Dissertation

Rational Parametrizations for Envelopes of Quadric Families

at the Institute of Geometry
University of Technology, Vienna

by

Martin Peternell

Zusammenfassung

Das ursprüngliche Ziel dieser Arbeit war das Studium von rationalen Flächen im euklidischen dreidimensionalen Raum, die rationale Parallelflächen (PN Flächen) besitzen. Ferner sollten diese Flächen von einem gewissen praktischen Interesse sein, das heißt, eine einfache geometrische Erzeugung gestatten. Angeregt wurde diese Arbeit einerseits durch die Arbeiten von Prof. Farouki über polynomiale PH Kurven, andererseits durch die Arbeiten meines Lehrers Prof. Pottmann, der unter Benützung der dualen Darstellung ebener Kurven und Flächen im Raum eine allgemeine Beschreibung rationaler PH Kurven und PN Flächen angegeben hat.

Die ersten Untersuchungen betrafen Kanalflächen mit rationaler Mittenlinie und rationaler Radiusfunktion. Der Beweis, daß diese Flächen rationale Parallelflächen besitzen, benützt aber nur, daß sie Einhüllende einer rationalen einparametrigen Schar von Drehkegeln, mit rationaler Radiusfunktion, sind. Damit ist eine wesentlich größere Flächenklasse mit obiger Eigenschaft gefunden.

Im ersten Kapitel werden jene Flächen im projektiven dreidimensionalen Raum studiert, die eine rationale einparametrige Schar von Kegelschnitten tragen. Es stellt sich heraus, daß diese Flächen rational sind. Dies bietet eine geometrische Grundlage für weitere Untersuchungen.

Das zweite Kapitel ist, aus projektiver Sicht, dual zum ersten. Aber für die Anwendungen ist es trotzdem nützlich die Hüllflächen einer rationalen einparametrigen Schar von quadratischen Kegeln zu studieren. Denn eine Unterfamilie bilden jene rationalen Kanalflächen, die durch rationale Kurven im vierdimensionalen Möbiusmodell der Kugeln des dreidimensionalen Raumes gegeben sind.

Das dritte Kapitel bietet eine kurze Einführung in die euklidische Laguerre Geometrie. Erstens, weil das Studium von PN Flächen einfacher ist, wenn man sie als Menge von orientierten Tangentialebenen auffaßt, und zweitens weil eine Fläche und ihre Parallelflächen laguerregeometrisch äquivalent sind.

Weiters wird gezeigt, daß alle PN Flächen Reflexionsantikaustiken von rationalen Flächen bei Parallelbeleuchtung sind.

Im vierten Kapitel werden dann PN Flächen studiert. Es gibt im wesentlichen zwei bemerkenswerte Ergebnisse. Das eine ist mehr theoretischer Natur und besagt, wie schon oben erwähnt, daß die Hüllflächen einer einparametrigen rationalen Schar von Drehkegeln

mit rationaler Radiusfunktion, PN Flächen sind. Damit zeigt man auch, daß die Hyperzykliden PN Flächen sind. Diese, von W. Blaschke angegebenen Flächen, beinhalten zum Beispiel die Parallelflächen der regulären Quadriken. Eine weitere Folgerung ist die Rationalität der Parallelflächen von rationalen Regelflächen.

Das zweite, eher praktischere Ergebnis bietet eine allgemeine Konstruktionsmöglichkeit für PN Flächen. Jede rationale Fläche im isotropen Modell der euklidischen Laguerre Geometrie bestimmt eine PN Fläche. Dies wurde dazu verwendet, ein Flächenmodellierungsschema, das PN Flächen benutzt, zu entwickeln.

Im fünften und letzten Kapitel werden gewisse Verallgemeinerungen im n–dimensionalen projektiven und euklidischen Raum studiert. Die Hüllflächen von Drehkegelscharen lassen sich auf diverse Arten verallgemeinern, eine davon ist hier beschrieben. Weiters stellt sich heraus, daß die Parallelhyperflächen von rational parametrisierbaren $k+1$ Regelflächen PN Flächen sind, falls die $k+1$ Regelfläche keine Hyperfläche ist, die konstante Tangentialräume längs der Erzeugenden besitzt.

Hier möchte ich meinem Lehrer, Prof. Helmut Pottmann für seine außerordentlich gewissenhafte Betreuung dieser Dissertation danken. Seine Ideen und Anregungen haben wesentlich zum guten Gelingen dieser Dissertation beigetragen.

Weiters gebührt mein Dank allen meinen Kollegen und Lehrern des Institutes für Geometrie an der Technischen Universität Wien, im besonderen Herrn Prof. Hellmuth Stachel. Teile diese Arbeit waren auch Inhalt eines vom Fonds zur Förderung der wissenschaftlichen Forschung (FWF) finanzierten Projekts über 'Rationale B–Spline Darstellungen funktioneller Formen für die CAD/CAM Technik' (P09790–MAT).

Wien, Oktober 1997 Martin Peternell

Contents

List of Figures

Preface

The original aim of this work was the study of rational surfaces in Euclidean 3–space, which possess rational offset surfaces (PN surfaces). Mainly such surfaces should be studied, which possess an almost simple geometric generation, such that they are of practical interest and use in CAD/CAM. This was mainly motivated by articles of Prof. Farouki about PH curves and by articles of my teacher, Prof. Pottmann, who used the dual representation of curves and surfaces to obtain a general representation of PH curves and PN surfaces.

First of all, canal surfaces determined by a rational center curve and a rational radius function were studied. The proof, that those are rational surfaces with rational offsets admitted the following generalization: envelopes of rational one parameter families of cones of revolution with rational radius function are PN surfaces.

The first chapter presents a much wider class of rational surfaces, which form the geometic background of this work. Those are surfaces generated by a real, rational one parameter family of conics.

The second chapter is mainly dual to the first one. Envelopes of a real, rational one parameter family of arbitrary quadratic cones are studied. A special subclass are rational canal surfaces, determined by a rational curve in the 4–dimensional Möbius model of spheres in Euclidean 3–space.

The third chapter mainly prepares for the study of PN surfaces. It is very helpful to use classical Laguerre geometry and several models. Many geometric properties of PN surfaces are easier understood in these models. This chapter also presents the general result, that PN surfaces are reflectional anticaustics of rational mirror surfaces under illumination with parallel light rays.

In the fourth chapter, PN surfaces are studied in detail. There are two remarkable results. The first one, more theoretically and also mentioned above, says that the envelope of a real rational one parameter family of cones of revolution with constant radius function is in general a PN surface. This also proves that hypercyclides, introduced by W. Blaschke, are PN surfaces. This family of surfaces include offsets of regular quadrics. A further consequence is that the offsets of rational ruled surfaces are PN surfaces.

The second one, more practically, says that any rational surface in the isotropic model of Euclidean Laguerre geometry determines a PN surface in Euclidean 3–space. This result was used to derive a surface modeling scheme, using PN surfaces.

The fifth chapter describes certain generalizations of results of previous chapters to Euclidean and projective n–space. PN surfaces are generalized to PN hypersurfaces. In particular, envelopes of a certain type of quadratic hypercones are discussed. Finally we have the result that offsets of (nearly all) rationally parameterizable ruled $k+1$–manifolds are PN hypersurfaces.

Here I want to thank Prof. Helmut Pottmann for supervising this work. His ideas, suggestions and comments were very helpful that this work turned out well. I also want to thank all members of the Institute of Geometry at the University of Technology in Vienna, especially Prof. Hellmuth Stachel. Additionally, this work was partly supported by the Austrian Science Foundation through project P09790–MAT.

Vienna, October 1997 Martin Peternell

Chapter 1

One Parameter Families of Conics

In this chapter, some important geometric properties of surfaces, generated by one parameter families of conics, so called *conic surfaces* shall be described. For a detailed description of conic surfaces see [9], [10], [11] and [48]. Further we will prove that a real rational family of conics generates a rational surface.

1.1 Geometric Properties of Conic Surfaces

Let $x = (x_0, \dots, x_3)$ be homogeneous coordinates of points and $X = (X_0, \dots, X_3)$ be homogeneous coordinates of planes with respect to an arbitrary, but fixed coordinate system in real projective 3–space $\mathbb{P}^3(\mathbb{R})$. Let $C = c_{jk}$ be a symmetric matrix. A quadratic equation

$$c : \sum_{j,k=0}^{3} c_{jk} X_j X_k = 0, \quad \text{with}\, c_{jk} = c_{kj}. \tag{1.1}$$

defines a (not necessarily regular) quadric in $\mathbb{P}^3$ as set of tangent planes. The quadric is called singular, if $\det(C) = 0$. In detail, if $\mathrm{rk}(C) = 3$, c is a conic as set of tangent planes (see Figure 1.1). The case $\mathrm{rk}(C) = 2$ characterizes a pair of bundles of planes and finally $\mathrm{rk}(C) = 1$ defines a double counted bundle of planes.

Since the coefficients c_{jk} are only determined up to a real common factor, they can be interpreted as homogeneous coordinates of a point C in $\mathbb{P}^9$. A mapping $\mathbb{P}^3 \to \mathbb{P}^9$ can be defined, which maps a quadric of $\mathbb{P}^3$ to a point of $\mathbb{P}^9$. Thus we identify points in $\mathbb{P}^9$ and symmetric matrices describing quadrics in $\mathbb{P}^3$. A line $\lambda C + \mu D$ in $\mathbb{P}^9$ corresponds to a pencil of quadrics $\lambda c + \mu d$ in $\mathbb{P}^3$, where c and d are quadrics determined by matrices c_{jk} and d_{jk}. The image points of singular quadrics, especially conics, are contained in an algebraic variety M of dimension 8 and degree 4, given by the equation $\det(C) = 0$. Since the rank of a matrix c_{jk} is invariant under projective mappings in $\mathbb{P}^3$ special subvarieties of M are of interest. They are defined by

$$D : \mathrm{rk}(C) \leq 2 \quad \text{and}\, V : \mathrm{rk}(C) = 1. \tag{1.2}$$

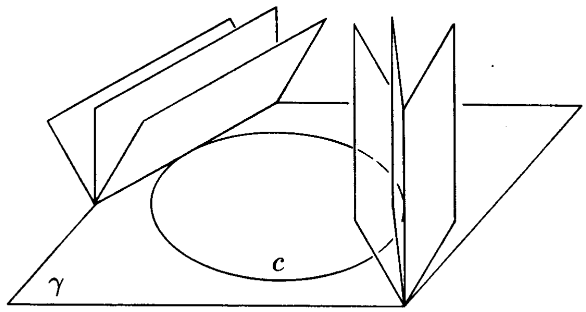

Figure 1.1: Conic as set of tangent planes

It can be shown that D consists of all singular points of M. Considering D as algebraic variety, V consists of all singular points of D.

It follows that V is a Veronese surface, namely the image of $\mathbb{P}^3$ under a quadratic Veronese map. The projective mappings in $\mathbb{P}^3$ induce in $\mathbb{P}^9$ automorphic projective mappings of these varieties. The set $M - D$ corresponds to conics c in $\mathbb{P}^3$.

Let $c(t)$ be a one parameter family of regular conics, defined by functions $c_{jk}(t)$ depending on a real parameter t, which varies in an interval $I \subset \mathbb{R}$. This defines a curve $C(t)$ on M and we require that $C(t)$ shall be sufficiently smooth. If the conics $c(t)$ are not contained in a fixed plane they generate a surface $\Phi \subset \mathbb{P}^3$. These surfaces are called *conic surfaces*, and the curves $c(t)$ are called *generating conics.* A conic surface Φ can be defined as envelope of common tangent planes of the pencils of dual quadrics $\lambda c(t) + \mu\dot{c}(t)$. These tangent planes are solutions of

$$c(t) : \sum_{j,k=0}^{3} c_{jk}(t) X_j X_k = 0 \text{ and } \dot{c}(t) : \sum_{j,k=0}^{3} \dot{c}_{jk}(t) X_j X_k = 0, \tag{1.3}$$

and define for a fixed $t \in \mathbb{R}$ a developable surface $D_c(t)$. Let $\gamma(t)$ be the plane containing $c(t)$. The homogeneous coordinates $(\gamma_0, \ldots, \gamma_3)(t)$ are solutions of the homogeneous linear system

$$\sum_{k=0}^{3} c_{jk}(t) X_k = 0, \qquad \text{for } i = 0, 1, 2, 3. \tag{1.4}$$

If $\gamma(t)$ and $\dot{\gamma}(t)$ are linearly independent, the envelope of $\gamma(t)$ is a developable surface B. Its generator lines are $g(t) = \gamma(t) \cap \dot{\gamma}(t)$. Depending on the number $0, 1$ or 2 of real intersection points of $c \cap g$, the corresponding conic surfaces are called *elliptic, parabolic* or *hyperbolic.* This is a local property, such that also the conic c itself shall be called *elliptic, parabolic* or *hyperbolic.* See Figures 1.2 and 1.3.

The plane γ satisfies $\sum_{k=0}^{3} c_{jk}\gamma_k = 0$ for each j and also $\sum_{j,k=0}^{3} c_{jk}\gamma_j\gamma_k = 0$ identically for all $t \in I$. The derivative with respect to t is

$$\sum_{j,k} \dot{c}_{jk}\gamma_j\gamma_k + \sum_{j,k} c_{jk}\dot{\gamma}_j\gamma_k + \sum_{j,k} c_{jk}\gamma_j\dot{\gamma}_k = 0. \tag{1.5}$$

Since the last two terms are zero, the identity

$$\sum_{j,k} \dot{c}_{jk}\gamma_j\gamma_k = 0 \tag{1.6}$$

holds, which implies that γ is tangent to all regular dual quadrics of the pencil $\lambda c + \mu\dot{c}$. The tangent planes of Φ in points of a generating conic c are common tangent planes of the pencil $\lambda c+\mu\dot{c}$. The developable surface $D_c(t)$ is rational of class 4. In the general case, namely case A of Thomsens classification (see below), $D_c(t)$ contains γ as double plane.

The derivative of (1.6) is

$$\sum_{j,k} \ddot{c}_{jk}\gamma_j\gamma_k + \sum_{j,k} \dot{c}_{jk}\dot{\gamma}_j\gamma_k + \sum_{j,k} \dot{c}_{jk}\gamma_j\dot{\gamma}_k = 0. \tag{1.7}$$

Let p be the point of contact between γ and any arbitrary regular dual quadric of $\lambda c+\mu\dot{c}$. Its homogeneous coordinates are

$$p_j = \sum_k \dot{c}_{jk}\gamma_k = -\sum_k c_{jk}\dot{\gamma}_k, \quad j = 0,1,2,3 \tag{1.8}$$

By the way, p is also the pole of g with respect to c, considered as a planar curve in γ, see Figure 1.2. Equation (1.7) can be read as

$$\sum_{j,k} \ddot{c}_{jk}\gamma_j\gamma_k + 2\sum_j p_j\dot{\gamma}_j = 0, \tag{1.9}$$

which leads to a characterization of parabolic conic surfaces.

Lemma 1.1 *Let c be a regular conic on $\Phi = \{c(t), t \in I\}$. Let c, $\dot{c}$ be linearly independent, and also let γ and $\dot{\gamma}$ be independent in I. The conic surface is parabolic in c if and only if the tangent line $T = \lambda C + \mu\dot{C}$ is an asymptotic line of M.*

Proof: If $p \in c$ it follows with (1.9) that γ is tangent to $\ddot{c}$, and further to all quadrics of the linear system $\lambda c + \mu\dot{c} + \nu\ddot{c}$. Translating this fact to $\mathbb{P}^9$ it follows that the linear subspace $E = C \vee \dot{C} \vee \ddot{C}$ is tangent to M in C. If E is a plane, the tangent line $T = C \vee \dot{C}$ is an asymptotic line of M. Otherwise $(C, \dot{C}, \ddot{C})$ are linearly dependent, such that C is an inflection point of $C(t)$ and T is again an asymptotic line of M.

Conversely, let T be an asymptotic line of M. Either $E = C \vee \dot{C} \vee \ddot{C}$ span a plane, which is tangent to M or these points are linearly dependent, such that C is an inflection point of $C(t)$. In both cases $\lambda C+\mu\ddot{C}$ is a tangent line of M. It follows that the developable defined by $\lambda c + \nu\ddot{c}$ contains γ and $\ddot{c}$ is tangent to γ. Formula (1.9) implies that $p \in c$. $\square$

The following local classification of conic surfaces is due to Thomsen [48]. It is based on the discussion of intersection points of the tangent line $T = \lambda C(t) + \mu \dot{C}(t)$ with M. Inserting a representation of T into an equation of M results in a polynomial φ of degree 4 in the homogeneous projective parameter $(\lambda : \mu)$. Since T is tangent to M, μ is a double zero of φ, such that T intersects M in general in two other points B_1 and B_2. Let us list the following cases, where A–C correspond to elliptic and hyperbolic conics, and D–F correspond to parabolic conics on a conic surface.

A The polynomial φ possesses two distinct further zeros. Corresponding to them are two different regular conics b_1 and b_2. The developable D_c is irreducible of class 4 and possesses γ as double plane.

B The polynomial φ possesses a further double zero, such that $b_1 = b_2 = b$. The conic b is always regular. Further D_c contains a rational developable of class 3 and a pencil of planes, passing through the line $\beta \cap \gamma$, where β is the plane containing b.

C The tangent line T intersects M in a further singular point, which corresponds to a singular conic b, namely to a pair of bundles of planes. The developable D_c contains a quadratic cone and two pencils of planes. These pencils pass through the tangent lines of c through p.

D The polynomial φ possesses μ as a triple zero and a further zero, which leads to a regular conic b in the pencil of quadrics $\lambda c + \mu \dot{c}$. The tangent line T is an asymptotic line of M. The point of contact p, at which the quadrics $\lambda c + \mu \dot{c}$ are tangent to γ is contained in c. The developable D_c is of class 4.

E The polynomial φ possesses μ as only zero. The tangent line $\lambda C + \mu \dot{C}$ is a special asymptotic line of M, such that $p \in c$.

F The polynomial φ is identically zero for all $(\lambda : \mu)$, such that the tangent line $\lambda C + \mu \dot{C}$ is contained in M. All quadrics of the pencil $\lambda c + \mu \dot{c}$ are conics, contained in a pencil of planes, passing through g. They possess the common point p and the common tangent line g.

The local differential geometry of conic surfaces is well studied, mainly with respect to the projective linear group $PGL(\mathbb{P}^3)$. For details on this topic the reader is referred to [48], [9] and [10]. In this report, surfaces generated by a real, rational one parameter family of conics $c(t)$ shall be studied from a more algebraic point of view.

Definition: A one parameter family of conics is called *rational*, if c_{jk} are rational functions. This implies that $C(t)$ is a rational curve on M.

Since c_{jk} are only determined up to a non zero real factor, let assume that these functions are polynomials $\in \mathbb{R}[t]$. Furthermore, a conic $c(t)$ is called *real*, if $c(t)$ contains real points for each real parameter t. The number of real points has to be ≥ 2.

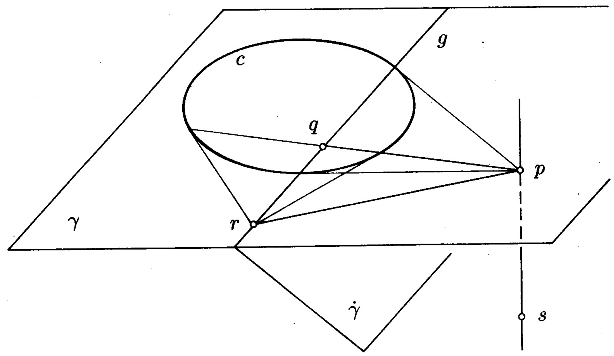

Figure 1.2: Local properties of an ell./hyp. conic surface

It shall be proved that these surfaces Φ possess rational parametrizations. In the first step it shall be shown that there exist rational curves $f(t)$ on Φ, which are different from the generating conics $c(t)$ and possess the following property. For all $t \in \mathbb{R}$ the point $f(t)$ is contained in the conic $c(t)$. In the second step each conic $c(t)$ can be rationally parametrized, for instance in a parameter u, using stereographic projection with center $f(t)$. This will be described in Section 1.4.

1.2 Local Coordinate Systems

In local differential geometry one studies objects with help of moving frames. For conic surfaces this is done in [9], [10]. Since we are only studying surfaces, defined by rational functions $c_{jk}(t)$, the coordinate transformations and the frames are restricted to be rational. Let us start with the elliptic and hyperbolic case. They shall be treated simultaneously since they only differ in the reality of the intersection points $c \cap g$.

1.2.1 Elliptic and Hyperbolic Conic Surfaces

Let Φ be an elliptic or hyperbolic conic surface, defined by a rational one parameter family of conics $c(t)$. Let $c(t)$ be an arbitrary, but fixed conic of Φ. The following construction is depending on the parameter value t, but we avoid to write this if it is not explicitly necessary.

