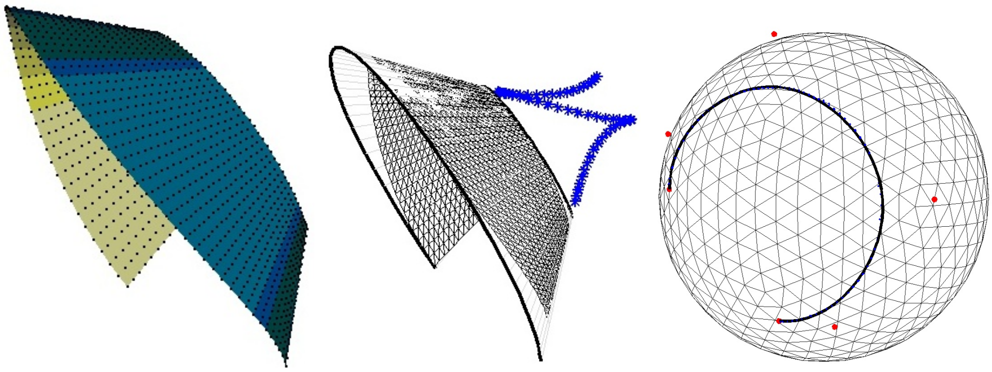

Fig. 4. Left: Developable of constant slope (math. model). Middle: Triangulated data points and approximation. Stars represent the singular curve. Right: Spherical image of the approximation with control points

For this classification we need to fix a threshold ε, to decide what *small* means. This value depends on the accuracy of the measurement device, the number of data points per area unit and the accuracy of the object. Some experience is necessary to choose this value for particular applications.

5 Reconstruction of developable surfaces from measurements

In this section we describe the construction of a best-fitting developable surface to data points $\mathbf{p}_i$ or to estimated tangent planes T_i. In addition we address some problems, in particular the control of the singular curve of the approximation. First we note some general demands on the surface D to be approximated.

(1) D is a smooth surface not carrying singular points. D is not necessarily exactly developable, but one can run the algorithm also for nearly developable surfaces (one small principal curvature).
(2) The density of data points $\mathbf{p}_i$ has to be approximately the same everywhere.
(3) The image $b(T_i)$ of the set of (estimated) tangent planes T_i has to be a simple, curve-like region on the Blaschke cylinder which can be injectively parameterized over an interval.

According to the made assumptions, the reconstruction of a set of measurement point $\mathbf{p}_i$ of a developable surface D can be divided into the following tasks:

(1) Fitting a curve $\mathbf{c}(t) \subset B$ to the curve-like region formed by the data points $b(T_i)$.
(2) Computation of the one-parameter family of planes $E(t)$ in $\mathbb{R}^3$ and of the generating lines $L(t)$ of the developable D^* which approximates measurements $\mathbf{p}_i$.
(3) Computation of the boundary curves of D^* with respect to the domain of interest in $\mathbb{R}^3$.

5.1 Curve fitting on the Blaschke cylinder B

We are given a set of unorganized data points $b(T_i) \in B$ and according to the made assumptions these points form a curve-like region on the Blaschke cylinder B. The aim is to fit a parametrized curve $\mathbf{c}(t) \subset B$ to these points. In order to satisfy the constraint $\mathbf{c}(t) \subset B$ we have to guarantee that

$$c_1(t)^2 + c_2(t)^2 + c_3(t)^2 = 1, \tag{27}$$

which says that the projection $\mathbf{c}'(t) = (c_1, c_2, c_3)(t)$ of $\mathbf{c}(t) = (c_1, c_2, c_3, c_4)(t)$ to $\mathbb{R}^3$ is a *spherical curve* (in S^2). The computation of a best fitting curve to unorganized points is not trivial, but there are several methods around. Estimation of parameter values or sorting the points are useful ingredients to simplify the fitting. We do not go into detail here but refer to the *moving least squares method* to estimate parameter values and to the approach by Lee [11] who uses a minimum spanning tree to define an ordering of the points. These methods apply also to thinning of the curve-like point cloud.