A projective frame (p, q, r, s, e) shall be defined, where p, q, r are contained in γ, $s \notin \gamma$ and e is an appropriate unit point. Instead of the point e one can use an appropriate normalization of the homogeneous coordinates of the points p, q, r and s. Let p be the pole of g with respect to c in γ, defined by formula (1.8), and further let p, q and r be points of a polar triangle of c. This means that q and r are contained in g and are conjugate with respect to c, see Figure 1.2. From formula (1.8) it follows that $p(t)$ is a rational curve. Further $q(t)$ and $r(t)$ can be chosen rationally.

The polarity with respect to any dual quadric of the pencil $\lambda c + \mu \dot{c}$ maps g to a line h, passing through p. If c is of Thomsen type A let b_1, b_2 be conics in this pencil. They intersect h in two, not necessarily distinct, points h_1 and h_2. The fourth vertex s of the frame is usually chosen to be contained in h and determined by $cr(p, s, h_1, h_2) = -1$. So also s is projectively invariant connected with the conic surface. Further s is uniquely determined even in the cases B, C of Thomsens classification (see [9]).

For our purposes this special choice of s is not necessary, such that we only require $s \notin \gamma$, but $s(t)$ rational. Let $\overline{X}_j$ and $\bar{x}_j$ be coordinate vectors of planes and points with respect to the frame (p, q, r, s, e). The conic c, interpreted as set of tangent planes is given by the equation

$$c_0\overline{X}_0^2 + c_1\overline{X}_1^2 + c_2\overline{X}_2^2 = 0. \tag{1.10}$$

The practical calculation reads as follows and shall be split up into two steps. Firstly let γ be the plane $\bar{x}_3 = 0$. Coordinate vectors of planes are transformed by $X = X'T$, where X' are the transformed coordinate vectors and the transformation matrix is

$$T = (t_{jk}) = \begin{bmatrix} \gamma_3 & 0 & 0 & 0 \\ 0 & \gamma_3 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}. \tag{1.11}$$

The conic c is represented by an equation $X'TCT^TX'^T = 0$, which is just a polynomial in X'_0, X'_1, X'_2. Note that this coordinate transformation restricted to γ is just the projection onto $x_3 = 0$ and of course does not work for zeros of γ_3.

In the second step the construction is restricted to the plane γ. A coordinate transformation is defined by choosing p, q and r to be frame points.

The vertex p is the pole of g with respect to c, and the remaining vertices q and r are contained in g. Let η be an arbitrary line in γ passing through p, for instance given by $(0, -p_2, p_1)$, where p_i are planar coordinates of p in γ. Let q be the pole of η with respect to c and $r = g \cap \eta$. Then $\zeta = p \times q$ is a conjugate line to η with respect to c and it is the polar line to r. The now described coordinate transformation is given by the matrix

$$S = (s_{jk}) = \begin{bmatrix} g_0 & g_1 & g_2 & 0 \\ \eta_0 & \eta_1 & \eta_2 & 0 \\ \zeta_0 & \zeta_1 & \zeta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.12}$$

Further the equation of the conic c is transformed to

$$c(t) : \overline{X}\,\overline{C}\,\overline{X}^T = c_0\overline{X}_0^2 + c_1\overline{X}_1^2 + c_2\overline{X}_2^2 = 0, \tag{1.13}$$

with $\overline{C} = (ST)C(ST)^T$. Since S and T possess only rational entries, c_j are assumed to be polynomials in t. Representation (1.13) is not unique, since q and r are just required to be conjugate points on g with respect to c. Additionally, (1.13) defines $c(t)$ as set of tangent lines in the plane γ, where lines in γ are represented by homogeneous coordinates $(\overline{X}_0, \overline{X}_1, \overline{X}_2)$. This interpretation will be used for the construction of rational parametrizations of Φ.

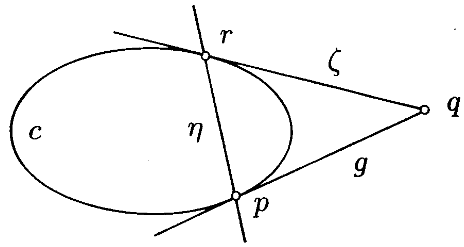

Figure 1.3: Local coordinate system of a parabolic conic surface

1.2.2 Parabolic Conic Surfaces

Let Φ be a conic surface, defined by polynomials $c_{jk}(t)$. Let c be a parabolic conic. We choose a frame (p, q, r, s, e) such that p, q, r are in γ, $s \notin \gamma$ and e is an appropriate unit point. Instead of e again we could use an appropriate normalization of the homogeneous coordinates of p, q, r and s.

Since $p \in c$, the points p q and r do not form a polar triangle. The first step which is the transformation given by T, is analogously to the elliptic/hyperbolic case. To determine S, one may proceed as follows. Let η be an arbitrary line passing through p, but $\eta \neq g$. The pole of η with respect to c is $q \in g$. Further let ζ be the second tangent line to c passing through q, (see Figure 1.3). The third vertex of the frame is $r = \eta \cap \zeta$. The matrix S corresponding to this coordinate transformation possesses equivalent entries as in (1.11), namely the rows of S are the coordinate vectors of g, η and ζ. This results in a representation of c of the form

$$c_{02}\overline{X}_2\overline{X}_0 + c_{11}\overline{X}_1^2 = 0. \tag{1.14}$$

The point representation, which looks like the representation of a parabola in an affine plane, but written in homogeneous coordinates, reads as

$$c_{11}\bar{x}_0\bar{x}_2 + c_{02}\bar{x}_1^2 = 0. \tag{1.15}$$

Note that the condition $p \in c$ can be true just for finitely many values t_j, but also for a non empty interval. Consider a conic surface Φ generated by a rational family of parabolic conics $c(t)$, such that $p \in c$ is an identity in an interval. Obviously Φ possesses rational curves, for instance $p(t)$. Rational parametrizations are constructed with help of a stereographic projection, described in Section 1.4.

1.3 Rational One Parameter Families of Conics in the Plane

Analogously to rational conics in $\mathbb{P}^3$ we need the following definition for planar families.

Definition: We call $c(t) \subset \mathbb{P}^2$ a *rational* one parameter family of conics, if a representation of $c(t)$ exists, such that the coefficients of the defining equation are rational functions.

Let X, Y and Z be homogeneous coordinates of points or lines in $\mathbb{P}^2$. By rational coordinate transformations we may assume that $c(t)$ is given by

$$c(t) : X^2 L(t) + Y^2 M(t) + Z^2 N(t) = 0, \tag{1.16}$$

where L, M and N are considered to be polynomials in $\mathbb{R}[t]$.

Under the assumption that $c(t)$ possesses real points and is an irreducible conic for almost all t except finitely many values, we will prove that there exist real polynomials $x(t)$, $y(t)$ and $z(t)$, which satisfy (1.16) identically and $(x, y, z)(t) \neq (0, 0, 0)$. Firstly, we prove the following.

Corollary 1.1 *Let L, M and N be polynomials in $\mathbb{R}[t] \setminus 0$ which satisfy the following assumptions.*

1. *The polynomials L, M and N define by (1.16) a one parameter family of conics, which possess real points ($\# \geq 2$) for all real t.*
2. *The polynomials L, M and N possess constant signs for all real t.*
3. *The polynomials L, M and N do not have multiple zeros. Further, neither L and M, nor L and N nor M and N possess common zeros.*

Then, there exist polynomials $x(t)$, $y(t)$ and $z(t)$ in $\mathbb{R}[t]$, which satisfy (1.16) identically and $(x, y, z)(t) \neq (0, 0, 0)$.

The assumptions say that $c(t)$ is a regular conic for almost all $t \in \mathbb{R}$. For finitely many, the conic $c(t)$ may degenerate to a pair of lines or a double line. The polynomials L, M and N possess even degrees, say $2l$, $2m$ and $2n$. The zeros of L, M and N shall be denoted by

$$\rho_1, \ldots, \rho_{2l}, \sigma_1, \ldots, \sigma_{2m} \text{ and } \tau_1 \ldots \tau_{2n} \in \mathbb{C} \setminus \mathbb{R}.$$

Let

$$\begin{aligned} x(t) &= x_0 + x_1 t + \ldots + x_{p-1} t^{p-1} + x_p t^p, \\ y(t) &= y_0 + y_1 t + \ldots + y_{q-1} t^{q-1} + y_q t^q, \\ z(t) &= z_0 + z_1 t + \ldots + z_{r-1} t^{r-1} + z_r t^r. \end{aligned} \tag{1.17}$$

with $(p+q+r)+3$ unknown real coefficients and unknown degrees p, q and r. We will see soon that the degrees shall be chosen to be

$$p = m + n, q = l + n, r = l + m. \tag{1.18}$$

For simplicity, let $\tau \in \{\tau_1, \dots, \tau_{2n}\}$ be a zero of N. Analogously, let ρ and σ be zeros of L and M, respectively. Evaluating (1.16) at these zeros leads to

$$x(\tau)^2 L(\tau) + y(\tau)^2 M(\tau) = 0, \tag{1.19}$$
$$y(\sigma)^2 M(\sigma) + z(\sigma)^2 N(\sigma) = 0, \tag{1.20}$$
$$x(\rho)^2 L(\rho) + z(\rho)^2 N(\rho) = 0. \tag{1.21}$$

If $\operatorname{sgn}(L(\tau)) = \operatorname{sgn}(M(\tau))$ the first equation (1.19) factorizes to

$$\left(x(\tau)\sqrt{L(\tau)} + \mathrm{i}y(\tau)\sqrt{M(\tau)}\right)\left(x(\tau)\sqrt{L(\tau)} - \mathrm{i}y(\tau)\sqrt{M(\tau)}\right) = 0. \tag{1.22}$$

Otherwise it factorizes to

$$\left(x(\tau)\sqrt{L(\tau)} + y(\tau)\sqrt{M(\tau)}\right)\left(x(\tau)\sqrt{L(\tau)} - y(\tau)\sqrt{M(\tau)}\right) = 0. \tag{1.23}$$

Analogously equations (1.20) and (1.21) can be factorized. Depending on the signs of L and M, equation (1.19) is satisfied if one of the factors in (1.22) or (1.23) is zero. To determine the coefficients x_i, y_j and z_k, we insert the expressions(1.17) into one of the factors of (1.22) or (1.23). And we do this for all zeros ρ_i, σ_j and τ_k of the given polynomials. These are $2(l+m+n)$ linear homogeneous equations in $2(l+m+n)+3$ unknowns $x_0, \dots, x_p$, $y_0, \dots, y_q$ and $z_0, \dots, z_r$.

The coefficients of this linear system are complex. But since the zeros appear in conjugate pairs $(\tau, \bar{\tau})$, for instance, we can form linear combinations to obtain a real linear system. Another possibility would be to split up each equation into its real and imaginary part.

The solutions of this linear system form an at least 3–dimensional linear space. In the case of maximal rank the solutions can be parametrized by

$$x_i = x_{i0}u_0 + x_{i1}u_1 + x_{i2}u_2, \text{ for } i = 0, \dots, p, \tag{1.24}$$
$$y_j = y_{j0}u_0 + y_{j1}u_1 + y_{j2}u_2, \text{ for } j = 0, \dots, q, \tag{1.25}$$
$$z_k = z_{k0}u_0 + z_{k1}u_1 + z_{k2}u_2, \text{ for } k = 0, \dots, r, \tag{1.26}$$

with parameters u_0, u_1 and u_2. Now, consider the polynomial

$$P(t) : x(t)^2 L(t) + y(t)^2 M(t) + z(t)^2 N(t).$$

It is of degree $\leq 2(l+m+n)$ in t and, as polynomial in t, it possesses $2(l+m+n)$ zeros at ρ_i, σ_j and τ_k. It clearly depends also on u_0, u_1, u_2. If $\deg(P) < 2(l+m+n)$ we are already done and $(x, y, z)(t)$ is a solution.

Otherwise, let L_0, M_0 and N_0 be the trailing coefficients of the polynomials L, M and N, and $(L_0, M_0, N_0) \neq (0,0,0)$. The trailing coefficient of P is

$$d = (\sum_{\alpha=0}^{2} x_{0\alpha}u_\alpha)^2 L_0 + (\sum_{\beta=0}^{2} y_{0\beta}u_\beta)^2 M_0 + (\sum_{\gamma=0}^{2} z_{0\gamma}u_\gamma)^2 N_0.$$

Consider the projective plane with coordinates u_0, u_1 and u_2. If $x_{0\alpha}$, $y_{0\beta}$ and $z_{0\gamma}$ are all zero, P possesses a further zero at $t = 0$, since x, y and z possess the common factor t. Otherwise, $d = 0$ is a quadratic curve and can be obtained by transforming the conic $c(t = 0)$ under

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} x_{00} & x_{01} & x_{02} \\ y_{00} & y_{01} & y_{02} \\ z_{00} & z_{01} & z_{02} \end{bmatrix} \cdot \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix},$$

which proves that $d = 0$ has real points. We choose (u_0, u_1, u_2) to be a real point on $d = 0$. It follows that $P(t)$ possesses a further zero at $t = 0$ such that it is identically zero. This implies that $(x, y, z)(t)$ is a non trivial solution of (1.16). □

What can be done if the assumptions of Corollary 1.1 are not satisfied. The first one is necessary, since it says that $c(t)$ is irreducible for almost all t and contains real points. If the polynomials do not possess constant signs, we can reparametrize. Let us study some examples. Let t be a quadratic function of a new parameter s. Then, $L(s)$, $M(s)$ and $N(s)$ are polynomials of even degree.

Let τ be a real zero of $N(t)$ with odd multiplicity. Substituting $t = \tau + s^2$ causes that $s = 0$ is a zero of even multiplicity for the polynomial $N(s)$.

Let $t = a$ and $t = b$ be real zeros of odd multiplicities of $N(t)$. The interval $[a, b] \neq \mathbb{R}$ can be reparametrized by

$$t = \frac{a + bs^2}{1 + s^2}, \tag{1.27}$$

such that $N(s)$ possesses 0 and ∞ as zeros with even multiplicity.

If the polynomials possess multiple or/and common zeros, we proceed as follows. Firstly, we discuss zeros of higher multiplicities. Let the constructions be demonstrated at zeros of N. Zeros of L and M can be treated analogously.

1. Let τ be a real zero of N with multiplicity $2k$, such that $N = (t - \tau)^{2k}\widetilde{N}$, where $\widetilde{N}$ is a polynomial of degree $2(n - k)$. This implies that $(t - \tau)^{2k}$ has to be a factor of $x^2L + y^2M$, such that we set

   $$x(t) = (t - \tau)^k \tilde{x}(t), \text{ and } y(t) = (t - \tau)^k \tilde{y}(t),$$

   where $\tilde{x}(t)$ and $\tilde{y}(t)$ are polynomials of degree $m + n - k$ and $l + n - k$, respectively. We determine $\tilde{x}(t), \tilde{y}(t)$ and $z(t)$ such that the polynomial

   $$\widetilde{P}(t) : \tilde{x}(t)^2 L(t) + \tilde{y}(t)^2 M(t) + z(t)^2 \widetilde{N}(t)$$

   is identically zero. This leads directly to a solution of the original problem.

2. Let $\tau \neq \bar{\tau}$ be zeros of N with even multiplicity $2k$, such that $N = (t - \tau)^{2k}(t - \bar{\tau})^{2k}\widetilde{N}$, where $\widetilde{N}$ is of degree $2n - 4k$. Choose x and y to be

   $$x(t) = (t - \tau)^k (t - \bar{\tau})^k \tilde{x}(t), \text{ and } y(t) = (t - \tau)^k (t - \bar{\tau})^k \tilde{y}(t).$$

   Using the same arguments as above, a solution $(x, y, z)(t)$ is constructed.

3. Let $\tau \neq \bar{\tau}$ be zeros of N of odd multiplicity $2k+1$, such that $N = (t-\tau)^{2k+1}(t-\bar{\tau})^{2k+1}\widetilde{N}$, where $\widetilde{N}$ is a polynomial of degree $2n-4k-2$. Choose

$$x(t) = (t-\tau)^k(t-\bar{\tau})^k\tilde{x}(t), \text{ and } y(t) = (t-\tau)^k(t-\bar{\tau})^k\tilde{y}(t),$$

where $\tilde{x}$, $\tilde{y}$ are of degree $m+n-k$ and $l+n-k$, respectively. We may determine $\tilde{x}(t), \tilde{y}(t)$ and $z(t)$ such that the polynomial $\widetilde{P}(t) : \tilde{x}^2L + \tilde{y}^2M + z^2(t-\tau)(t-\bar{\tau})\widetilde{N}$ is identically zero.

It remains to discuss common zeros of two polynomials, for instance L and M.

1. Let τ be a real zero of L and M with multiplicity $2k$, such that

$$L(t) = (t-\tau)^{2k}\widetilde{L}(t) \text{ and } M(t) = (t-\tau)^{2k}\widetilde{M}(t),$$

where $\widetilde{L}$ and $\widetilde{M}$ are polynomials of degree $2(l-k)$ and $2(m-k)$, respectively. Let $z(t) = (t-\tau)^k\tilde{z}(t)$, such that $\tilde{z}(t)$ is of degree $l+m-k$. We calculate $x(t)$, $y(t)$ and $\tilde{z}(t)$ such that

$$\widetilde{P}(t) : x(t)^2\widetilde{L}(t) + y(t)^2\widetilde{M}(t) + \tilde{z}(t)^2N(t)$$

is identically zero. With analogous arguments as above, $(x, y, z)(t)$ is a solution of the original problem.

2. Let $\tau \neq \bar{\tau}$ be common zeros of L and M with even multiplicity $2k$. Let again

$$L(t) = (t-\tau)^{2k}(t-\bar{\tau})^{2k}\widetilde{L}(t), \text{ and } M(t) = (t-\tau)^{2k}(t-\bar{\tau})^{2k}\widetilde{M}(t).$$

Choose $z(t) = (t-\tau)^k(t-\bar{\tau})^k\tilde{z}(t)$ and the rest is clear.

3. Let $\tau \neq \bar{\tau}$ be common zeros of $L(t)$ and $M(t)$ with odd multiplicity $2k+1$. Let

$$L(t) = (t-\tau)^{2k+1}(t-\bar{\tau})^{2k+1}\widetilde{L}(t) \text{ and } M(t) = (t-\tau)^{2k+1}(t-\bar{\tau})^{2k+1}\widetilde{M}(t),$$

where $\widetilde{L}(t)$ and $\widetilde{M}(t)$ are polynomials of degree $2l-4k-2$ and $2m-4k-2$, respectively. Multiply equation (1.16) by $(t-\tau)(t-\bar{\tau})$, such that $L(t)$ and $M(t)$ possess τ and $\bar{\tau}$ as zeros of even multiplicity $2k+2$. Choose $z(t) = (t-\tau)^{k+1}(t-\bar{\tau})^{k+1}\tilde{z}(t)$, where $\tilde{z}(t)$ is a polynomial of degree $l+m-k-1$. We determine $x(t)$, $y(t)$ and $\tilde{z}(t)$ such that the polynomial

$$\widetilde{P}(t) : x(t)^2\widetilde{L}(t) + y(t)^2\widetilde{M}(t) + \tilde{z}(t)^2(t-\tau)(t-\bar{\tau})N(t)$$

is identically zero. Using the same arguments as above a solution $(x, y, z)(t)$ is constructed.

It could be necessary to repeat applying the substitutions discussed above. For instance in the case where $\tau \neq \bar{\tau}$ is of multiplicity $2j$ for L and of multiplicity $2k+1$ for M. But in any case the equation $X^2L(t) + Y^2M(t) + Z^2N(t) = 0$ can be reduced such that it satisfies the conditions of Corollary 1.1.

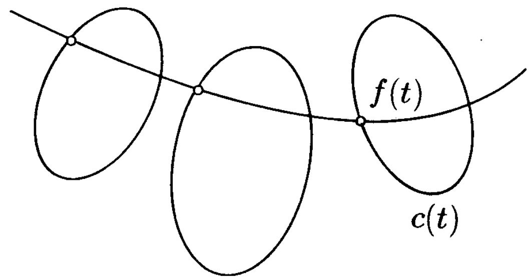

Figure 1.4: Parameterization problem of a rational family of conics

Theorem 1.1 *Let $c(t)$ be a rational one parameter family of conics in $\mathbb{P}^2$. If $c(t)$ contain real points for all t, there exist polynomials $x(t)$, $y(t)$ and $z(t)$, such that the point with homogeneous coordinates $(x, y, z)(t)$ is contained in the conic $c(t)$ for all t.*

This will be the key idea to prove the rationality and to construct rational parameterizations of conic surfaces, determined by a rational one parameter family of conics in $\mathbb{P}^3$ (see Section 1.5). A further technique is the stereographic projection of quadrics or conics, which shall be discussed in the following section.

1.4 Stereographic Projection

Let c be a conic in $\mathbb{P}^2$ and f a real point on c. Let a frame by chosen such that c as point set is represented by the equation

$$c_0 x_0^2 + c_1 x_1^2 + c_2 x_2^2 = 0,$$

where (x_0, x_1, x_2) are homogeneous coordinates of points in the plane and $f = (f_0, f_1, f_2)$. Projecting c from one of its points, for instance f, to a line g with $f \notin g$, is a birational map. The inverse map $\sigma : g \to c$ is called *stereographic projection* (see Figure 1.5). With help of the map σ one can derive rational parametrizations of conics.

Choose a line g not passing through f, for instance by $g : x_0 = 0$. The line g shall be parametrized by $q(u) = (0, 1, u)$, where u is an inhomogeneous projective parameter, thus $u \in \mathbb{R} \cup \infty$. Let $a(u)$ be the pencil of lines passing through f, determined by

$$a(u) : \lambda f + \mu q(u), \tag{1.28}$$

where (λ, μ) is a homogeneous projective parameter on the line $a(u)$. The intersection points of $a(u) \cap c$ are obtained for parameter values $\mu = 0$ and

$$\lambda_0 = c_1 + c_2 u^2, \mu_0 = -2(c_1 f_1 + c_2 f_2 u).$$

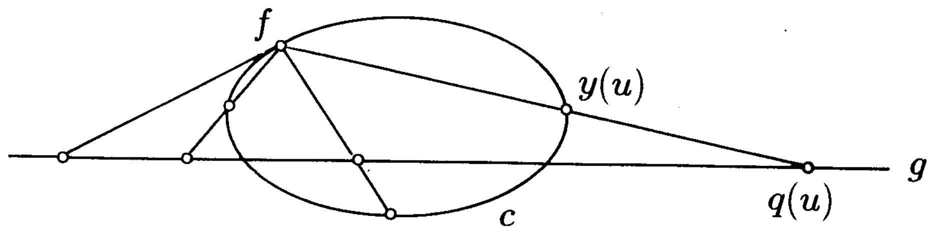

Figure 1.5: Stereographic Projection

The point f corresponds to $\mu = 0$. Inserting (λ_0, μ_0) in (1.28) leads to

$$c : y(u) = \lambda_0 f + \mu_0 q(u), \tag{1.29}$$

which is a quadratic parametrization of c in the parameter u.

Clearly, stereographic projections can be done for quadrics of arbitrary dimension. In the one dimensional case, the projective line or a pencil of lines are rationally equivalent to the points of a conic c, which lead to a parametrization of c over the projective line $\mathbb{P}^1$. In the higher dimensional cases it is as follows.

Let Q be a hyperquadric in $\mathbb{P}^n$, defined by its real projective normal form

$$Q : x_0^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_r^2 = 0. \tag{1.30}$$

The highest dimension of subspaces being contained in Q is $n - p$ and the dimension of the singular set is $n - r - 1$. Let $\mathbb{P}^r$ be the projective space defined by $x_{r+1} = \ldots = x_n = 0$ and let $q = Q \cap \mathbb{P}^r$. Then, q is a regular quadric in $\mathbb{P}^r$ and its equation is (1.30).

We discuss a stereographic projection of q. Let $f = (1, 0, \ldots, 1)$ be a point in $\mathbb{P}^r$, we call it northpole of q. Its tangent hyperplane τ in $\mathbb{P}^r$ is defined by $x_0 - x_r = 0$. Further let $\pi : x_r = 0$. Instead of π we could choose any hyperplane not passing through f. The tangent hyperplane τ intersects q in a singular quadric $\eta = q \cap \tau$ with vertex f. The set $\tilde{q} = q \setminus \eta$ is called an affine quadric and our intention is to parametrize $\tilde{q}$ by stereographic projection. Let $\alpha = \pi \setminus (\pi \cap \tau)$ be the affine space corresponding to π.

Let $p \in \alpha$ and $g(p)$ be the line, joining f and p. The stereographic projection

$$\sigma : \alpha \to \tilde{q} \tag{1.31}$$

with center f maps points $p \in \alpha$ to points $g(p) \cap \tilde{q}$ and σ is bijective. This construction leads to the following rational parametrization of an affine quadric $\tilde{q}$. We choose affine coordinate functions $u := (u_1, \ldots, u_{r-1})$ in α. Let x_i be homogeneous coordinates, the parametrization of $\tilde{q}$ is

$$\begin{aligned}(x_0, x_1, \ldots, x_r)(u) = \;& (u_1^2 + \ldots + u_p^2 - u_{p+1}^2 - \ldots - u_{r-1}^2 - 1, \\ & -2u_1, \ldots, -2u_{r-1}, u_1^2 + \ldots + u_p^2 - u_{p+1}^2 - \ldots - u_{r-1}^2 + 1).\end{aligned} \tag{1.32}$$

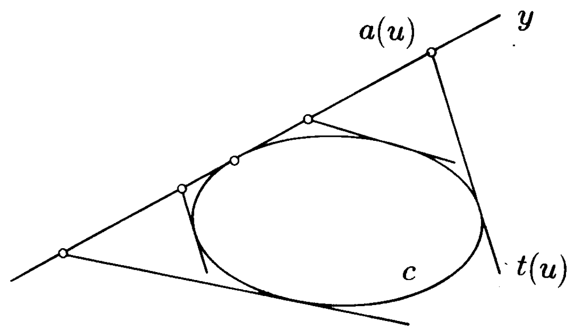

Figure 1.6: Dual stereographic projection

The parametrization does not work for points in the cone η. Usually, one forms a conformal closure $\alpha \cup \eta$ of the affine space α and extends the stereographic projection to a bijective map $\bar{\sigma} : \alpha \cup \eta \to q$. To have an example, see for instance Section 3.4.

Let us return to the parametrization of $\tilde{q}$. We wanted to construct a parametrization of Q, but clearly we also have to consider an affine part $\tilde{Q}$ of Q, and we do this corresponding to $\tilde{q}$. What remains is to extend the parametrization $(x_0, x_1, \ldots, x_r)$ to

$$(x_0(u), x_1(u), \ldots, x_r(u), v_1, \ldots, v_{n-r}),$$

where $v_1, \ldots, v_{n-r}$ are affine parameters to parametrize the subspaces, contained in $\tilde{Q}$.

Returning to conics, the principle of duality in the projective plane $\mathbb{P}^2$ can be applied, to define parametrizations of conics as set of tangent lines (see Figure 1.6).

Let c be a conic in $\mathbb{P}^2$, y a real tangent line. Let a frame be chosen, such that c is represented as set of tangent lines by the equation

$$c_0 X_0^2 + c_1 X_1^2 + c_2 X_2^2 = 0,$$

where X_j are homogeneous coordinates of lines in $\mathbb{P}^2$. Completely dual to the above discussed construction one chooses a point p not contained in y, for instance by $p : X_0 = 0$. The pencil of lines with vertex p shall be parametrized by line coordinates $g = (0, 1, u)$, where u again is an inhomogeneous projective parameter. Let $a(u)$ be points contained in y, defined by

$$a(u) : \lambda y + \mu g(u). \tag{1.33}$$

The tangent lines of c passing through $a(u)$ are obtained for parameters $\mu = 0$ and $\lambda_0 = c_1 + c_2 u^2$, $\mu_0 = -2(c_1 y_1 + c_2 y_2 u)$. Inserting them leads to a quadratic parametrization of c as set of tangent lines

$$c : t(u) = \lambda_0 y + \mu_0 g(u). \tag{1.34}$$

1.5 Rational Parametrizations of Conic Surfaces

Now we are able to construct rational parametrizations of conic surfaces $\Phi = c(t)$ determined by rational functions $c_{jk}(t)$. There are two different ways to study the problem. One way is to construct a rational curve $f(t)$ on Φ, where for all real t the curve point $f(t)$ is contained in the conic $c(t)$. The rest is done by stereographic projection. This results in a point representation of Φ. The other way is to construct a rational developable surface $F(t)$ of Φ, where for all real t the plane $F(t)$ is a tangent plane of the developable surface $D_c(t)$. This implies that $F(t)$ is a tangent plane of Φ. Here the rest can be done by a dual stereographic projection. This construction represents Φ as set of tangent planes. Both ways shall be described.

Firstly, a rational curve $f(t)$ and further a point representation of Φ will be constructed. Let Φ be a real conic surface of elliptic or hyperbolic type. A moving frame is constructed and each conic possesses a local representation (1.13). Let $y(t) = (y_0, y_1, y_2)(t)$ be a solution of (1.13), which represents a tangent line of $c(t)$, contained in γ. For each fixed t_0, the polarity with respect to $c(t_0)$ maps the tangent line $y(t_0)$ to a point $f(t_0)$. Thus, $f(t)$ is a rational curve on Φ.

Let Φ be a parabolic conic surface, that means $c(t)$ is parabolic for all $t \in \mathbb{R}$. Then $p(t)$, the pole of $g(t)$, already defines a rational curve on Φ.

Theorem 1.2 *A conic surface defined by a real, rational family of conics $c(t)$ possesses a rational curve $f(t)$ with the property that for all $t \in \mathbb{R}$ the curve point $f(t)$ is contained in the conic $c(t)$.*

To obtain a rational parametrization of Φ, one just has to apply a stereographic projection to each conic $c(t)$ with center $f(t)$.

Let us study the second way and let Φ be of elliptic or hyperbolic type. This construction is illustrated in Figure 1.7. Again let $y(t) = (y_0, y_1, y_2)(t)$ be a solution of (1.13), which represents a tangent line of the conic $c(t)$, contained in the plane γ. Considering $c(t)$ as set of tangent planes, there passes a pencil of tangent planes through $y(t)$, which is denoted by $Y(t, \lambda)$. We want to construct a tangent plane $F(t)$ for all t, which is contained in this pencil and is tangent to Φ. This implies that $F(t)$ has to be tangent to the quadric $\dot{c}(t)$. That means, calculate the tangent planes to $\dot{c}(t)$, passing through the line $y(t)$. Since one solution, namely γ is already known, the remaining $F(t)$ is linearly constructed in λ and thus rational in t. Further, $F(t)$ is contained in the developable $D_c(t)$, for all $t \in \mathbb{R}$.

Let Φ be a parabolic conic surface. Let $g(t)$ be the tangent line of $c(t)$ in $p(t)$. Since $\dot{c}(t)$ is tangent to $\gamma(t)$, there is no further tangent plane to $\dot{c}(t)$ through $g(t)$. So we take an arbitrary tangent line $\eta(t)$ of $c(t)$, which is rational in t and $p \notin \eta$. Then $\eta(t)$ carries two tangent planes of $\dot{c}(t)$. One of them is $\gamma(t)$ and the remaining tangent plane $F(t)$ to $\dot{c}(t)$ is rational in t and for all $t \in \mathbb{R}$, the plane $F(t)$ is tangent to $D_c(t)$.

Theorem 1.3 *A conic surface Φ defined by a real, rational family of conics $c(t)$ possesses a rational one parameter family of tangent planes $F(t)$, which form a developable surface. The tangent plane $F(t)$ is contained in $D_c(t)$, for all $t \in \mathbb{R}$.*

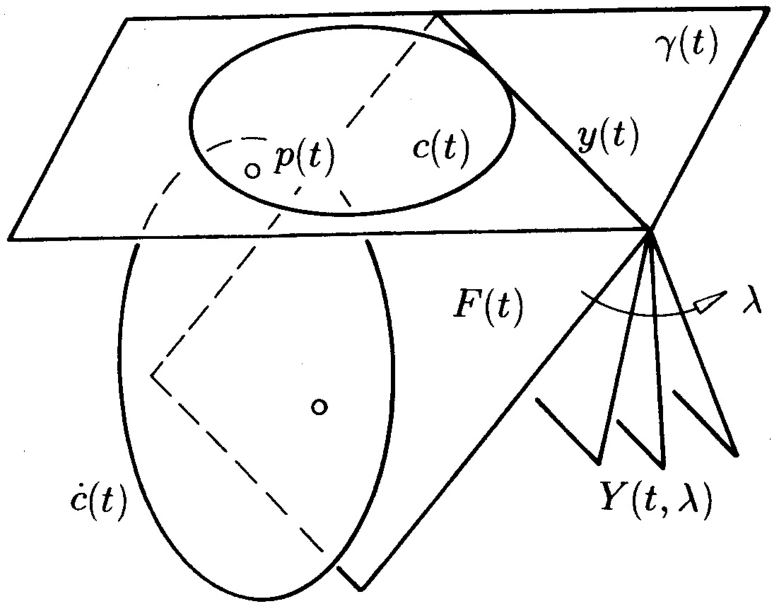

Figure 1.7: Construction of a rational tangent plane

To obtain the entire parametrization of Φ as set of tangent planes, one just has to apply the dual stereographic projection, to parametrize $c(t)$ as set of tangent lines in the plane γ. Let these tangent lines be denoted by $y(t,u)$. Further, the tangent planes of Φ passing through $y(t,u)$ are constructed, analogously denoted by $F(t,u)$, as described above. This results in a rational parametrization of Φ as set of tangent planes.

It depends on the kind of the application, which way will be preferable.

Theorem 1.4 *A conic surface Φ defined by a real, rational one parameter family of conics $c(t)$ possesses rational parametrizations over an interval, where $c(t)$ is of fixed type. Since Φ is an algebraic surface, the entire surface Φ possesses rational parametrizations.*

Remark: In algebraic geometry it is known that a rational one parameter family of (irreducible) conics is a rational surface (see [45], p.73 f). This follows from Tsen's Theorem which states that an equation $F(x_1,\ldots,x_n)=0$ of degree $m<n$ in x_i, whose coefficients are polynomials in one variable t, has a polynomial solution $x_i=p_i(t)$, $i=1,\ldots,n$.

But since this solution $p_i(t)$ is in general not a real curve, this method will not lead to real parametrizations of real conic surfaces, immediately. A reparametrization, which should produce a real parametrization of a real surface, seems to be not easy.

The construction we gave above always produces real parametrizations for real surfaces. Further, in case of Tsen's Theorem one has to solve a possibly large polynomial system, whereas in our case the problem is split up. Firstly, one calculates the roots of polynomials of simpler form and secondly one solves a linear system.

Example: Let a conic surface Φ be given by $XC(t)X^T = 0$, where $C(t)$ is the 4×4 matrix

$$\left[\begin{array}{cccc} (1+t^2)^2 & (1+t^2)^2 & 2(1+t^2)(t^2-1)^2 & (1+t^2)t(-3+t^2) \\ (1+t^2)^2 & 0 & 2(1+t^2)(t^2-1)^2 & (1+t^2)t(-3+t^2) \\ 2(1+t^2)(t^2-1) & 2(1+t^2)(t^2-1)^2 & 3(t-1)^4(t+1)^4 & 2(t^2-1)^2t(-2+t^2) \\ (1+t^2)t(-3+t^2) & (1+t^2)t(-3+t^2) & 2(t^2-1)^2t(-2+t^2) & t^2(t^2-1)(t^2-5) \end{array}\right],$$

and $X = [X_0, X_1, X_2, X_3]$ are homogeneous coordinates of planes in $\mathbb{P}^3$. This surface is generated by transforming the conic

$$-X_0^2 + X_1^2 - 4X_0X_2 - 3X_2^2 = 0$$

under a transformation, given by

$$X = X' \left[\begin{array}{cccc} 1+t^2 & 1+t^2 & 0 & t(1+t^2) \\ 0 & 1+t^2 & 0 & 0 \\ 0 & 0 & (1-t^2)^2 & -2t \\ 0 & 0 & 2t & (1-t^2)^2 \end{array}\right].$$

Here, X' are coordinates of the transformed plane. Since, for small t this transformation is near a rotation, the displayed surface of Figure (1.8) looks similar to a torus. The plane γ carrying the conic c is $\gamma = [-t(t^2-1)^2, 0, 2t, (t^2-1)^2]$. The next step is to change the coordinate system. So we apply a transformation, according to (1.11), which is given by

$$T = \left[\begin{array}{cccc} (t^2-1)^2 & 0 & 0 & 0 \\ 0 & (t^2-1)^2 & 0 & 0 \\ 0 & 0 & (t^2-1)^2 & 0 \\ -t(t^2-1)^2 & 0 & 2t & (t^2-1)^2 \end{array}\right].$$

This yields that TCT^T is of the form

$$TCT^T = \left[\begin{array}{cccc} (1+t^2)^2 & (1+t^2)^2 & 2(t^4-1)^2 & 0 \\ (1+t^2)^2 & 0 & 2(t^4-1)^2 & 0 \\ 2(t^4-1)^2 & 2(t^4-1)^2 & 3(t^2-1)^4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The intersection line $g = \gamma \cap \dot{\gamma}$ possesses the plane homogeneous coordinates

$$g = \left[(t^2-1)^3, 0, 2(3t^2+1)\right].$$

Further we calculate two conjugate lines η, ζ to g. Let, for instance, η be

$$\eta = \left[2(t^2-1)^2(t^4+12t^2+3), -2(t^2-1)^2(3t^2+1), -(1+t^2)(t^4+12t^2+3)\right].$$

We apply the transformation, given by the matrix S (see (1.12)) and obtain the diagonal form (1.13). We prefer here to convert to the point representation of c. Let x_0, x_1, x_2 be homogeneous coordinates of points in γ, we obtain

$$-x_0^2(1+t^2)^2(t^4+12t^2+3)^2 + x_1^2 + x_2^2(t^4+18t^2+5)(t^4+6t^2+1) = 0. \qquad (1.35)$$

To solve this equation, we substitute

$$\begin{aligned} x_0 &= \tilde{x}_0 \\ x_1 &= \tilde{x}_1(1+t^2)(t^4+12t^2+3) \\ x_2 &= \tilde{x}_2(1+t^2)(t^4+12t^2+3) \end{aligned}$$

and it remains the already factorized equation

$$(\tilde{x}_0+\tilde{x}_1)(\tilde{x}_0-\tilde{x}_1) = (t^4+18t^2+5)(t^4+6t^2+1)\tilde{x}_2^2.$$

We see that

$$x_0 = 1, x_1 = (1+t^2)(6t^2+2), x_2 = 1+t^2$$

is a solution of (1.35). Applying a stereographic projection (1.29) we obtain polynomials $(y_0, y_1, y_2)(t, u)$, which are a parametrization of the one parameter family of conics (1.35). Transforming back to the original coordinate system we get

$$(ST)^{-1}(y_0, y_1, y_2, 0)(t, u),$$

which is a parametrization of the conic surface Φ in homogeneous point coordinates. Figure (1.8) shows parameter lines of a segment of the surface. The parameter t varies in the interval $[-0.5, 0.5]$.

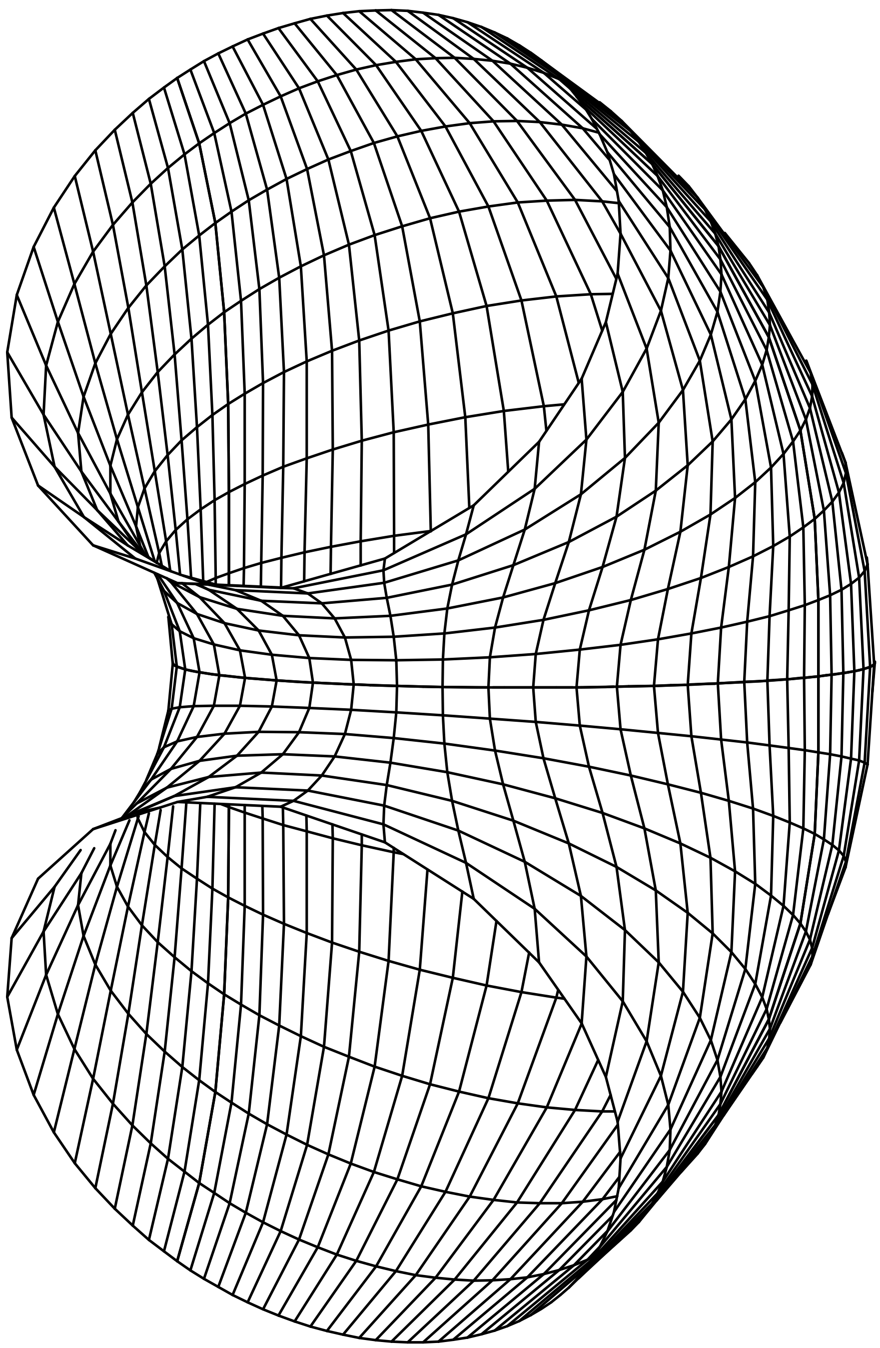

Figure 1.8: Conic surface

Chapter 2

One Parameter Families of Quadratic Cones

In this chapter we study geometric properties of surfaces which are envelopes of one parameter families of quadratic cones. In particular we will study real rational families and will prove that their envelope is a rational surface.