After this preparation we perform standard curve approximation with B-splines and project the solution curve to the Blaschke cylinder B in order to satisfy the constraint (27). If the projection $\mathbf{c}' \subset S^2$ of $\mathbf{c}(t)$ is contained in a hemisphere of S^2 and if additionally the fourth coordinate $c_4(t)$ does not vary to much, it is appropriate to perform a *stereographic projection* so that we finally end up with a rational curve $\mathbf{c}(t)$ on B. For practical purposes it will often be sufficient to apply a projection to B with rays orthogonally to u_4, the axis of B.

Figure 5 shows a curve-like region in S^2 with varying width, an approximating curve $\mathbf{c}'(t)$ to this region and the approximation $c_4(t)$ of the support function to a set of image points $b(T_i) \subset B$.

We mention here that the presented curve fitting will fail in the case when inflection generators occur in the original developable shape, because inflection generators correspond to singularities of the Blaschke image. Theoretically, we have to split the data set at an inflection generator and run the algorithm for the parts separately and join the partial solutions. In practice, however, it is not so easy to detect this particular situation and it is not yet implemented.

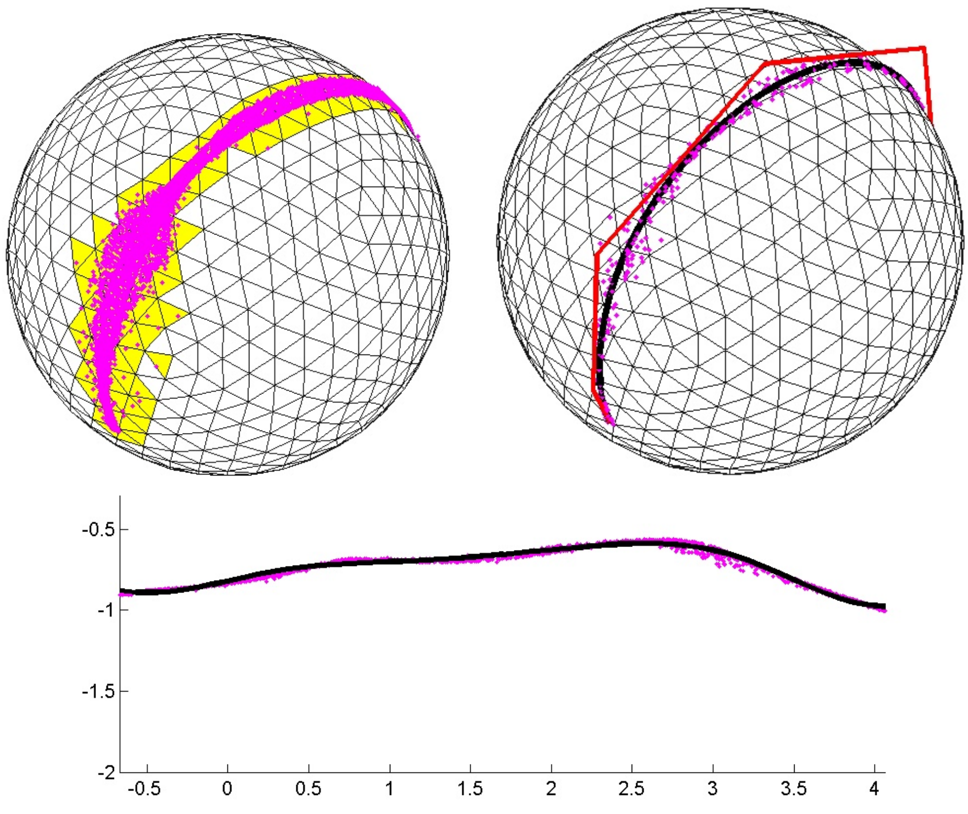

Fig. 5. Blaschke image (left) (projected onto S^2), approximating curve to thinned point cloud (right) and support function (fourth coordinate)

5.2 Biarcs in the space of planes

We like to mention an interesting relation to biarcs. Biarcs are curves composed of circular arcs with tangent continuity and have been studied at first in the plane, see e.g. [12]. It is known that the G^1-Hermite interpolation problem of Hermite elements (points plus tangent lines) P_1, V_1 and P_2, V_2 possesses a one-parameter solution with biarcs which can be parameterized over the projective line. Usually one can expect that suitable solutions exist, but for some configurations there are no solutions with respect to a given orientation of the tangent lines V_j.