2.1 Geometric Properties of Envelopes of Quadratic Cones

Let $\mathbb{P}^3$ be projective 3–space. A regular duality or correlation in $\mathbb{P}^3$ maps planes to points and the set of lines of $\mathbb{P}^3$ to itself. If we apply such a mapping to the tangent planes of a conic (in the sense of chapter 1), we obtain a quadratic cone as point set. This proves that envelopes of a one parameter set of quadratic cones are dual to conic surfaces in the sense of projective geometry and they can be treated completely analogously to conic surfaces.

With respect to a fixed coordinate system, points x and planes X are represented by their coordinate vectors $(x_0, \ldots, x_3)$ and $(X_0, \ldots, X_3)$, respectively. A quadratic equation

$$c : \sum_{j,k=0}^{3} c_{jk} x_j x_k = 0, \quad c_{jk} = c_{kj}, \tag{2.1}$$

defines a quadratic cone, if the matrix $C = c_{jk}$ possesses rank 3. Rank 2 defines a pair of planes, rank 1 a double plane.

Considering c_{jk} as functions of a parameter t, a one parameter family of quadratic cones is defined. We can map them to a curve $C(t)$, contained in a manifold M in $\mathbb{P}^9$. Let $C(t)$ is a smooth curve on M, such that the corresponding quadratic cones envelop a surface Φ. Each quadratic cone $c(t)$ is tangent to the envelope Φ in points of the intersection curves $d_c(t)$ of the pencils of quadrics $\lambda c(t) + \mu \dot{c}(t)$. The curves $d_c(t)$ are called *characteristic curves.*

Let $v(t)$ be the vertex of $c(t)$. The homogeneous coordinates $(v_0, \ldots, v_3)$ are solutions of the homogeneous linear system

$$\sum_{k=0}^{3} c_{jk} x_k = 0, \quad \text{for } i = 0, 1, 2, 3. \tag{2.2}$$

If v and $\dot{v}$ are linearly independent, the vertices v define a curve. Its tangent lines are $g = v \vee \dot{v}$. Depending on the number $0, 1$ or 2 of real tangent planes of c, which pass through g, the corresponding envelopes and the corresponding quadratic cones are called *elliptic, parabolic* and *hyperbolic*. Note that this is a local property of a cone c and its derivative quadric $\dot{c}$.

The vertex v satisfies $\sum_k c_{jk} v_k = 0$, for $j = 0, \ldots, 3$, and also $\sum_{jk} c_{jk} v_j v_k = 0$ identically in t. The derivative with respect to t is

$$\sum_{jk} \dot{c}_{jk} v_j v_k + \sum_{jk} c_{jk} \dot{v}_j v_k + \sum_{jk} c_{jk} v_j \dot{v}_k = 0. \tag{2.3}$$

Since the last two terms are zero, the identity

$$\sum_{jk} \dot{c}_{jk} v_j v_k = 0 \tag{2.4}$$

holds, which implies that v is a common point of all quadrics of the pencil $\lambda c + \mu \dot{c}$. The derivative of (2.4) is

$$\sum_{jk} \ddot{c}_{jk} v_j v_k + \sum_{jk} \dot{c}_{jk} \dot{v}_j v_k + \sum_{jk} \dot{c}_{jk} v_j \dot{v}_k = 0. \tag{2.5}$$

Let

$$\varphi_j = \sum_{k} \dot{c}_{jk} v_k = -\sum_{k} c_{jk} \dot{v}_k \tag{2.6}$$

be the homogeneous coordinates of the plane φ, which is tangent to each arbitrary regular quadric of the pencil $\lambda c + \mu \dot{c}$ in the point v. Inserting (2.6) into (2.5) leads to

$$\sum_{jk} \ddot{c}_{jk} v_j v_k + 2 \sum_{j} \varphi_j \dot{v}_j = 0. \tag{2.7}$$

Completely dual to Lemma (1.1) one finds that the parabolic quadratic cones correspond to the asymptotic tangent lines of the manifold M.

A classification according to the singularities and multiplicities of intersection points of the tangent line $T(t) = \lambda C(t) + \mu \dot{C}(t)$ with M is analogous to that one for conic surfaces. The developable D_c just is replaced by the curve d_c, pencils of planes by points of a line, and so on.

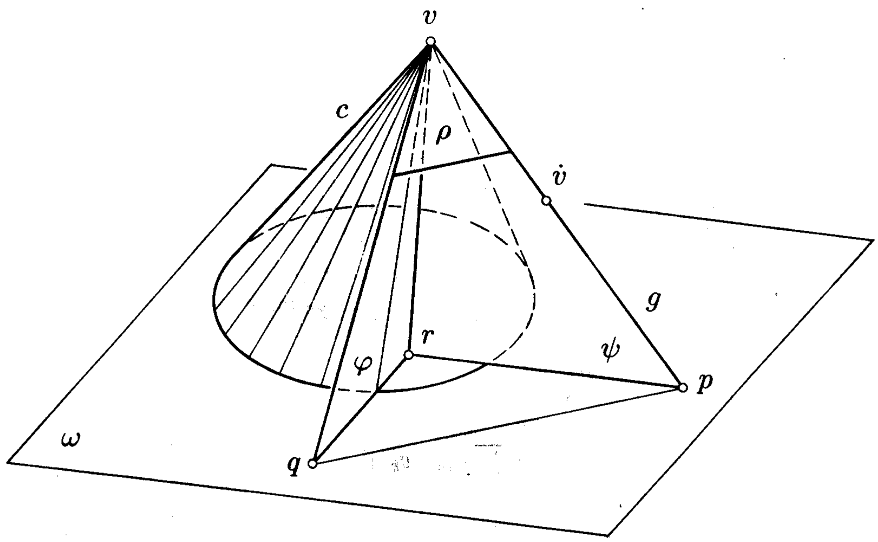

Figure 2.1: Local frame of a one parameter family of ell./hyp. quadratic cones

2.2 Local Coordinate Systems

As in the previous chapter it is again necessary to transform $c(t)$ to an appropriate coordinate system. Completely analogously one finds a frame, defined by planes φ, ψ, ρ, σ and ε, which is connected with Φ invariantly under transformations of $PGL(\mathbb{P}^3)$. In particular we want to study rational families of quadratic cones.

Definition: We call $c(t) \subset \mathbb{P}^3$ a rational one parameter family of quadratic cones if a representation exists such that the defining equation possesses only rational coefficients.

So we use only those rational projective transformations which preserve rational equations.

2.2.1 Elliptic and Hyperbolic Types

A projective frame will be determined, where φ, ψ and ρ pass through v and of course, $v \notin \sigma$. Further, ε is an appropriate unit plane. The choice of ε can be replaced by normalizing the coordinate vectors of φ, ψ, ρ and σ.

Let v be the origin of the frame. Further let φ, ψ and ρ be pairwise conjugate planes with respect to c, which means that, for instance, the intersection line $\varphi \cap \psi$ is the polar line to ρ. Let $\bar{x}_j$ and $\bar{X}_j$ be homogeneous coordinates of points and planes with respect to this frame. Then the quadratic cone c is given by the equation

$$c_1\bar{x}_1^2 + c_2\bar{x}_2^2 + c_3\bar{x}_3^2 = 0. \tag{2.8}$$

This coordinate transformation shall be described by two steps. Firstly, let v be the origin of the new frame. This is done by a transformation $T = t_{jk}$, such that the coordinate vectors of points are transformed as $x = Tx'$, and T is

$$T = (t_{jk}) = \begin{bmatrix} v_0 & 0 & 0 & 0 \\ v_1 & v_0 & 0 & 0 \\ v_2 & 0 & v_0 & 0 \\ v_3 & 0 & 0 & v_0 \end{bmatrix}. \tag{2.9}$$

It follows that $x'^T T^T C T x' = 0$ is just a quadratic polynomial in x'_1, x'_2 and x'_3. Of course, for zeros of v_0 the transformation is not defined.

In the second step the construction is restricted to the bundle v. Let φ be the polar plane to $g = v \vee \dot{v}$, given by (2.6). The remaining planes ψ and ρ have to pass through g. Since $v_0 \neq 0$ one may restrict the construction to the plane $\omega : x_0 = 0$. Let $p = g \cap \omega$ and q, r be vertices of a polar triangle of $c(t) \cap \omega$. Clearly $p = (0, \dot{v}_1, \dot{v}_2, \dot{v}_3)$ and let q be contained in $\varphi \cap \omega$, for instance by the choice $q = (0, -\dot{v}_2, \dot{v}_1, 0)$. Further r is constructed as intersection point of the polar lines to p and q with respect to $c(t) \cap \omega$, see Figure 2.1.

The second step of the coordinate transformation is given by the matrix

$$S = (s_{jk}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r_1 & q_1 & p_1 \\ 0 & r_2 & q_2 & p_2 \\ 0 & r_3 & q_3 & p_3 \end{bmatrix},$$

and coordinates are transformed by $x' = S\bar{x}$. Further the equation of the quadratic cone c is transformed to

$$c : \bar{x}^T \bar{C} \bar{x} = c_1 \bar{x}_1^2 + c_2 \bar{x}_2^2 + c_3 \bar{x}_3^2 = 0, \tag{2.10}$$

where $\bar{C} = (TS)^T CTS$. Since S and T possess only rational entries, c_j are assumed to be polynomials in t. Equation (2.10) can be considered as equation of a conic $\tilde{c}$, which is contained in $\omega : \bar{x}_0 = 0$ and obtained for $\tilde{c} = \omega \cap c$. This interpretation will be used when we construct rational parametrizations of rational one parameter families of quadratic cones.

2.3 Rational Parametrizations

Analogously to conic surfaces, one constructs rational parametrizations of surfaces Φ, enveloped by a real, rational one parameter family of quadratic cones $c(t)$. Let $\Phi = c(t)$ be of elliptic or hyperbolic type. Let $v(t)$ be the vertices of the cones $c(t)$. Further, a local frame is used, constructed by transformations T and S according to Section 2.2, such that each quadratic cone $c(t)$ is represented with respect to this frame by an equation

$$c(t) : \bar{x}_1^2 c_1(t) + \bar{x}_2^2 c_2(t) + \bar{x}_3^2 c_3(t) = 0. \tag{2.11}$$

One may interpret $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ as homogeneous projective coordinate vectors of lines in the bundle v, or of points in plane $\omega : \bar{x}_0 = 0$. Applying the construction demonstrated in

Section (1.3) results in polynomials $(y_1, y_2, y_3)(t)$ which satisfy (2.11) identically. These polynomials define a generating line $y(t)$ of each quadratic cone $c(t)$, we call it a *rational* generator. But the solution $(y_1, y_2, y_3)(t)$ also defines a point in the conic $c(t) \cap \omega$ and we will call it a *rational* point.

Analogously to the case of conic surfaces there are two ways to construct rational parametrizations of Φ. Firstly, the rational generator line $y(t)$ is mapped under the polarity with respect to $c(t)$ to a rational tangent plane $Y(t)$ of the quadratic cone $c(t)$. This one parameter family of tangent planes $Y(t)$ form a rational developable surface, which is tangent to the envelope Φ.

Corollary 2.1 *An envelope of a real, rational one parameter family of quadratic cones $c(t)$ possesses a rational developable surface determined by a one parameter family of tangent planes $Y(t)$. For each fixed $t \in \mathbb{R}$ the tangent plane $Y(t)$ is tangent to the cone $c(t)$.*

Let us derive the entire parametrization of Φ. Each quadratic cone possesses a rational tangent plane $Y(t)$. One has to apply a stereographic projection to $c(t)$, to obtain a rational parametrization. This can be done as follows. The intersection lines of tangent planes $Y(t)$ with ω are tangent lines of the conic $c(t) \cap \omega$. Apply the dual stereographic projection (1.34) to construct a rational parametrization of $c(t) \cap \omega$ as set of tangent lines, denoted by $z(t, u)$. Connecting these tangent lines with the vertices $v(t)$ results in tangent planes $Z(t, u)$. These planes are already a dual rational parametrization of Φ.

In the second way one intents to construct a point representation of Φ. The cone $c(t)$ is tangent to Φ in points of the characteristic curve $d_c(t) = c(t) \cap \dot{c}(t)$. Let $y(t)$ be a rational generator line of $c(t)$ which intersects $\dot{c}(t)$ in $v(t)$ and a further point $f(t)$. This implies that $f(t)$ defines a rational curve on the envelope Φ, which possesses the property that for each $t \in \mathbb{R}$ the curve point $f(t)$ lies on the characteristic curve $d_c(t)$.

Corollary 2.2 *An envelope of a real, rational one parameter family of quadratic cones $c(t)$ possesses a rational curve $f(t)$. For all $t \in \mathbb{R}$, the curve point $f(t)$ is contained in the characteristic curve $d_c(t)$.*

To obtain a parametrization of the entire envelope, one intersects the rational generator line $y(t)$ with ω and applies a stereographic projection with center $y(t) \cap \omega$ to construct a rational parametrization of the conic $c(t) \cap \omega$ in a real parameter u. This results in a rational representation of all generator lines $y(t, u)$ of the cones $c(t)$. The intersection points $z(t, u) = y(t, u) \cap \dot{c}(t)$ are the points of contact of $c(t)$ and Φ. This implies that $z(t, u)$ is a rational parametrization of Φ as set of points.

Theorem 2.1 *The envelope Φ of a real, rational one parameter family of quadratic cones $c(t)$ possesses rational parametrizations over an interval, where $c(t)$ is of fixed type. Since Φ is an algebraic surface, the entire surface Φ possesses rational parametrizations.*

Next, we will study a special subclass of conic surfaces in projective and Euclidean 3–space, denoted by $\mathbb{P}^3$ and $\mathbb{R}^3$, respectively.

Let Φ be a conic surface of Thomsen type C in $\mathbb{P}^3$, which is characterized by the property that the developable surface tangent to Φ in points of a generating conic $c \subset \gamma$ contains a quadratic cone D and further two pencils of planes. Let p be the point of contact of an arbitrary regular quadric of the dual pencil $\lambda c + \mu \dot{c}$ with the plane γ. The two pencils of planes from above pass through the tangent lines to c through p. The vertex v of the cone D lies on h, the polar line to g with respect to $\dot{c}$. So we can consider Φ also as envelope of the one parameter set of quadratic cones $D(t)$. The characteristic curves are the conics $c(t)$ and additionally two generator lines of $D(t)$. These two families of lines generate two developable surfaces, but we do not study them here in this context.

Further, such a surface Φ of type C possesses an inscribed quadric for each conic $c(t)$ and cone $D(t)$. This implies that Φ can also be interpreted as envelope of a one parameter family of quadrics $q(t)$, where $q(t) \cap \dot{q}(t)$ contains $c(t)$ and further a quadratic rest. These surfaces were firstly studied by Blutel [5]. See also Degen [10].

Here we only want to study the special Euclidean types of envelopes of quadrics, namely envelopes of one parameter families of spheres, so called *Euclidean canal surfaces*. To do so we will use a model, where such surfaces are represented by curves.

2.4 Euclidean Möbius Geometry

Let $\mathbb{R}^n$ be Euclidean n–dimensional space, let $\mathbb{R}^{n+1}$ be $n+1$–dimensional Euclidean space, such that $\mathbb{R}^n$ is embedded as hyperplane $y_{n+1} = 0$,where $y_1, \dots, y_{n+1}$ are Cartesian coordinates in $\mathbb{R}^{n+1}$. Further $\mathbb{R}^{n+1}$ shall be projectively extented to $\mathbb{P}^{n+1}$, where we use homogeneous coordinates $x_0, \dots, x_{n+1}$.

Let Q be the unit hypersphere in $\mathbb{R}^{n+1}$ represented by the equation

$$Q : -x_0^2 + x_1^2 + \ldots + x_{n+1}^2 = 0. \tag{2.12}$$

The geometry of hyperplanar intersections and points of Q is called *n–dimensional Euclidean Möbius geometry.* Any intersection of Q by a linear subspace is called a Möbius sphere.

Another model of *Euclidean Möbius geometry* can be obtained by applying an extended stereographic projection

$$\sigma : Q \to M^n = \mathbb{R}^n \cup \infty$$

with center $N = (1, 0, \dots, 0, 1)$, often referred as northpole. The image of N under σ is ∞ and M^n is called *conformal closure of* $\mathbb{R}^n$. Then, in M^n, Euclidean Möbius geometry is the geometry of hyperspheres and hyperplanes on the one hand and points on the other hand. The hyperplanes stem from hyperplanar intersections of Q, which pass through N. In M^n, spheres and linear subspaces are called Möbius spheres. So we have two models of Euclidean Möbius geometry, where Q is referred as quadric model and M^n is referred as standard model. A detailed description of Möbius and other sphere geometries can be found in [8].

We take a closer look to the map σ. Let $p \in \mathbb{P}^{n+1}$ be a point in the exterior of Q and let π be its polar hyperplane with respect to Q. Since any hyperplanar intersection $\pi \cap Q$

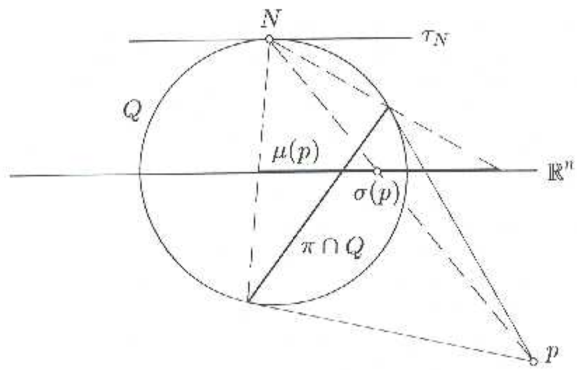

Figure 2.2: Models of Euclidean Möbius geometry

can be uniquely represented by p, we identify hyperspheres and hyperplanes in M^n with points in $\mathbb{P}^{n+1} \setminus Q$. Note that real hyperplanar intersections correspond to points in the exterior of Q. Let τ_N be the tangent space to Q in N and let p be in the exterior of Q. We define a map $\mu : \mathbb{P}^{n+1} \to M^n$, by

$$\begin{array}{rcll} \mu(p) & = & \sigma(p) & \text{if } p \in Q \setminus N \\ \mu(p) & = & \pi \cap \mathbb{R}^n & \text{if } p \in \tau_N \setminus N \\ \mu(p) & = & \sigma(\pi \cap Q) & \text{otherwise, but } p \neq N. \end{array}$$

At last we define $\mu(N) = \infty$. So, μ describes the transfer from the quadric model to the standard model. If p is in the interior of Q, the sphere $\mu(p)$ possesses a real center but the squared radius is negative, see (2.13).