The construction of biarcs can be carried out on quadrics too, see [13], in particular on the sphere S^2 or on the Blaschke cylinder B. If we consider a biarc (elliptic) $\mathbf{c} = b(D) \subset B$ then the corresponding developable surface D in $\mathbb{R}^3$ is composed of cones or cylinders of revolution with tangent plane continuity along a common generator, see [14,15]. To apply this in our context we sample Hermite elements P_j, V_j, $j = 1, \ldots, n$ from an approximation $\mathbf{c}(t) \subset B$ of the set $b(T_i)$. Any pair of Hermite elements P_j, V_j and P_{j+1}, V_{j+1} is interpolated by a pair of elliptic arcs on B with tangent continuity. Applying this concept, the final developable surface is composed of smoothly joined cones of revolution. This has the advantage that the development (unfolding) of the surface is elementary.

5.3 A parametrization of the developable surface

Once we have computed a curve $\mathbf{c}(t) \subset B$ that approximates the image points $b(T_i)$ well, the one-parameter family $E(t)$ determining the approximating developable surface D^* is already given by

$$E(t) : c_4(t) + c_1(t)x + c_2(t)y + c_3(t)z = 0.$$

The generating lines $L(t)$ of D^* are the intersection lines $E(t) \cap \dot{E}(t)$. We assume that there exist two bounding planes H_1 and H_2 of the domain of interest in a way that all generating lines $L(t)$ intersect H_1 and H_2 in proper points. The intersection curves $\mathbf{f}_i(t)$ of $L(t)$ and $H_i, i = 1, 2$ are computed by

$$\mathbf{f}_i(t) = E(t) \cap \dot{E}(t) \cap H_i, \tag{28}$$

and the final point representation of D^* is

$$\mathbf{x}(t, u) = (1 - u)\mathbf{f}_1(t) + u\mathbf{f}_2(t). \tag{29}$$

Figures 3, 4, 6 and 7 show developable surfaces which approximate data points (displayed as dots).

The deviation or *distance* between the given surface D and the approximation D^* can be defined according to distances between estimated planes T_i, $i = 1, \ldots, N$ (with corresponding parameter values t_i) and the approximation $E(t)$ by

$$d^2(D, D^*) = \frac{1}{N} \sum_i \mathrm{dist}^2(T_i, E(t_i)). \tag{30}$$

If more emphasis is on the deviation of the measurements $\mathbf{p}_i$ from the developable D^*, one can use

$$d^2(D, D^*) = \frac{1}{N} \sum_i \mathrm{dist}^2(\mathbf{p}_i, E(t_i)), \tag{31}$$

with respect to orthogonal distances between points $\mathbf{p}_i$ and planes $E(t_i)$.

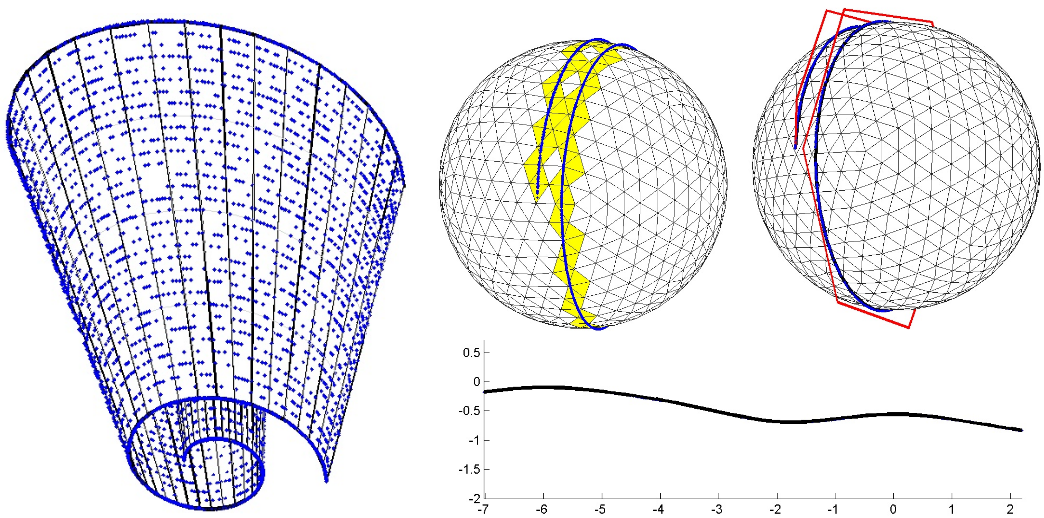

Fig. 6. Left: Developable surface approximating the data points. Right: Projection of the Blaschke image onto S^2, approximating curve with control polygon and support function.