The stereographic projection σ can also be considered as central projection

$$\sigma : \mathbb{P}^{n+1} \to \mathbb{R}^n,$$

thus, σ can be applied also to points of $\mathbb{P}^{n+1}$ not contained in Q. Let $p = (p_0, \ldots, p_{n+1}) \neq N$. The polar hyperplane π is given by the linear equation

$$\pi : -p_0 x_0 + p_1 x_1 + \ldots + p_{n+1} x_{n+1} = 0.$$

If $p_0 \neq p_{n+1}$, which means that $p \notin \tau_N$, the hypersphere $\mu(p)$ in $\mathbb{R}^n$ possesses $\sigma(p)$ as center and is given by the equation

$$\mu(p) : (y_1 - \frac{p_1}{p_0 - p_{n+1}})^2 + \ldots + (y_n - \frac{p_n}{p_0 - p_{n+1}})^2 = \frac{p_1^2 + \ldots + p_{n+1}^2 - p_0^2}{(p_0 - p_{n+1})^2}, \qquad (2.13)$$

where $y_1 = x_1/x_0, \ldots, y_n = x_n/x_0$ are affine coordinates in $\mathbb{R}^n$. If $p \in \tau_N$, but $p \neq N$, $\mu(p)$ corresponds to the hyperplane

$$\pi \cap \mathbb{R}^n : -p_0 + p_1 y_1 + \ldots p_n y_n = 0.$$

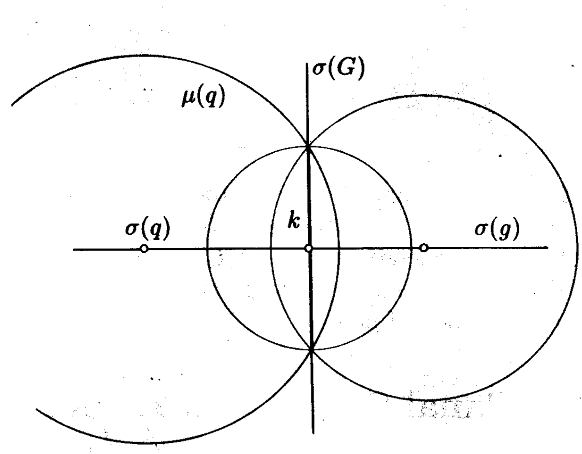

Figure 2.3: Pencil of spheres in $\mathbb{R}^3$ with real circle k.

On the one hand we have studied the elements of Möbius geometry, but on the other hand it is necessary to study transformations. A Möbius transformation in the standard model $\mathbb{R}^n$ is defined as bijective map both on points and on Euclidean Möbius hyperspheres. Additionally, a Möbius transformation respects the incidence of Möbius spheres and points.

It is the main theorem of Euclidean Möbius geometry that Möbius transformations appear in the quadric model $Q \subset \mathbb{P}^{n+1}$ as automorphic projective collineations of Q. We give a special example. Let z be a point in the exterior of Q. The perspective automorphic collineation with center z and axis ζ, which is the polar to z with respect to Q, corresponds to an *inversion* at the hypersphere $\mu(z)$, if $z \notin \tau_N$. Otherwise, if $z \in \tau_N$, one obtains a Euclidean reflection at the hyperplane $\mu(z)$ in the standard model.

Now we will restrict to 3–dimensional Euclidean Möbius geometry. Let $c(t)$ in $\mathbb{P}^4$ be a smooth curve. Then, the mapping μ defines a one parameter family of spheres $\mu(c(t))$, planes included. Our aim is to study such one parameter families and firstly, let $c(t) = g$ be a straight line in $\mathbb{P}^4$. The image $\mu(g)$ is called a pencil of spheres and depending on the number of real intersection points of $g \cap Q$ we have to distinguish three cases.

1. The line g intersects Q in two conjugate complex points $p, \bar{p}$. The polar plane G to g with respect to Q intersects Q in a conic, which contains real points. Further $k = \sigma(G \cap Q)$ is a circle, containing real points. The pencil of spheres $\mu(g)$ consists of all spheres, whose centers are in $\sigma(g)$ and which pass through the circle k.

2. The line g intersects Q in two real points p, q. The polar plane G to g with respect to Q intersects Q in a conic, without any real points. Further $k = \sigma(G \cap Q)$ is a circle, without real points. The pencil of spheres $\mu(g)$ consists of all spheres, whose centers are in $\sigma(g)$. These spheres induce in the plane $\sigma(G)$ the same polarity, which defines the circle k. Further, $\mu(g)$ contains two points $\sigma(p)$ and $\sigma(q)$ as limit cases of spheres with radius zero.

3. The line g is tangent to Q in p. The pencil $\mu(g)$ consists of all spheres with centers in $\sigma(g)$ which are tangent to $\sigma(G)$ in $\sigma(p)$.

Let $c(t)$ be a smooth curve in $\mathbb{P}^4$ ant t_0 a fixed parameter value. Then, $\mu(c(t))$ possesses an envelope Φ in a neighbourhood of the sphere $\mu(c(t_0))$, if the tangent $g = c(t_0) \vee \dot{c}(t_0)$ does not intersect Q. The characteristic circle $k(t_0) \subset \mu(c(t_0))$ is given by $\mu(g)$. If g is tangent to Q, $k(t_0)$ degenerates to a single point. Otherwise, $\mu(c(t))$ does not possess a real envelope in a neighbourhood of $\mu(c(t_0))$.

2.5 Rational Canal Surfaces in Euclidean 3–Space

Let $\mathbb{R}^3$ be Euclidean 3–space. Points y are represented by their coordinate vectors (y_1, y_2, y_3) with respect to a fixed, but arbitrary frame. Let a canal surface Φ be defined as envelope of a one parameter family of spheres $S(t)$.

Definition: A family of spheres shall be called *rational*, if the defining equation of $S(t)$ possesses only rational coefficients.

The envelope Φ of a rational one parameter set of spheres is defined by the equations

$$S(t) : \sum_1^3 (y_j - m_j(t))^2 - r(t)^2 = 0 \text{ and } \dot{S}(t) : \sum_1^3 (y_j - m_j(t))\dot{m}_j(t) + r(t)\dot{r}(t) = 0, \quad (2.14)$$

where $m_j(t)$ are coordinate functions of the rational center curve $m(t)$ of $S(t)$. The radius function $r(t)$ of the spheres is not necessarily rational, but a square root of a rational function.

The generating conics of Φ are so–called characteristic circles $c = S \cap \dot{S}$. The circles contain real points, if and only if $\dot{m}^2 - \dot{r}^2 \geq 0$. Equality holds, if the plane $\dot{S}$ is tangent to S and c degenerates to a point. Further Φ is enveloped by a one parameter family of cones of revolution $D(t)$, which are tangent to Φ in points of c. The axis of D is the tangent line of the center curve $m(t)$.

Let $\lambda S(t) + \mu \dot{S}(t)$ be a pencil of quadrics, where $\dot{S}(t)$ is considered to be a double plane. The cone of revolution $D(t)$ is a further singular quadric, contained in this pencil. The determinant $\det(\lambda S + \mu \dot{S})$ possesses a threefold zero at λ, which implies that $D(t)$ is given by an equation with rational coefficients. This proves the following Theorem.

Theorem 2.2 *The real envelope Φ of a rational one parameter family of spheres $S(t)$ can be generated as envelope of a real rational one parameter family of cones of revolution $D(t)$, in the sense of Theorem 2.1.*

We mention that the envelope of the one parameter family of cones of revolution $D(t)$ also contains two, not necessarily real developable surfaces, which are not considered in the above Theorem. From Theorem 2.1 it is clear that Φ is a rational surface.

If we return to Möbius geometry and take the transfer mapping μ into account, we obtain the following.

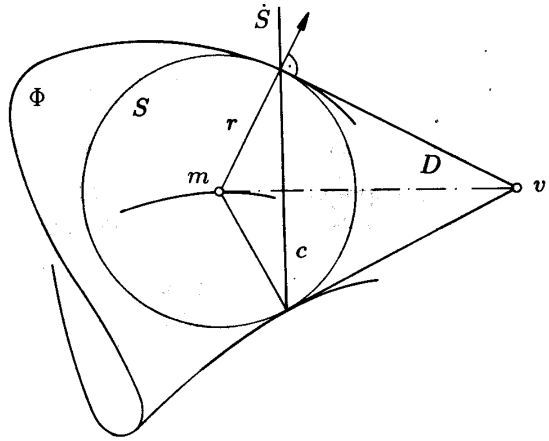

Figure 2.4: Local properties of a canal surface

Theorem 2.3 *Let $c(t)$ in $\mathbb{P}^4$ be a rational curve whose tangent lines do not intersect the Möbius quadric Q in real points. The one parameter family of spheres $\mu(c(t))$ is a rational family and envelops a rational canal surface.*

The special case of conics c in $\mathbb{P}^4$ is discussed in [7]. The envelope of the one parameter family of spheres $\eta(c)$, corresponding to the conic c, is a rational canal surface of order ≤ 4. In the case of order 4, the surface passes two times through the absolute conic of Euclidean 3–space. These canal surfaces are contained in the family of so–called *Darboux cyclides* or only *cyclides*. A general Darboux cyclide is the envelope of a two parameter family of spheres $\eta(q)$, where q is a 2–dimensional quadric in $\mathbb{P}^4$.

Later, in Chapter 4, we will discuss a special case of Theorem 2.3, namely that the radius function, defined by the family $\mu(c(t))$, is rational too. For this, it is better to use another, more suitable, model, namely Euclidean Laguerre space. We will prove in Theorem 4.5 that families of spheres with rational radius function envelop surfaces Φ which possess rational offset surfaces. Further we will see that Φ possesses rational unit normals.

Remark: Analogously to 3–dimensional case we can study canal hypersurfaces in n–dimensional Euclidean space. Let $c(t)$ be a rational curve in $\mathbb{P}^{n+1}$. The one parameter family of hyperspheres $\mu(c(t))$, with centers on the rational curve $\sigma(c(t))$, possesses a radius function $r(t)$, where $r^2(t)$ is rational. This means that $\mu(c(t))$ is represented by an equation with rational coefficients. If the tangent lines of $c(t)$ do not intersect the Möbius quadric Q in real points, the family $\mu(c(t))$ envelops a real canal hypersurface Φ. The parametrization method, described in Section 5.1, which is a straightforward generalization of that one given in Section 1.3, proves that Φ possesses rational parametrizations.

Corollary 2.3 *A real canal hypersurface in $\mathbb{R}^n$, determined by a rational curve $c(t)$ in $\mathbb{P}^{n+1}$ whose tangent lines do not intersect the Möbius quadric Q, possesses rational parametrizations.*

Chapter 3

Fundamentals of Laguerre Geometry

3.1 Euclidean Laguerre Space

Let $\mathbb{R}^n$ be Euclidean n–space. With respect to an arbitrary Cartesian coordinate system each point x is represented by a coordinate vector $x = (x_1, \ldots, x_n)$. A hyperplane e is given by a linear equation

$$e_0 + e_1 x_1 + \ldots + e_n x_n = 0.$$

The coefficients e_i are called hyperplane coordinates, and are homogeneous coordinates in the projective extension $\mathbb{P}^n$ of $\mathbb{R}^n$. The vector $(e_1, \ldots, e_n)$ is a normal vector of e. Each coordinate vector $\rho(e_0, \ldots, e_n)$ with $\rho \in \mathbb{R} \setminus \{0\}$ defines the same hyperplane.

Euclidean Laguerre geometry in $\mathbb{R}^n$ is the geometry of *oriented hyperplanes* and *oriented hyperspheres*, including points as hyperspheres of radius zero. An oriented hyperplane is described by a homogeneous coordinate vector $\rho(e_0, \ldots, e_n)$, but $\rho \in \mathbb{R}^+$. Taking the normalization $e_1^2 + \ldots + e_n^2 = 1$ into account, any oriented hyperplane e is uniquely determined by $(e_0, \ldots, e_n)$. We will define the set of oriented hyperplanes as

$$H := \{e = (e_0, e_1, \ldots, e_n), e_i \in \mathbb{R}, e \neq (0, \ldots, 0), e_1^2 + \ldots + e_n^2 = 1\}. \tag{3.1}$$

Oriented hyperspheres and points are collected to one set C and are called *cycles*. A cycle $c : (m_1, \ldots, m_n; r)$ is defined by its center $m = (m_1, \ldots, m_n)$ and its signed radius r. Points are obtained for $r = 0$. Cycles often are interpreted as set of oriented hyperplanes, such that a cycle c is the set

$$c = \{\rho(e_0, \ldots, e_n), \rho \in \mathbb{R}^+, e_1^2 + \ldots + e_n^2 = 1, e_0 + e_1 m_1 + \ldots e_n m_n + r = 0\}. \tag{3.2}$$

Both interpretations shall be used. The carrier of each cycle c is a hypersphere defined by

$$(x_1 - m_1)^2 + \ldots + (x_n - m_n)^2 = r^2.$$

The basic relation in Euclidean Laguerre geometry is that of *oriented contact* of cycle and oriented hyperplane. A cycle c and an oriented hyperplane e are in oriented contact, if they are tangent and the unit normals coincide at the point of contact. For a point and an oriented hyperplane, oriented contact equals incidence. The property of oriented contact is defined by

$$e_0 + e_1 m_1 + \ldots + e_n m_n + r = 0, \tag{3.3}$$

which, of course, occurs in (3.2). Two oriented hyperplanes are in oriented contact, if their unit normals coincide. When no ambiguity can arise, oriented hyperplanes will simply be called hyperplanes.

Laguerre geometry is the study of properties which are invariant under Laguerre transformations. A *Laguerre transformation* α consists of two maps

$$\alpha_C : C \to C, \alpha_H : H \to H.$$

These maps are bijective on C and H, respectively. Additionally, α preserves oriented contact and non–contact between cycles $c \in C$ and oriented hyperplanes $\varepsilon \in H$. For a coordinate representation of Laguerre transformations see Section 3.2 and 3.3.

A simple, but important example of a Laguerre transformation is a *dilatation* λ, which adds a constant $d \neq 0$ to the signed radius of each cycle and leaves its center unchanged, such that we have

$$c = (m_1, \ldots, m_n; r) \mapsto \lambda(c) = (m_1, \ldots, m_n; r + d). \tag{3.4}$$

Note that λ does not preserve points. Considering a hypersurface as envelope of its oriented tangent hyperplanes, a dilatation λ maps the hypersurface onto its one–sided offset at distance d. This already indicates the advantage of using Laguerre geometry in connection with surfaces and offset surfaces.

The model of Euclidean Laguerre geometry just described is also referred to as *standard model*. We will study some models of Euclidean Laguerre geometry to get a better insight. In particular, if oriented hyperplanes are of interest, the standard model is not preferable. Additionally, the group of Laguerre transformations is easier to be understood in other models.

3.2 The Cyclographic Model

To obtain useful coordinate representations of cycles c and oriented hyperplanes e and to get a better insight to Laguerre transformations we do the following. Let Euclidean n–space $\mathbb{R}^n$ be embedded in $\mathbb{R}^{n+1}$ as hyperplane $x_{n+1} = 0$. For now, $\mathbb{R}^{n+1}$ shall be considered to be an affine space. Each oriented hyperplane $e \in H$ is mapped by ζ^* to the hyperplane $E = \zeta^*(e)$ in $\mathbb{R}^{n+1}$, defined by its homogeneous coordinates

$$\zeta^*(e) = E = (e_0, e_1, \ldots, e_n, 1), \text{ with } e_1^2 + \ldots e_n^2 = 1. \tag{3.5}$$

Note that for each $E = \zeta^*(e)$ the Euclidean angle $\angle(E, \mathbb{R}^n)$ equals $\gamma = \pi/4$. Thus, E is called a γ–hyperplane in $\mathbb{R}^{n+1}$. A cycle $c = (m_1, \ldots, m_n; r)$ with center m and signed radius r is mapped to the point

$$\zeta(c) = C = (m_1, \ldots, m_n, r) \text{ in } \mathbb{R}^{n+1}. \tag{3.6}$$

Interpret c as set of tangent hyperplanes, the mapping ζ is determined by the mapping ζ^*. All oriented hyperplanes e, which are tangent to c, are mapped to γ–hyperplanes E, passing though the point C. That means that the property of oriented contact between a cycle c and an oriented hyperplane e equals incidence between C and E in this model. The coordinate representation of this property is given by (3.3). The γ–hyperplanes $\zeta^*(e)$ that pass through $\zeta(c)$ envelope a quadratic hypercone $\Gamma(c)$ with vertex $\zeta(c)$. Its generator lines form the angle γ with $\mathbb{R}^n$, see Figure 3.1.

Let $\mathbb{P}^{n+1}$ be projective extension of $\mathbb{R}^{n+1}$ and let $\omega : x_0 = 0$ be the ideal plane. Let Ω be the quadric defined by

$$\Omega : x_0 = 0, x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = 0, \tag{3.7}$$

where $x_0, \ldots, x_{n+1}$ are homogeneous coordinates in $\mathbb{P}^{n+1}$ and we see that γ–hyperplanes are tangent to Ω. Further, the intersection $\Gamma(c) \cap \omega$ is the quadric Ω. Let a, b be two vectors in $\mathbb{R}^{n+1}$. The polarity with respect to Ω determines a *pseudo–Euclidean* (pe) scalar product in the vector space $\mathbb{R}^{n+1}$. It is defined by

$$\langle a, b\rangle_{pe} = a^T E_{pe} b,$$

where $E_{pe} = \operatorname{diag}(1, \ldots, 1, -1)$. By

$$d_{pe}(a, b) = \sqrt{\langle a, b\rangle_{pe}}$$

a pseudo–metric or pseudo–Euclidean distance is induced in the affine space $\mathbb{R}^{n+1}$. The so obtained $n + 1$–dimensional model of n–dimensional Laguerre geometry is called *cyclographic model.* A vector space $\mathbb{R}^{n+1}$ with a pseudo–Euclidean scalar product is often called a *Lorentz space.* A vector a is said to be *timelike*, *spacelike* or *lightlike*, depending on whether $\langle a, a\rangle_{pe}$ is negative, positive or zero.

A line g with timelike direction vector a is called an *elliptic line.* The angle $\angle(g, \mathbb{R}^n)$ exceeds $\gamma = \pi/4$. A lightlike direction vector determines a γ–line or *parabolic line* and a spacelike direction vector characterizes a *hyperbolic line.* Since the pe scalar product is determined by Ω, we find equivalently to this characterization that g is an elliptic line, if $g \cap \omega$ is in the interior of Ω. A γ–line g is characterized by $g \cap \omega \in \Omega$. Finally g is a hyperbolic line, if $g \cap \omega$ is in the exterior of Ω.

Let π be a subspace of $\mathbb{R}^{n+1}$ of dimension $k \geq 2$. A characterization of π is given by the type of the polarity in $\pi \cap \omega$ induced by the quadric Ω. The subspace π is called *Euclidean,* if $|\Omega \cap \pi| = 0$. If $|\Omega \cap \pi| = 1$, that means $\pi \cap \omega$ is tangent to Ω, π is said to be *parabolic* or called γ–space, and otherwise π is called *pseudo–Euclidean.*

Let a and b be two points in $\mathbb{R}^{n+1}$, contained in a hyperbolic line. Any common oriented tangent hyperplane of the two cycles $\zeta^{-1}(a)$ and $\zeta^{-1}(b)$ touches the cycles at points y_a and y_b. The Euclidean distance of $d(y_a, y_b)$ equals the pe distance $d_{pe}(a, b)$ and is called *tangential distance* of the two cycles $\zeta^{-1}(a)$ and $\zeta^{-1}(b)$.

For two points a, b contained in a parabolic line g, one gets $d_{pe}(a,b) = 0$ and the corresponding cycles $\zeta^{-1}(a)$ and $\zeta^{-1}(b)$ are tangent. Further, all cycles $\eta(y)$ with $y \in g$ are tangent to a common plane in a common point which means that they possess a common surface element.

Two points a, b on an elliptic line correspond to two cycles without common oriented tangent hyperplanes. A tangential distance of the cycles $\zeta^{-1}(a)$, $\zeta^{-1}(b)$ as well as a pe distance of the points a, b are not defined. But one may use the invariant $d^2_{pe}(a,b) < 0$ of two points on an elliptic line.

It is an important theorem of Euclidean Laguerre geometry that a Laguerre transformation α appears in the cyclographic model as a special affine map. Using the projective extension $\mathbb{P}^{n+1}$ of $\mathbb{R}^{n+1}$, α maps the quadric $\Omega \subset \omega$ onto itself. Since Ω determines the pe scalar product in $\mathbb{R}^{n+1}$, Laguerre transformations are of the form

$$\alpha(x) = a + \lambda A \cdot x, \tag{3.8}$$

where $a, x \in \mathbb{R}^{n+1}$, $\lambda \in \mathbb{R} \setminus \{0\}$ and A is a pe orthogonal matrix. These mappings are often called *Lorentz transformations*. The matrix A satisfies

$$A^T \cdot E_{pe} \cdot A = E_{pe}. \tag{3.9}$$

Such an affine mapping in $\mathbb{R}^{n+1}$ is called *pe similarity*. If $|\lambda| = 1$, α is called *pe congruence*. It preserves the squared pe distance $d^2_{pe}(.,.)$ of any two points and corresponds to a tangential distance preserving Laguerre transformation α in $\mathbb{R}^n$.

Let us look at a special example, where Laguerre transformations are used. Let π be a Euclidean hyperplane in the projective extension $\mathbb{P}^{n+1}$ of the cyclographic model $\mathbb{R}^{n+1}$. It shall be shown that there exists a Laguerre transformation α, which maps π to $\mathbb{R}^n$. Firstly, let $\mathbb{P}^n$ be the projective extension of Euclidean $\mathbb{R}^n$. If π and $\mathbb{R}^n$ are parallel, α is just a translation in $\mathbb{R}^{n+1}$, which is obviously linear and maps Ω to itself, since α restricted to the plane at infinity ω is the identity.

Otherwise, we take a closer look to the situation in ω. The hyperplanes $\pi \cap \omega$ and $\mathbb{P}^n \cap \omega$ in ω are in the exterior of Ω. Since the real projective automorphic collineation group $PGL(\Omega)$ of a hyperquadric Ω of index 0 is transitive on the points in the interior and in the exterior, there exists a collineation $\in PGL(\Omega)$, which maps $\pi \cap \omega$ to $\mathbb{P}^n \cap \omega$. We can extend this collineation to a projective map in $\mathbb{P}^{n+1}$, which fixes $\pi \cap \mathbb{R}^n$, and the corresponding Laguerre transformation has the desired properties.

A Laguerre transformation α maps γ–subspaces onto γ–subspaces and pe subspaces are mapped onto pe subspaces.

3.3 The Blaschke Model

If more emphasis is on oriented hyperplanes rather than on cycles, one might be interested in a model where oriented hyperplanes appear as points. This can easily be done by

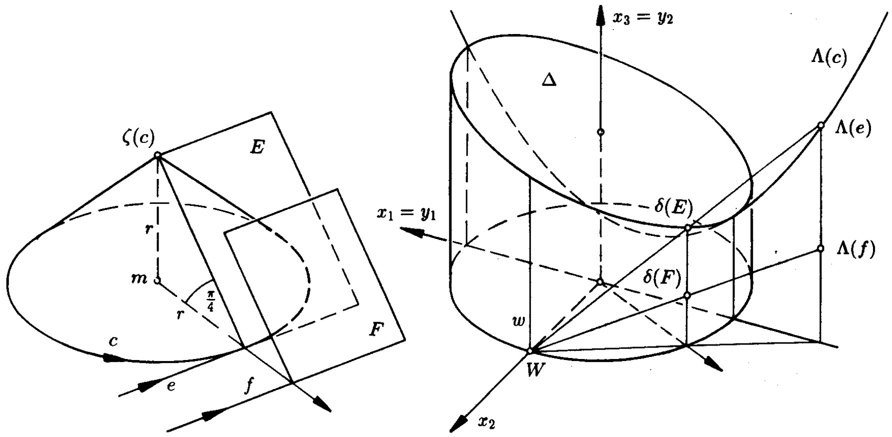

Figure 3.1: Models of Euclidean Laguerre geometry

applying a duality $\delta : \mathbb{R}^{n+1*} \longrightarrow \mathbb{R}^{n+1}$ (Blaschke map), which maps hyperplanes to points. Let E be a γ–hyperplane. With respect to an appropriate coordinate system, the Blaschke map δ is given a homogeneous coordinate representation

$$E = (e_0, e_1, \dots, e_n, 1) \mapsto \delta(E) = (1, e_1, \dots, e_n, e_0), \text{ with } e_1^2 + \ldots + e_n^2 = 1. \tag{3.10}$$

Taking the normalization in formula (3.1) into account, the set of γ–hyperplanes is mapped to points contained in a quadratic hypercylinder $\Delta \subset \mathbb{R}^{n+1}$ with equation

$$\Delta : x_1^2 + \cdots + x_n^2 = 1.$$

This quadric possesses one dimensional generator lines, parallel to the x_{n+1}–axis. The images of cycles, interpreted as set of oriented tangent planes appear in Δ as hyperplanar sections of Δ. The points of Δ together with its hyperplanar sections is called *Blasche model* or *Blaschke cylinder* of Euclidean Laguerre geometry.

Let e and f be parallel oriented hyperplanes in $\mathbb{R}^n$, then $E = \zeta^*(e)$ and $F = \zeta^*(f)$ are parallel γ–hyperplanes in $\mathbb{R}^{n+1}$. With formula (3.10) it follows that the image points $\delta(E)$ and $\delta(F)$ are contained in the same generator line of Δ (see Figure 3.1).

Since a Laguerre transformation appeared in the cyclographic model as pe–similarity, it may uniquely be extended to a projective map acting on $\mathbb{P}^{n+1}$, which maps Ω onto itself. This implies that δ transforms a pe–similarity to a projective automorphic map of Δ. It is an important advantage of the Blaschke model and also of the cyclographic model, that Laguerre transformations are represented by special projective maps in $\mathbb{P}^{n+1}$.

One can obtain an affine space by applying a stereographic projection to Δ. Sometimes this is preferable and is discussed in the following section.

3.4 The Isotropic Model

Let w be the generator line of Δ containing the point W with affine coordinates $(0,\ldots,0,1,0)$ in $\mathbb{R}^{n+1}$. Furthermore, let $\overline{\mathbb{R}}^n$ be the hyperplane $x_n = 0$ in $\mathbb{R}^{n+1}$, parallel to w. In $\overline{\mathbb{R}}^n$ a coordinate system shall be chosen, such that the origins of $\mathbb{R}^{n+1}$ and $\overline{\mathbb{R}}^n$ coincide and

$$y_1 = x_1, \ldots, y_{n-1} = x_{n-1}, y_n = x_{n+1}.$$

Let $\sigma : \Delta \setminus w \to \overline{\mathbb{R}}^n$ be a stereographic projection with center W. Together with the cyclographic mapping ζ^* and the Blaschke mapping δ one obtains

$$\sigma \circ \delta \circ \zeta^* : H \to \overline{\mathbb{R}}^n,$$

which maps oriented hyperplanes in $\mathbb{R}^n$ to points in $\overline{\mathbb{R}}^n$. In coordinates it reads as

$$\sigma \circ \delta \circ \zeta^*(e) = \frac{1}{1-e_n}(e_1, \ldots, e_{n-1}, e_0). \tag{3.11}$$

Interpreting cycles as sets of tangent hyperplanes, we may state an important result.

Lemma 3.1 *The set of tangent hyperplanes of a cycle c is mapped with $\sigma \circ \delta \circ \zeta^*$ to the set of points of a paraboloid of revolution or a hyperplane Ψ satisfying*

$$\sigma\circ\delta\circ\zeta^*(c) = \Psi : 2y_n + (y_1^2 + \ldots + y_{n-1}^2)(r+m_n) + 2y_1m_1 + \ldots + 2y_{n-1}m_{n-1} + r - m_n = 0. \tag{3.12}$$

The surfaces, defined by (3.12) are called *isotropic spheres*. For the low dimensional case $n = 2$ this is illustrated in Figure 3.1. Isotropic spheres are elementary parabolas or straight lines. So far, oriented hyperplanes in $\mathbb{R}^n$ with unit normal $(0, \ldots, 0, 1)$ do not have an image point in $\overline{\mathbb{R}}^n$. This shall be improved by the so–called *isotropic conformal closure* $I^n := \overline{\mathbb{R}}^n \cup \mathbb{R}$ of $\overline{\mathbb{R}}^n$. Additionally the map $\sigma \circ \delta \circ \zeta^*$ is extended to the map

$$\Lambda := \overline{\sigma} \circ \delta \circ \zeta^*,$$

which maps the oriented hyperplane $(e_0, 0, \ldots, 0, 1) \subset \mathbb{R}^n$ onto the real number e_0. To fix the problem of missing images of exceptional oriented hyperplanes in Lemma 3.1, we have to add the real number $r+m_n$ to the paraboloids Ψ, which equals 0 for a hyperplane Ψ. The resulting model of Euclidean Laguerre space, where oriented hyperplanes are represented by points and cycles appear as isotropic spheres, is called *isotropic model*.

Very important for our applications is the transformation Λ which describes the change from the standard model to the isotropic model. Also the inverse map Λ^{-1} is of particular interest. Let $y = (y_1, \ldots, y_n)$ be a point in I^n, the preimage in Euclidean $\mathbb{R}^n$ is the oriented hyperplane given by its normalized coordinates

$$\Lambda^{-1}(y) = \frac{1}{y_1^2 + \ldots y_{n-1}^2 + 1}(2y_n, 2y_1, \ldots, 2y_{n-1}, y_1^2 + \ldots y_{n-1}^2 - 1). \tag{3.13}$$

In I^n, y_n–parallel lines (called *isotropic lines*) represent parallel oriented hyperplanes of the standard model. For simplicity, let us now restrict to $n = 3$. In I^3, non–isotropic lines as well as ellipses, whose normal projection onto y_3=0 are circles, and parabolas with isotropic axis are called *isotropic Möbius circles*. They may be obtained as intersection of two isotropic spheres. Thus, they are the Λ–image of the common tangent planes of two cycles, that means the planes of a pencil or the tangent planes of a cone of revolution. This also shows that isotropic (y_3–parallel) planes as well as isotropic cylinders with a circular cross section in $y_3 = 0$ represent planes parallel to the planes of a pencil or to the tangent planes of a cone of revolution.

In the isotropic model, Laguerre transformations are realized as special quadratic transformations, so–called *isotropic Möbius transformations*. Let us consider two special cases. A translation in the standard model, represented in hyperplane coordinates by

$$(e_0, \dots, e_n) \mapsto (e_0 + a_1 e_1 + \dots + a_n e_n, e_1, \dots, e_n)$$

yields in I^n the transformation

$$(y_1, \dots, y_n) \mapsto \left(y_1, \dots, y_{n-1}, y_n + a_1 y_1 + \dots + a_{n-1} y_{n-1} + \frac{a_n}{2}(y_1^2 + \dots + y_{n-1}^2 - 1)\right). \tag{3.14}$$

A dilatation $(e_0, \dots, e_n) \mapsto (e_0 + d, e_1, \dots, e_n)$ appears as isotropic Möbius transformation

$$(y_1, \dots, y_n) \mapsto \left(y_1, \dots, y_{n-1}, y_n + \frac{d}{2}(y_1^2 + \dots + y_{n-1}^2 + 1)\right). \tag{3.15}$$

3.5 A Dual Isotropic Model

We pick a fixed oriented hyperplane a in $\mathbb{R}^n$, say $x_n = 0$ with normal $(0, \dots, 0, 1)$. An arbitrary oriented hyperplane e determines a γ–hyperplane $E = \zeta^*(e) = (e_0, \dots, e_n, 1)$. Its intersection with $\zeta^*(a) = (0, \dots, 0, 1, 1)$ is now projected orthogonally onto $\mathbb{R}^n$ and yields a (non oriented) hyperplane

$$\Lambda^*(e) = (e_0, \dots, e_{n-1}, e_n - 1). \tag{3.16}$$

It is the Euclidean bisector of e and the fixed oriented hyperplane a, namely the locus of points possessing the same signed distance to e and a. Currently, a has no image and the remaining oriented hyperplanes parallel to a are mapped to the hyperplane at infinity, which shows that the projective closure is not the right one for the image space. In fact, Λ^* generates a *dual isotropic model*. To convert to the previous isotropic model, we apply the polarity π with respect to the isotropic sphere

$$x_1^2 + \dots + x_{n-1}^2 - 2x_n = 0.$$

$\Lambda^*(e)$ is mapped to the point with inhomogeneous coordinates

$$\pi \circ \Lambda^*(e) = \frac{1}{1 - e_n}(e_1, \dots, e_{n-1}, -e_0).$$

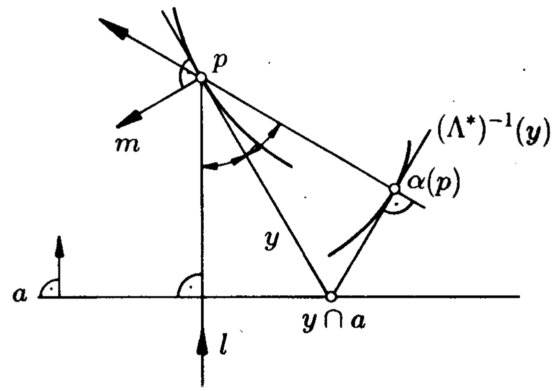

Figure 3.2: Anticaustic of reflection of a surface for parallel light rays

Up to a reflection at $x_n = 0$, we get exactly $\Lambda(e)$ as in (3.11). Hence, we can deduce the properties of the present dual isotropic model from the isotropic model.

In applications discussed later, the inverse transformation $(\Lambda^*)^{-1}$ is of particular interest. Taking the normalization $e_1^2 + \ldots + e_n^2 = 1$ in (3.16) into account, we find with non–normalized homogeneous coordinates $(y_0, y_1, \ldots, y_n)$ of the hyperplane y to be transformed,

$$(\Lambda^*)^{-1}(y) = \left(y_0, \ldots, y_{n-1}, \frac{y_n^2 - y_1^2 - \ldots - y_{n-1}^2}{2y_n} \right). \tag{3.17}$$

For more details on Laguerre geometry and its relation to Einstein's theory of special relativity, we refer to [2], [3], [4] and [8].

3.6 The Anticaustic Map

Closely related to the dual isotropic model is the anticaustic map. Let $\mathbb{R}^n$ be Euclidean n–space, and a be a fixed oriented hyperplane, say $x_n = 0$. Let Φ be a C^1–hypersurface, which is parametrized by $p(u) = (p_1, \ldots, p_n)(u)$ and the parameter u is $u = (u_1, \ldots, u_{n-1}) \in \mathbb{R}^{n-1}$. Let $(m_1, \ldots, m_n)(u)$ be a not necessarily normalized normal vector of the hypersurface Φ. Consider light rays perpendicular to a, which shall be reflected at the hypersurface Φ. Any hypersurface, which is perpendicular to the reflected light rays, is called an *anticaustic of reflection* to the given illumination. Then the anticaustics of reflection $\alpha(\Phi)$ of the hypersurface Φ with respect to the given light rays are parametrized by

$$\alpha(\Phi) : (p_1, \ldots, p_{n-1}, 0) - \frac{2(p_n - c)m_n}{m_1^2 + \ldots m_n^2}(m_1, \ldots, m_n), \tag{3.18}$$

with an arbitrary constant c. The connection to the dual isotropic model is the following. Let $(y_0, y_1, \ldots, y_n)(u)$ be homogeneous coordinates of the tangent hyperplane of Φ, and $(m_1, \ldots, m_n)(u)$ be the normal vector of Φ as above. Then it follows immediately that

$(\Lambda^*)^{-1}(y)$ are homogeneous coordinates of the tangent hyperplanes of an anticaustic $\alpha(\Phi)$, see Figure 3.2.

It is not necessary that Φ is a hypersurface. Let $(u_1, \ldots, u_k) = u \in \mathbb{R}^k$ such that $p(u)$ parametrizes a smooth k-dimensional manifold Ψ in $\mathbb{R}^n$. What means reflecting the light rays, perpendicular to a, at Ψ. Let Ψ be considered as set of tangent hyperplanes that means we interpret Ψ as dual hypersurface. It can be parametrized by homogeneous plane coordinates $y(u, v)$, where $v = (v_1, \ldots, v_{n-k-1})$ parametrizes the set of tangent hyperplanes passing through a fixed point Ψ. Let $m(u, v)$ be the surface normals of Ψ. Then we can use formula (3.18), which maps the k-dimensional surface Ψ to an anticaustic hypersurface $\alpha(\Psi)$. Additionally, formula (3.17) parametrizes an anticaustic as set of tangent hyperplanes. An example in 3–space is the special case, discussed in Section 4.2, illustrated in Figure 4.1.

3.7 The Cyclographic Map

Let $\mathbb{R}^n$ be Euclidean n–space, $\mathbb{R}^{n+1}$ be the corresponding cyclographic model. Let C be the set of cycles in $\mathbb{R}^n$. We had the map $\zeta : \mathbb{R}^n \to \mathbb{R}^{n+1}$ in Section 3.2. Now we will study the inverse map

$$\eta : \mathbb{R}^{n+1} \to \mathbb{R}^n. \tag{3.19}$$

It maps a point $x \in \mathbb{R}^{n+1}$ to a cycle $\eta(x)$ with center $(x_1, \ldots, x_n)$ and signed radius x_{n+1} and is called *cyclographic map*. Additionally we have a map

$$\begin{array}{llcl} \eta^* : & \gamma\text{–hyperplanes} & \to & H, \\ & E = (e_0, e_1, \ldots, e_n, e_1^2 + \ldots e_n^2) & \mapsto & e = (e_0, e_1, \ldots, e_n) \end{array} \tag{3.20}$$

which maps γ–hyperplanes E to oriented hyperplanes e in $\mathbb{R}^n$. Clearly, η^* determines η and vice versa.

It is an important question, how points p and q behave, if the cycles $\eta(p)$ and $\eta(q)$ are in oriented contact.

Lemma 3.2 *All points x, contained in a γ–cone $\Gamma(p)$ with vertex p are mapped to cycles $\eta(x)$, which are in oriented contact to the cycle $\eta(p)$ and vice versa. Further, the set of γ–hyperplanes E_p, tangent to a γ–cone $\Gamma(p)$ are mapped to oriented tangent planes $\eta^*(E_p)$ of the cycle $\eta(p)$.*

Let φ be a smooth k–dimensional manifold in $\mathbb{R}^{n+1}$. Applying η, we obtain a k–parametric set of cycles in $\mathbb{R}^n$. The envelope of all cycles $\eta(x)$, where $x \in \varphi$ is called the cyclographic image $\eta(\varphi)$ of φ.

Consider the cyclographic image $\eta(\varphi)$ as set of oriented tangent planes. Let $\Gamma(\varphi)$ be the γ–surface of φ, that is the envelope of all γ–hyperplanes, passing through tangent k-spaces of φ. It follows from Lemma 3.2 that $\eta(\varphi)$ is the intersection of $\Gamma(\varphi) \cap \mathbb{R}^n$. In this sense we can denote the cyclographic image also by $\eta^*(\varphi)$.

We mainly concentrate on $\eta : \mathbb{R}^4 \to \mathbb{R}^3$. The sets φ which are mapped are smooth curves and smooth surfaces.

3.8 The Cyclographic Image of Curves

Let $\mathbb{R}^4$ be cyclographic model of 3–dimensional Laguerre space $\mathbb{R}^3$. Let the orthogonal projection of a set $x \subset \mathbb{R}^4$ onto $\mathbb{R}^3$ be denoted by x'. As first and simple case we start with the cyclographic image of a line. Each point $p \in g$ is mapped to a cycle $\eta(p)$, with center $p' \in g'$.

Let $g \subset \mathbb{R}^4$ be a hyperbolic line. The image cycles $\eta(p)$ of points $p \in g$ possess a cone of revolution Φ as real envelope with axis g' and vertex $g \cap \mathbb{R}^3$. If g is parallel to $\mathbb{R}^3$, the envelope is a line or a cylinder, depending on whether the x_4–coordinate of points in g is zero or not.

The one parameter set of γ–hyperplanes $E(t)$ passing through g intersect $\mathbb{R}^3$ in a set of oriented planes $e(t)$. These oriented planes are tangent to each cycle $\eta(p)$ with $p \in g$ and are oriented tangent planes of the cone of revolution Φ.

If g is a parabolic line in $\mathbb{R}^4$, there is exactly one γ–hyperplane E passing through g. Let $s = g \cap \mathbb{R}^3$ and $e = E \cap \mathbb{R}^3$. The line g' is perpendicular to the oriented plane e. Each point $p \in g$ is mapped to a cycle, which is tangent to e in s.

If g is an elliptic line, the set of cycles $\eta(p)$, with $p \in g$ has no real envelope Φ. One may use the complex extension of Euclidean 3–space. The envelope Φ is an imaginary cone of revolution, with real vertex $g \cap \mathbb{R}^3$ and real axis g', such that Φ is given by a real equation. But for our intentions this is not of particular interest.

Let $C(t)$ be a regular C^1–curve in $\mathbb{R}^4$, with only hyperbolic tangent lines, except isolated parabolic ones. The cyclographic image Φ is the envelope of a one parameter family of spheres $\eta(C(t))$, thus a real *canal surface*. Each cycle $c(t)$ is tangent to Φ at the characteristic circle $k(t)$, which can be obtained as intersection $(c \cap \dot{c})(t)$. The hyperbolic tangent lines $C + \lambda\dot{C}$ are mapped to cones of revolution, which are tangent to Φ in points of $k(t)$. A parabolic tangent line is mapped to a surface element $(e(t), s(t))$, where $e(t)$ is a tangent plane and $s(t) = (C + \lambda\dot{C})(t) \cap \mathbb{R}^3$ is the point of contact.

Let $C(t)$ be a C^1–curve with only parabolic tangent lines. The cyclographic image $\eta(C)$ is a one parameter family of surface elements $(e(t), s(t))$ which is called *surface strip*.