5.4 Fitting developable surfaces to nearly developable shapes

The proposed method can be applied also to fit a developable surface to data which comes from a nearly developable shape. Of course, we have to specify what *nearly developable* means in this context. Since the fitting is performed by fitting a one-parameter family of tangent planes, we will formulate the requirements on the data $\mathbf{p}_i$ in terms of the Blaschke image of the estimated tangent planes T_i.

If the data points $\mathbf{p}_i$ are measurements of a developable surface D and if the width in direction of the generators does not vary too much, the Blaschke image $b(D) = R$ will be a tubular-like (curve-like) region on B with nearly constant thickness. Its boundary looks like a pipe surface.

Putting small distortions to D, the normals of D will have a larger variation near these distortions. The Blaschke image $b(D)$ possesses a larger width locally and will look like a canal surface. As long as it is still possible to compute a fitting curve to $b(D)$, we can run the algorithm and obtain a developable surface approximating D. Figures 5 and 7 illustrate the projection of $b(D)$ onto the unit sphere S^2.

The analysis of the Blaschke image $b(D)$ gives a possibility to check whether D can be approximated by a developable surface or not. By using the cell structure of the cleaned Blaschke image $b(D)$, we pick a cell C and an appropriately chosen neighborhood U of C. Forming the intersection $R = U \cap b(D)$, we compute the ellipsoid of inertia (or a principal component analysis) of R. The existence of *one significantly larger eigenvalue* indicates that R can be approximated by a curve in a stable way. Thus, the point set corresponding to R can be fitted by a developable surface.

Since approximations of nearly developable shapes by developable surfaces are quite useful for practical purposes this topic will be investigated in more detail in the future.

5.5 Singular points of a developable surface

So far, we did not pay any attention to singular points of D^*. The control and avoidance of the singular points within the domain of interest is a complicated topic because the integration of this into the curve fitting is quite difficult.

If the developable surface D^* is given by a point representation, formula (4) represents the singular curve $\mathbf{s}(t)$. If D^* is given by its tangent planes $E(t)$, the singular curve $\mathbf{s}(t)$ is the envelope of the generators $L(t)$ and so it is computed by

$$\mathbf{s}(t) = E(t) \cap \dot{E}(t) \cap \ddot{E}(t). \tag{32}$$

Thus, the singular curve $\mathbf{s}(t)$ depends in a highly nonlinear way on the coordinate functions of $E(t)$.