Lemma 3.3 *Let $C(t)$ be a C^1–curve in $\mathbb{R}^4$ and let $\Gamma(C(t))$ be the γ–hypersurface of $C(t)$. The cyclographic image $\eta(C(t))$ is the intersection $\Gamma(C(t)) \cap \mathbb{R}^3$.*

A good and non trivial example for cyclographic images of curves are Dupin cyclides. Because of their plenty of geometric properties, they appeared quite often in the literature. In particular, nowadays these surfaces are of certain interest for surface modeling, see for instance [6], [30], [42], [43], [46], [47]. A detailed description of rational curves and surface patches on Dupin cyclides is given in [29].

3.8.1 Dupin Cyclides

A conic C in $\mathbb{R}^4$ is called a *pe circle*, if its intersection points $C \cap \omega$ are contained in the absolute quadric Ω. This is an appropriate generalization of circles from the Euclidean point of view. We can distinguish between three types of pe circles, depending on the type of the plane $\pi \supset C$. If π is Euclidean, C is an elementary ellipse, in particular if π is parallel to $\mathbb{R}^3$, C is an elementary circle. If π is pseudo–Euclidean, C is an elementary hyperbola, and finally if π is a γ–plane, C is an elementary parabola.

A Dupin cyclide Φ can be generated as the envelope of a set of spheres, which are tangent to three fixed spheres a_i. Using an arbitrary orientation for the spheres a_i, the cycles a_i are mapped to points $\zeta(a_i) = A_i$. By Lemma 3.2 it follows that all cycles, which are tangent to a_i are mapped to the intersection of the γ–cones

$$\Gamma(A_1) \cap \Gamma(A_2) \cap \Gamma(A_3) = C,$$

where C is a pe circle.

Lemma 3.4 *The cyclographic image of a pe circle C with hyperbolic tangent lines, $C \notin \mathbb{R}^3$, is a Dupin cyclide.*

Consider a Dupin cyclide Φ as envelope of its oriented tangent planes. From Lemma 3.3 we know that $\Phi = \Gamma(C) \cap \mathbb{R}^3$ and $\Gamma(C)$ is formed by the common tangent hyperplanes of the pencil of dual hyperquadrics $\lambda C + \mu\Omega$. The Dupin cyclide Φ can be generated as cyclographic image $\eta(q)$ of any singular hyperquadric q of this pencil. Thus, we calculate these singular hyperquadrics which correspond to the zeros of the polynomial $\det(\lambda C + \mu\Omega)$. Two cases have to be discussed.

Firstly, let C be a pe circle in a Euclidean plane. We may apply a Laguerre transformation α, such that $\alpha(C)$ is given by the equation

$$\alpha(C) : -aX_0^2 + X_1^2 + X_2^2 = 0, a \in \mathbb{R}^+.$$

The polynomial $\det(\lambda\alpha(C) + \mu\Omega)$ possesses $\lambda = 0$ as double zero and $\mu = 0$ as one fold zero. A further double zero is $\lambda = -\mu$ and the corresponding singular quadric is the pe circle

$$\alpha(D) : -aX_0^2 - X_3^2 + X_4^2 = 0$$

Since $\alpha(D)$ is contained in a pe plane, the same is valid for D. The γ–hypersurface $\Gamma(C) = \Gamma(D)$ is of class 4. The cyclographic image Φ in $\mathbb{R}^3$ is also of class 4 and of order 4.

Proposition 3.1 *The cyclographic image of a pe circle C with hyperbolic tangent lines, not contained in $\mathbb{R}^3$ or in a γ–plane, is a Dupin cyclide Φ of order 4. Additionally, Φ is the cyclographic image of the pe circle D, which is a further singular quadric in the pencil $\lambda C + \mu\Omega$, different from Ω. The pe circle D is contained in a plane, pe perpendicular to the plane containing C, their centers coincide and they possess same pe radii.*

Secondly, let C be a pe circle contained in a γ–plane. Thus, C is an elementary parabola. We may assume that C is given by the equation

$$C : X_0(X_3 + X_4) + (b - a)X_2^2 + a(X_4^2 - X_3^2) = 0,$$

such that as set of points, C is contained in the plane

$$x_1 = 0, ax_0 + x_3 - x_4 = 0,$$

and possesses the point $[0, 0, 0, 1, 1]$ at infinity. Taking a closer look to the pencil of dual quadrics $\lambda C + \mu\Omega$, we find further singular quadrics, contained in this pencil, namely the pe circle D

$$D : X_0(X_3 + X_4) + (a - b)X_1^2 + b(X_4^2 - X_3^2) = 0.$$

It is contained in the plane

$$x_2 = 0, bx_0 + x_3 - x_4 = 0,$$

and possesses the same point at infinity as C. A further singular quadric of rank three is

$$P : X_0(X_3 + X_4) + aX_1^2 + bX_2^2 = 0.$$

As set of points, it is a paraboloid contained in the γ–hyperplane $x_3 - x_4 = 0$ and possesses the same point at infinity as C. The γ–hypersurface $\Gamma(c)$ is of class 4, but of order 3. In Section 4.4 a further generation of a Dupin cyclide of order 3 is discussed.

Proposition 3.2 *The cyclographic image of a pe circle C, contained in a γ–plane is a parabolic Dupin cyclide Φ. It can be generated as cyclographic image of the pe circle D or the paraboloid P, both contained in the pencil $\lambda C + \mu\Omega$.*

This concept can be used to model canal surfaces composed of Dupin cyclide pieces. In [39] a technique is described how to generate a smooth blend surface between two given cones of revolution with two given circles on each of them. The blend surface is a pair of Dupin cyclides and is tangent to each cone at the given circles.

3.9 Cyclographic Image of Surfaces

The cyclographic mapping of surfaces is analogous to the curve case. Let $\varphi = s(u, v)$ be a smooth surface in $\mathbb{R}^4$. In general one can define the cyclographic image $\Phi = \eta(\varphi)$ as envelope of the two parameter set of cycles $c(u, v) = \eta(s(u, v))$. This envelope can contain real and imaginary parts. Analogously to the curve case the cyclographic image can be obtained with help of γ–hypersurfaces.

Lemma 3.5 *Let φ be a smooth two dimensional surface in $\mathbb{R}^4$ and let $\Gamma(\varphi)$ be the envelope of all γ–hyperplanes, passing through tangent planes of φ. The cyclographic image $\eta(\varphi)$ is the intersection $\Gamma(\varphi) \cap \mathbb{R}^3$.*

Let us start with a plane φ, and firstly φ is Euclidean. If φ is not parallel to $\mathbb{R}^3$, then let $f = \varphi \cap \mathbb{R}^3$. The cyclographic image of φ is the pair of planes, which pass through f and which are tangent to cycles $\eta(x)$ for all points $x \in \varphi$. If φ is parallel to $\mathbb{R}^3$, the cyclographic image consists of parallel planes with opposite orientation.

Let φ be a γ–plane and $f = \varphi \cap \mathbb{R}^3$. There is exactly one γ–hyperplane Γ passing through φ. Then $\eta(\varphi) = \Gamma \cap \mathbb{R}^3$ and all cycles $\eta(x)$ for $x \in \varphi$ are tangent to $\eta(\varphi)$ in f.

If φ is a pe plane, $\eta(\varphi)$ consists of a conjugate complex pair of planes passing through $f = \varphi \cap \mathbb{R}^3$.

Let φ be a general smooth surface. We proceed as above and map its surface elements (x, τ), where τ is the tangent plane at $x \in \varphi$. We fix one surface element. If τ is Euclidean, then the cyclographic image consists of two surface elements (x_1, τ_1) and (x_2, τ_2), where (τ_1, τ_2) is the cyclographic image of τ. The points x_1, x_2 are points of contact of the planes τ_1 and τ_2 with the cycle $\eta(x)$. The normals of these surface elements (x_1, τ_1) and (x_2, τ_2) are the lines x_1x' and x_2x'. If all tangent planes are Euclidean, the entire cyclographic image consists of two sheets

$$\eta(\varphi) = \eta(\varphi)_1 \cup \eta(\varphi)_2$$

corresponding to the surface elements (x_1, τ_1) and (x_2, τ_2). Compare Section 3.10 and see Figure 3.3.

A particular discussion is necessary in the case of a surface φ all whose tangent planes are γ–planes. Let φ be called a γ–surface. At each of its points, there is a unique parabolic surface tangent line. Integrating this field of γ–tangent lines one obtains a family of γ–curves on φ. Their cyclographic images are principal curvature strips of the cyclographic image surface $\eta(\varphi)$. The projection φ' of s is locus of principal curvature centers, that means one sheet of the focal surface of $\eta(\varphi)$. The surface $\eta(\varphi)$ possesses a second sheet of the focal surface corresponding to the other family of principal curvature lines. The γ–tangent lines of φ form a *γ–hypersurface* $\Gamma(\varphi)$ in $\mathbb{R}^4$, which may also be considered as cyclographic preimage $\eta^{-1}(y)$ of all cycles y, which are in oriented contact to the oriented surface $\eta(\varphi)$. The γ–hypersurface $\Gamma(\varphi)$ possesses φ as set of singular points.

Let π be a hyperplane in $\mathbb{R}^4$, parallel to $\mathbb{R}^3$. The top views of all intersections $\Gamma(\varphi) \cap \pi$ are offset surfaces of $\eta(\varphi)$ and possess edges of regression at $\varphi \cap \pi$. This yields the well–known fact that edges of regression of the offset surfaces to a surface are contained in its focal surface.

3.9.1 Cyclographic Image of Quadrics

Let q be a 2–dimensional pe sphere, which can be defined to be the intersection of a hyperplane π with a γ–cone $\Gamma(a)$. For a γ–hyperplane π, the complete γ–hypersurface $\Gamma(q)$ through q is $\pi \cup \Gamma(a)$ and therefore $\eta(q)$ is a *cycle* $\Gamma(a) \cap \mathbb{R}^3$ and an *oriented plane* $\pi \cap \mathbb{R}^3$. Otherwise, there exists a pe reflection σ at π which maps $\Gamma(a)$ to a second γ–cone $\Gamma(\sigma(a))$ through q. The cyclographic image $\eta(q)$ is the *union of two cycles* $\eta(a)$ and $\eta(\sigma(a))$, which are the intersections of $\Gamma(a)$ and $\Gamma(\sigma(a))$ with $\mathbb{R}^3$.

A further discussion of the cyclographic image of quadrics q, different from pe spheres is given in Section 4.7. There we will prove that the cyclographic image $\eta(q)$ of regular quadrics q is rational and possess rational offset surfaces.

3.9.2 Cyclographic Image of Ruled Surfaces

As mentioned in a further section, the cyclographic image of a straight line $g \in \mathbb{R}^4$ is a cone of revolution $\Delta = \eta(g)$, and degenerated cases. Let φ be a ruled surface, generated by a one parameter family of lines $g(t)$. The cyclographic image $\eta(\varphi)$ is the envelope of a one parameter family of cones of revolution $\Delta(t) = \eta(g(t))$, including degenerate cases. Let almost all generator lines be hyperbolic, except finitely many parabolic generator lines.

Firstly, let $\varphi = g(t)$ be a non developable ruled surface. Let g_0 be a fixed generator line. The surface elements (x_0, τ_0) of φ along this fixed generator line g_0 are mapped to surface elements of the cyclographic image $\eta(\varphi)$. They define the curve d_0 on the cone $\Delta_0 = \eta(g_0)$, and $\eta(\varphi)$ is in oriented contact to Δ_0 in points of d_0. We note here that in general d_0 is a rational quartic space curve, but can take degenerate forms. In the non developable case the curve d_0 is different from a generator line of Δ_0, and $\eta(\varphi)$ is never developable.

If all tangent planes τ_0 along the generator line g_0 are Euclidean, then all surface elements of φ in points of g_0 possess a real cyclographic image. Otherwise there is a non empty segment on g_0, which possesses a real cyclographic image.

Secondly, let φ be a developable surface. Along each generator line g_0 there is a fixed tangent plane τ_0. Let τ_0 be Euclidean, then there are two γ–hyperplanes passing through it. The γ–surface $\Gamma(\varphi)$ is enveloped by two one parameter sets of γ–hyperplanes. With Lemma 3.5 it follows that the cyclographic image $\eta(\varphi)$ consists of two developable surfaces. The cone $\Delta_0 = \eta(g_0)$ is tangent to $\eta(\varphi)$ in points of two generator lines of Δ_0. Clearly, if τ is a γ–plane, we obtain only one generator line on Δ_0 and if τ is pe, there is no real envelope.

Consider the case, where the line of regression r of φ is a regular curve. As assumed, the tangent lines of r are almost all hyperbolic, except finitely many γ–lines. The cyclographic image $\eta(r)$ is a real canal surface. As set of tangent planes, $\eta(r)$ contains the developables $\eta(\varphi)$, or in other words, $\eta(\varphi)$ is tangent to the canal surface $\eta(r)$.

We will return to cyclographic images of ruled surfaces, when discussing envelopes of one parameter sets of cones of revolution in Section 4.5. There we will prove that a non developable rational ruled surface φ possesses a rational cyclographic image surface $\eta(\varphi)$.

3.10 Optical Interpretation

Let $\mathbb{R}^3$ be Euclidean Laguerre space. Let φ be a smooth surface in the cyclographic model $\mathbb{R}^4$ and let $x \in \varphi$ be a point with Euclidean tangent plane τ. The cyclographic image of the surface element (x, τ) consists of two surface elements (x_1, τ_1) and (x_2, τ_2), as discussed above. The normals of these elements are $x'x_1$ and $x'x_2$, which form the same angle with the tangent plane τ' to φ' at x' (see Figure 3.3).

We can now formulate an *optical interpretation*. Let φ' be the top view of φ in $\mathbb{R}^3$, and φ' shall be a smooth surface. Consider the normals of one sheet of the cyclographic image, say $\eta(\varphi)_1 =: \varphi_1$, as light rays. Then, $\eta(\varphi)_2 =: \varphi_2$ is a reflectional anticaustic to the mirror surface φ'. If the surface φ is contained in a γ–hyperplane Γ, $\eta(\varphi)_1$ is $a := \Gamma \cap \mathbb{R}^3$ and

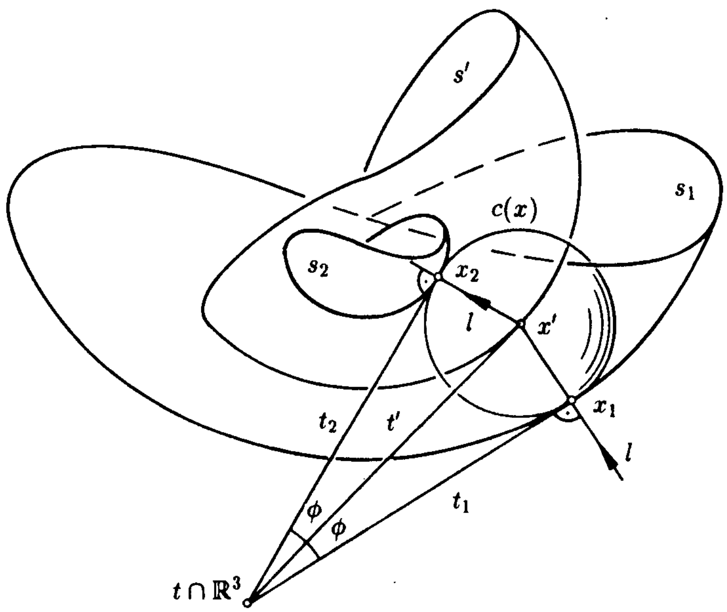

Figure 3.3: Cyclographic image of a surface and optical interpretation

$\eta(\varphi)_2$ is an anticaustic to φ' for parallel light rays, perpendicular to a, see Figure 3.2. If φ is contained in a γ–cone, we obtain an anticaustic for central illumination.

Note the relation between the construction of anticaustics in the case of parallel light rays, orthogonal to a plane a and the transformation Λ^* from the standard model of Laguerre space to its dual isotropic model (see Section 3.5). An anticaustic for the given illumination to a mirror surface φ', which shall be interpreted as set of tangent planes, is

$$(\Lambda^*)^{-1}(\varphi')$$

and computed with formula (3.17). The point representation of an anticaustic is given by formula (3.18). Varying the plane a in a pencil of parallel planes, one obtains all anticaustics, which are offset surfaces of each other.

Chapter 4

Rational Surfaces with Rational Offsets

There is a lot of literature on rational curves with rational offset curves, for instance see ([19], [16], [35], [26], [36]). Concerning rational surfaces with rational offsets we refer to the articles [35], [25], [27], [38], [24]. In [35] the dual representation of curves and surfaces is used to describe rational surfaces with rational offsets. Here, some geometric ideas to generate rational surfaces with rational offsets shall be presented.

4.1 PN Surfaces in Euclidean 3–Space

Definition: Let φ be a rational surface in $\mathbb{R}^3$, parametrized by $x(u,v) = (x_1, \ldots, x_3)(u,v)$. The parametrization $x(u,v)$ is called a *PN parametrization*, if and only if the unit normal vectors $n(u,v)$ of φ are rational. Further, φ is called a *PN surface*, if and only if it possesses a PN parametrization.

Let φ be a PN surface, then all one sided offset surfaces φ_d at an arbitrary oriented distance d, which are parametrized by $x_d = x + dn$, are rational. The notation PN comes from *Pythagorean Normal Vector Field* and is the analogon to *Pythagorean Hodograph curves*, which were introduced in [19].

We do not discuss the trivial case where φ is a plane. To get a better insight to rational PN surfaces, let a surface φ be interpreted as envelope of its oriented tangent planes (see [35])

$$(e_0 + e_1 x_1 + e_2 x_2 + e_3 x_3)(u,v) = 0.$$

A dual parametrization of φ is given by (non normalized) homogeneous plane coordinates $e(u,v) = (e_0, \ldots, e_3)(u,v)$. Let φ be a PN surface. As developed in [35], φ admits a dual parametrization of the form

$$(e_0, e_1, e_2, e_3) = (g, 2acf, 2bcf, (a^2 + b^2 - c^2)f), \tag{4.1}$$

where a, b, c, g, f are polynomials in $\mathbb{R}[u, v]$, and can be assumed to have no common factor. The not normalized surface normal is represented by (e_1, e_2, e_3). Often it is useful to work with oriented normalized plane coordinates

$$e(u, v) = \left(h, \frac{2ac}{a^2 + b^2 + c^2}, \frac{2bc}{a^2 + b^2 + c^2}, \frac{a^2 + b^2 - c^2}{a^2 + b^2 + c^2}\right), \tag{4.2}$$

where h is an arbitrary rational function in $\mathbb{R}(u, v)$ and a, b, c are polynomials in $\mathbb{R}[u, v]$ without a common factor. We see that substituting h by $h + d$ in (4.2) parametrizes the one sided offset surface φ_d at oriented distance d, which is again a PN surface. If φ is regular as set of tangent planes, the conversion to a point representation $p(u, v)$ is done by intersecting the tangent plane e with its first derivative planes e_u and e_v and we obtain

$$p(u, v) = (e \cap e_u \cap e_v)(u, v).$$

The explicit representation in terms of a, b, c and h may be found in [35].

4.2 PN Surfaces in the Cyclographic Model

How do PN surfaces appear in the cyclographic model? Let φ be a PN surface, represented in Cartesian point coordinates $p = (p_1, p_2, p_3)(u, v)$. Let $n = (n_1, n_2, n_3)(u, v)$ be the normalized surface normal. Considering φ as set of oriented tangent planes

$$e(u, v) : (x - p) \cdot n = 0,$$

we can apply the map ζ^* and obtain in the cyclographic model a rational two–parameter family of γ–hyperplanes

$$\zeta^*(e)(u, v) = E(u, v) = (-n_1 p_1 - n_2 p_2 - n_3 p_3, n_1, n_2, n_3, 1)(u, v). \tag{4.3}$$

The envelope of $E(u, v)$ is a *rational γ–hypersurface* $\Gamma(\varphi)$, and can also be considered as preimage of φ with respect to the map η. Further $\Gamma(\varphi)$ contains a 2–parameter family of generating γ–lines

$$g(u, v) = (E \cap E_u \cap E_v)(u, v),$$

where E_u and E_v are first partial derivatives of E with respect to u and v, respectively. The singular set ψ of $\Gamma(\varphi)$, which is a γ–surface, is not necessarily rational, but the generating lines $g(u, v)$ depend rationally on the parameters u and v. This implies that the intersection of $\Gamma(\varphi)$ with any hyperplane is a rational surface. In particular, the intersection of $\Gamma(\varphi)$ with an arbitrary γ–hyperplane A is a rational surface $m(u, v)$ in $\mathbb{R}^4$. With Lemma 3.5 it follows that $\Gamma(\varphi) \cap \mathbb{R}^3$ is contained in the cyclographic image $\eta(m(u, v))$. Let $A \cap \mathbb{R}^3 =: a$ be the intersection plane of A with $\mathbb{R}^3$, then $\eta(m(u, v)) = \varphi \cup a$.