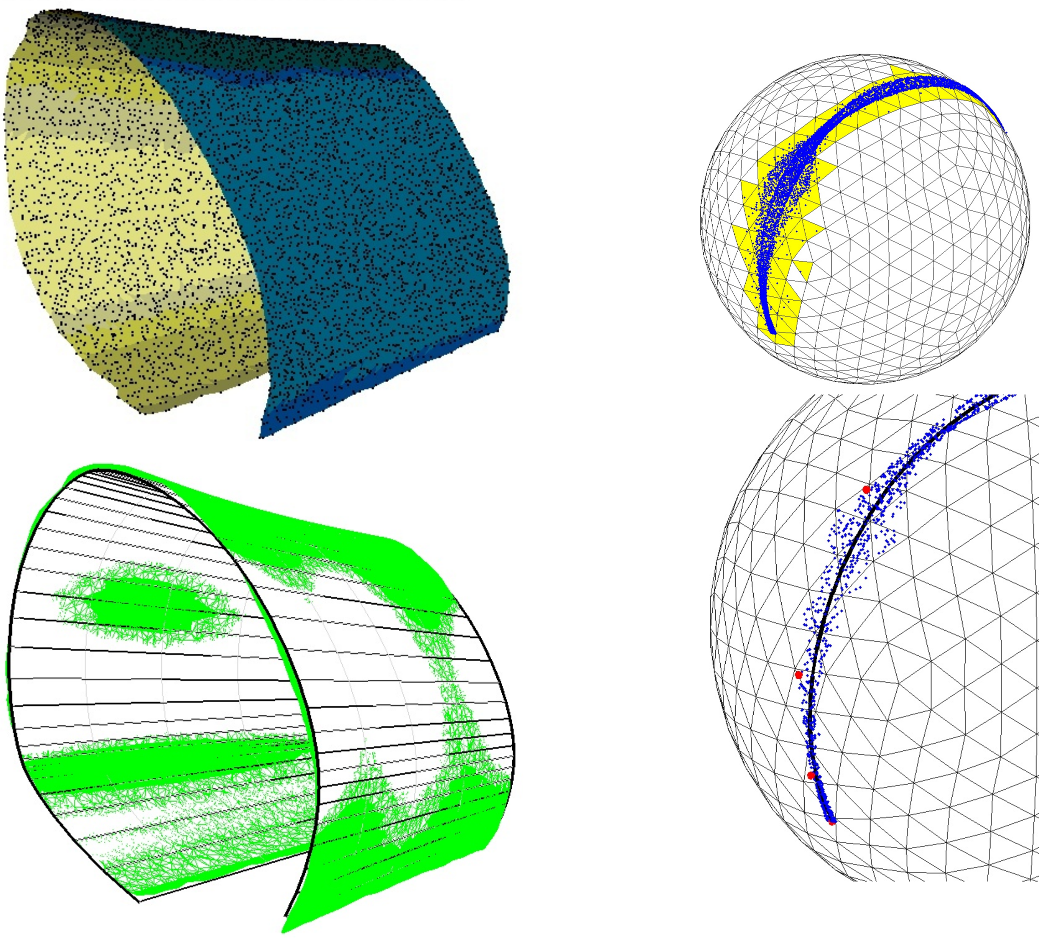

Fig. 7. Left: Nearly developable surface and developable approximation. Right: Projection of the original Blaschke image onto S^2 and thinned Blaschke image with approximating curve.

In order to compute the singular curve $\mathbf{s}(t)$, let $\mathbf{n} = \mathbf{c} \wedge \dot{\mathbf{c}} \wedge \ddot{\mathbf{c}}$, where $\wedge$ denotes the vector product in $\mathbb{R}^4$. The Cartesian coordinates of the singular curve are then found by

$$\mathbf{s}(t) = \frac{1}{n_4(t)}(n_1(t), n_2(t), n_3(t)). \tag{33}$$

Zeros of the function n_4 correspond to points at infinity of $\mathbf{s}(t)$. In Section 4.2 we have assumed that all coordinates of data points are bounded by $\pm c$ such that we have $\|\mathbf{p}_i\| \leq 1$. In order to approximate the data with singularity-free developable surfaces, we have to guarantee

$$\|\mathbf{s}(t)\| > 1, \tag{34}$$

when we fit curves $\mathbf{c}(t)$ to image points $b(T_i) \in B$ of estimated tangent planes T_i. Since the data comes from a developable surface without singularities, we can expect that there exist solutions satisfying (34).

Assuming that the curve $\mathbf{c}(t) \in B$ fitted to the data $b(T_i)$ is composed of biarcs, we obtain the following: For two consecutive Hermite elements P_j, V_j and P_{j+1}, V_{j+1} there exists a one-parameter family of interpolating pairs of arcs and condition (34) leads to a quadratic inequality. Thus, solutions can be computed explicitly. However, as we have mentioned in Section 5.2, there is no guarantee that feasible solutions exist and the construction clearly depends on the choice of the Hermite elements which have been sampled from an initial solution of the curve fitting.

5.6 Conclusion

We have proposed a method for fitting a developable surface to data points coming from a developable or a nearly developable shape. The approach applies curve approximation in the space of planes to the set of estimated tangent planes of the shape. This approach has advantages compared to usual surface fitting techniques, like

- avoiding the estimation of parameter values and direction of generators,
- guaranteeing that the approximation is developable.

The detection of regions containing inflection generators, and the avoidance of singular points on the fitted developable surface have still to be improved. The approximation of nearly developable shapes by developable surface is an interesting topic for future research. In particular we will study the segmentation of a non-developable shape into parts which can be well approximated by developable surfaces. This problem is relevant in certain applications (architecture, ship hull manufacturing), although one cannot expect that the developable parts will fit together with tangent plane continuity.

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