The use of the cyclographic map leads to an optical interpretation. Let $m' \subset \mathbb{R}^3$ be the top view of m. Consider light rays perpendicular to a and consider the reflection at the mirror surface m'. Then, the top view ψ' of ψ is the caustic and φ is an anticaustic. This

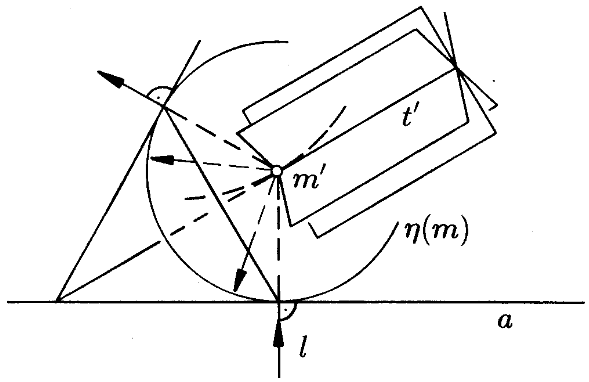

Figure 4.1: Anticaustic of reflection of a curve for parallel light rays

proves that a PN surface is anticaustic to a rational mirror surface m' for an illumination perpendicular to a.

Conversely, let a rational mirror surface $m' \subset \mathbb{R}^3$ be given by a dual parametrization in homogeneous plane coordinates

$$m' : (y_0, y_1, y_2, y_3)(u, v).$$

The map $(\Lambda^*)^{-1}$ maps tangent planes of m' to oriented tangent planes of an anticaustic for light rays perpendicular to a. Choose $a : x_3 = 0$. Since $y_0, \ldots, y_3$ are rational functions, formula (3.17) proves that $(\Lambda^*)^{-1}(y)$ is a dual PN parametrization, such that the anticaustic φ is a PN surface. A point representation of the anticaustic is given by Formula 3.18.

It is important to note the following special case. It may happen that $m = \Gamma(\varphi) \cap A$ is just a rational *curve* m. Its projection m' in $\mathbb{R}^3$ is interpreted as 2–parameter set of tangent planes, which are the pencils passing through each tangent line t' of m'. We may also perform the reflection of light rays perpendicular to a. The anticaustic φ is a rational canal surface with the constant tangent plane a. Again, $\varphi \cup \alpha$ is the cyclographic image of m. Figure 4.1 shows the reflection of light rays perpendicular to a at the planes of a pencil passing through t'. The reflected light rays are perpendicular to the cycle $\eta(m)$ in points of the characteristic circle.

In [22] is is proved that rational curves with rational offsets are the anticaustics to rational mirror curves for parallel light rays. Here we have the spatial analogon.

Theorem 4.1 *PN surfaces are exactly the anticaustics of rational curves or surfaces for parallel light rays. An anticaustic to a rational mirror curve is a rational canal surface which is in line contact with a plane orthogonal to the light rays.*

Since the cyclographic map as well as the anticaustic map (3.18) and $(\Lambda^*)^{-1}$ in (3.17) are formulated for arbitrary dimension n, Theorem 4.1 can be generalized to k–dimensional surfaces in $\mathbb{R}^n$. Compare the statements in Section 3.6.

4.3 Blaschke Cone and Isotropic Model

We study PN surfaces $\varphi \subset \mathbb{R}^3$ in the Blaschke model and the isotropic model of Euclidean Laguerre space, respectively. A generalization to $\mathbb{R}^n$ is straightforward.

Using the cyclographic model, a PN surface $\varphi \subset \mathbb{R}^3$ is mapped to a γ–hypersurface $\zeta^*(\varphi) = \Gamma(\varphi)$ and we have representation (4.3). Mapping $\Gamma(\varphi)$ to the Blaschke cone Δ results in a surface $\delta \circ \zeta^*(\varphi)$ on $\Delta \subset \mathbb{P}^4$ with representation in homogeneous point coordinates

$$\delta \circ \zeta^*(\varphi) = \delta(E(u,v)) = (1, n_1, n_2, n_3, -n_1x_1 - n_2x_2 - n_3x_3)\,(u,v).$$

Note that $\delta(E(u,v)) \cap x_4 = 0$ is a parametrization of the Gaussian image of φ. Laguerre transformations appear as projective maps in the Blaschke model and projective maps preserve the rationality of a surface. This proves the following result.

Theorem 4.2 *Laguerre transformations in $\mathbb{R}^3$ map PN surfaces (as sets of oriented tangent planes) onto PN surfaces.*

Remark: PN surfaces are also invariant under Möbius transformations in 3-space. For the proof of this fact, it is sufficient to show invariance under inversions. Together with the invariance under Laguerre transformations we can state that *PN surfaces are invariant under Lie transformations*. For details on Lie geometry, we refer to [4] and [8].

The extended stereographic projection $\bar{\sigma}$ defines a bijective map between rational surfaces on Δ and rational surfaces in the isotropic model I^3. This gives us a simple construction of PN surfaces.

Theorem 4.3 *Let Λ^{-1} be the geometric transformation which describes the change from the isotropic model I^3 of 3–dimensional Laguerre space to the standard model $\mathbb{R}^3$. The Λ^{-1}–image of a rational curve or surface in I^3 is a developable or non–developable PN surface in $\mathbb{R}^3$, respectively. Every PN surface may be obtained in this way.*

A degenerate case should be mentioned. Let g be an isotropic line in I^3, that means g is parallel to the y_3–axis of the chosen coordinate system. It corresponds to a pencil of parallel oriented planes in $\mathbb{R}^3$. A rational cylinder in I^3 with isotropic generators belongs to a 2–parameter set of planes that touch a rational curve at infinity. Moreover, if the rational surface $\Psi \subset I^3$ possesses a real curve along which it is touched by an isotropic cylinder, the corresponding PN surface in $\mathbb{R}^3$ contains a real curve at infinity.

The mapping $(\Lambda^*)^{-1}$ from the dual isotropic model to the standard model $\mathbb{R}^3$ is a geometric transformation that maps a surface in I^3 onto an anticaustic for parallel illumination. Therefore, Theorem 4.3 is dual to the interpretation of PN surfaces as anticaustics in Theorem 4.1.

A PN surface appears in the isotropic model as

$$y(u,v) = (\frac{a}{c}, \frac{b}{c}, \frac{g}{2c^2 f})(u,v), \tag{4.4}$$

which follows from Formula (3.11) and (4.1). If the PN surface $y(u,v)$ is derived from a quadratic parametrization of the unit sphere, the polynomials a, b and c are linear in u and v. By an appropriate reparametrization we can assume $a = u$, $b = v$ and $c = 1$. This implies that non–developable PN surfaces, derived from a quadratic spherical representation, are mapped to graphs of rational functions in I^3.

Theorem 4.4 *A rational PN surface, derived from a dual quadratic spherical representation appears in the isotropic model as graph of a rational function.*

4.4 Modeling with PN Surfaces

Let $y(u,v) \subset I^3$ be the graph of a quadratic polynomial. Lemma 3.1 says that an isotropic sphere y corresponds to a cycle in $\mathbb{R}^3$. Otherwise we can assume the normal form

$$y(u,v) = \alpha u^2 + \beta v^2, \text{ with } \alpha \neq \beta,$$

where α or β could also be zero. The preimages $\Lambda^{-1}(y)$ of these quadratic functions are well known PN surfaces, namely *parabolic Dupin cyclides*, which possess a dual normalized parametric representation of the form

$$e(u,v) = \frac{1}{u^2 + v^2 + 1}(\alpha u^2 + \beta v^2, 2u, 2v, u^2 + v^2 - 1).$$

This can be used to develop a surface modeling scheme using parabolic Dupin cyclides.

Let scattered data elements (A_i, α_i) be given in $\mathbb{R}^3$, where A_i are vertices, coinciding with the oriented planes α_i. In [31] it is discussed how to construct a C^1–PN surface, which interpolates the given data and which is composed of triangular patches from parabolic Dupin cyclides. The concept is the following.

The data (A_i, α_i) are mapped by Λ to I^3 and the images are scattered data elements, say (B_i, β_i), with $\Lambda(\alpha_i) = \beta_i$. The data (B_i, β_i) will be interpolated by a C^1 function Ψ, which is piecewise quadratic, using the method of Powell–Sabin [41]. Returning to the standard model we obtain a C^1 interpolating surface $\Lambda(\Psi)$, composed of parabolic Dupin cyclides. We note that in general the triangular cyclide pieces are tangent to each other along cubics and not along circles. This already indicates that this method is rather different from other surface modeling schemes, using (parabolic) Dupin cyclides, as [42], [46] and others.

Parabolic Dupin cyclides may also be generated with help of the optical interpretation in Theorem 4.1. To do so, we have to convert them to the dual isotropic model. The polarity of an isotropic sphere maps points of a paraboloid y to tangent planes of a paraboloid y^*, in general. A special case occurs if y is a parabolic cylinder y, whose points are mapped onto planes, which are tangent to a parabola with y_3–parallel axis. Using Theorem 4.1, we recognize that parabolic Dupin cyclides are *anticaustics of paraboloids or parabolas* for light rays parallel to the axis of the mirror curve or surface. This is proved in [49]. This optical interpretation comes into play if we generated parabolic Dupin cyclides as cyclographic images of pe circles in Section 3.8.1.

The presented results can be used for modeling with PN surfaces based on the geometric transformation Λ. Further, Λ preserves the order of geometric contact, but one has to take into account that we are working with the dual representation in $\mathbb{R}^3$. Parabolic points of a surface in $\mathbb{R}^3$ correspond to singularities of the Gauss map and therefore to singularities in the isotropic model. Conversely, having a regular surface in the isotropic model, then the preimage in $\mathbb{R}^3$ is regular as set of oriented tangent planes, but not necessarily as point set. Note also that in Laguerre geometry a surface and its offset surfaces are equivalent, since they are related by Laguerre transformations (dilatations, see Formula (3.4)). But clearly, the offset surfaces of a regular surface are not necessarily regular.

4.5 Envelopes of Cones of Revolution

In this section we study surfaces, enveloped by one parameter families of cones of revolution. In particular, PN surfaces will be constructed as envelopes of certain rational families of cones.

Definition: A one parameter family of cycles $c(t) = (m_1, m_2, m_3; r)(t)$ is called rational, if the center curve $m(t)$ and the radius function $r(t)$ are rational in the same parameter t.

A rational one parameter family of cycles corresponds to a rational curve in the cyclographic image. If two cycles possess real common oriented tangent planes, they form a unique real cone of revolution. Let two rational families of cycles $c_1(t), c_2(t)$ be given, parametrized by a common parameter t.

Definition: The common oriented tangent planes of two rational families of cycles $c_1(t)$ and $c_2(t)$ define a *rational one parameter family of cones of revolution* $\Delta(t)$ *with rational radius function.*

We have to point out the important difference to *rational one parameter families of cones* in the sense of Chapter 2, where we only required that the cones are given by equations with rational coefficients. See also the difference between *rational one parameter families of spheres* in Section 2.5 and the above given definition of rational families of cycles.

Each cone $\Delta(t)$ shall be considered as envelope of its oriented tangent planes. Degenerate cases are included, for instance, let both functions $r_1(t)$ and $r_2(t)$ be identically zero. Then, each cone $\Delta(t)$ degenerates to the line, connecting the centers of the cycles c_1 and c_2. More precisely, $\Delta(t)$ consists of a pencil of planes, passing through this line.

A family $\Delta(t)$ is called *real*, if $\Delta(t)$ possesses real tangent planes for all $t \in \mathbb{R}$. If $\Delta(t)$ is real only for parameter values t contained in a real interval $[a, b]$, one can reparametrize, for instance by the quadratic function

$$t = \frac{a + bs^2}{1 + s^2}.$$

It follows that $\Delta(s)$ is real for all $s \in \mathbb{R}$. For finitely many parameter values t_i, the degeneracy of $\Delta(t_i)$ to a surface element is allowed, which occurs if $\Delta(t_i)$ is determined by cycles $c_1(t_i)$ and $c_2(t_i)$ which are in oriented contact.

A real rational one parameter family of cones of revolution $\Delta(t)$ with rational radius function is the cyclographic image of a rational ruled surface $\varphi = g(t) \subset \mathbb{R}^4$ with only hyperbolic generator lines $g(t) = \zeta(\Delta(t))$, except finitely many parabolic generator lines.

The cyclographic image of the entire surface is the envelope of $\Delta(t)$. In general, $\Delta(t)$ is tangent to the envelope in points of a rational quartic curve $\Delta(t) \cap \dot{\Delta}(t)$. We omit the discussion of all cases here, but note that the quartic can be reducible, for instance consists of a conic and two, not necessarily real generating lines of $\Delta(t)$. In this case the envelope is reducible and consists of a surface generated by a one parameter family of conics and additionally two developable surfaces.

A rational one parameter family $c(t)$ of cycles envelops a canal surface. This canal surface is also part of the envelope of a rational set of cones of revolution with rational radius function. The canal surface is real, if

$$\dot{m}(t)^2 - \dot{r}(t)^2 \geq 0$$

for all t, where equality shall hold only for finitely many parameter values. Real canal surfaces are cyclographic images of curves $C(t)$ without elliptic and with only finitely many parabolic tangents. The cones of revolution $\Delta(t)$ occur as cyclographic images of the tangent lines of the curves $C(t)$.

We require that the envelope contains a surface, which is regular as set of tangent planes. This means that the tangent planes and its first derivative planes are linearly independent, or equivalently that the Gaussian image is 2–dimensional and does not degenerate to a curve.

The technique which is used, is based on the dual representation, that means the representation of surfaces as set of tangent planes. The parametrization only works for the part of the envelope, which possesses a two parameter set of tangent planes. The possibly occurring developables possess tangent planes contained in this two parameter family.

Theorem 4.5 *Let Φ be the envelope of a real rational one–parameter family of cones of revolution $\Delta(t)$ with rational radius function, degenerate cases are allowed. If Φ possesses a regular dual parametrization, it is a PN surface.*

Proof: The family $\Delta(t)$ is determined by two rational families of cycles $c_i(t)$ with centers $(m_{i,1}(t), m_{i,2}(t), m_{i,3}(t))$ and radii $r_i(t)$, $i = 1, 2$. By Lemma 3.1 the cycles, as sets of oriented tangent planes, appear in the isotropic model as isotropic spheres,

$$\begin{aligned} \Lambda(c_1(t)) = \Psi_1(t) &: 2y_3 + (y_1^2 + y_2^2)(r_1 + m_{1,3}) + 2y_1 m_{1,1} + 2y_2 m_{1,2} + r_1 - m_{1,3} = 0, \\ \Lambda(c_2(t)) = \Psi_2(t) &: 2y_3 + (y_1^2 + y_2^2)(r_2 + m_{2,3}) + 2y_1 m_{2,1} + 2y_2 m_{2,2} + r_2 - m_{2,3} = 0. \end{aligned} \tag{4.5}$$

We required that $\Delta(t)$ possesses real oriented tangent planes for all t. These planes are mapped by Λ to the intersection curve $d(t)$ of the two isotropic spheres $\Psi_1(t)$ and $\Psi_2(t)$.

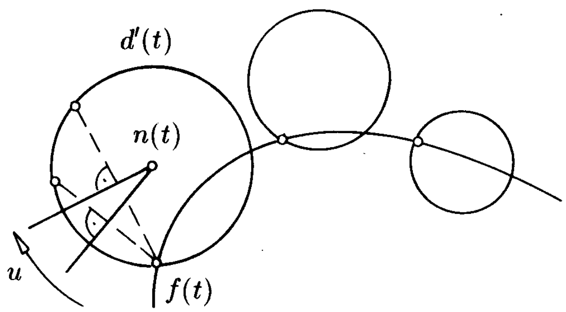

Figure 4.2: Parametrization problem in isotropic space I^3

But this intersection is a conic, a so called isotropic Möbius circle. The projection $d'(t)$ of $d(t)$ onto $y_3 = 0$ is

$$d'(t): \;\; (y_1^2 + y_2^2)(R + M_3) + 2y_1M_1 + 2y_2M_2 + R - M_3 = 0, \tag{4.6}$$

where $M_j := m_{1,j} - m_{2,j}$ and $R := r_1 - r_2$. If $R + M_3 \neq 0$, $d'(t)$ is a Euclidean circle. Its center is

$$(n_1, n_2)(t) = \frac{-1}{R + M_3}(M_1, M_2). \tag{4.7}$$

The radius function $\rho(t)$ is not rational, however we have

$$\rho^2 = \frac{1}{(R + M_3^2)^2}(M_1^2 + M_2^2 + M_3^2 - R^2). \tag{4.8}$$

Due to our assumptions on a real rational family $\Delta(t)$, we have $\rho^2 \geq 0$. It follows that all $d(t)$ possess real points. The surface $d(t)$ contains a one parameter family of isotropic circles, and additionally it follows from formula (4.6) that it is a rational conic surface in the sense of Chapter 1. We already know that a rational parametrization exists and how to find it. But here it is better to consider the special properties of a rational one parameter family of isotropic Möbius circles.

Firstly, we construct two rational functions $\rho_1(t)$ and $\rho_2(t)$, satisfying

$$\rho_1^2(t) + \rho_2^2(t) = \rho^2(t). \tag{4.9}$$

This is done by factorization over the complex field, and is equivalent to determine all roots of the nonnegative polynomial ρ^2, which is of even degree. We note that in several applications there is not only a numerical solution of the factorization, but ρ_1 and ρ_2 can be calculated explicitly. A solution of (4.9) defines the planar rational curve

$$f(t) = (n_1 + \rho_1, n_2 + \rho_2). \tag{4.10}$$

For each parameter value, the point $f(t)$ is contained in the circle $d'(t)$, see Figure 4.2. For fixed t, let the normals of the diameters of the circle $d'(t)$ be parametrized by $c(u) =$

$(u, 1)$. To get all diameters, u has to be considered to be a inhomogeneous projective parameter. Appropriate sampling techniques for $\mathbb{P}^1$ are described in [13]. We reflect $f(t)$ at all diameters and obtain a rational parametrization of the top view $d'(t)$ of the conic surface $d(t)$ as

$$(y_1, y_2)(t, u) = f(t) + 2\frac{(n(t) - f(t)) \cdot c(u)}{c(u) \cdot c(u)} c(u). \tag{4.11}$$

Further, by inserting (y_1, y_2) into a suitable equation of (4.5), we obtain

$$2y_3(t, u) = -r_i(y_1^2 + y_2^2 + 1) - m_{i,3}(y_1^2 + y_2^2 - 1) - 2m_{i,1}y_1 - 2m_{i,2}y_2. \tag{4.12}$$

This is a rational parameterization $y(t, u)$ of the envelope in the isotropic model I^3. The special case $R + M_3 \equiv 0$ is much simpler, since the rational one parameter family of lines $d'(t)$ is easily parametrized in rational form and again we obtain a rational parametrization for the Λ–image of the envelope Φ. With $\Lambda^{-1}(y)$ by formula (3.13) we calculate normalized plane coordinates of Φ and by Theorem 4.3, the envelope of $\Delta(t)$ is a PN surface. $\square$

Remark: It follows also from Theorem 2.1 that $\Delta(t)$ envelops a rational surface. Additionally, we could form the offsets $\Delta_d(t)$ of all cones $\Delta(t)$ at distance d. This is again a rational family of cones of revolution with rational radius function. This implies that the envelope of $\Delta_d(t)$ is again a rational surface. But note, Theorem 4.5 proves that the envelope possesses a rational unit normal vector field. This is the main purpose why we used laguerre geometric aspects and defined rational families of cycles and rational families of cones of revolution with rational radius function.

We translate Theorem 4.5 into the language of Laguerre geometry.

Corollary 4.1 *If a rational, non–developable ruled surface $\varphi \subset \mathbb{R}^4$ possesses a real cyclographic image $\eta(\varphi)$, then this is a PN surface.*

A special case occurs if $\Delta(t)$ is a one parameter family of generating lines, interpreted as pencils of planes. This family can be parametrized by inserting radius functions $r_i = 0$. The envelope of $\Delta(t)$ is a ruled surface Φ. If Φ is a non–developable ruled surface, Theorem 4.5 generates a PN parametrization of Φ. This result can also be found in [38], where it is proved in a different way. A generalization is discussed in Section 5.5.

Corollary 4.2 *Non–developable rational ruled surfaces are PN surfaces. Further, Laguerre transforms of non–developable rational ruled surfaces are PN surfaces.*

However, for a developable surface, we parametrize just the edge of regression l as set of tangent planes. The offsets are pipe surfaces with spine curve l. Laguerre transforms of pipe surfaces are canal surfaces. Figure (4.3) shows rational parameter lines of a canal surface, determined by a polynomial cubic spine curve and a polynomial cubic radius function. The parametrization is of degree 5 in the parameter of the spine curve. For more details on rational parametrizations of canal surfaces see [32]. Another proof of the rationality of pipe surfaces may be found in [24].

Corollary 4.3 *Real canal surfaces with rational spine curve and rational radius function are rational PN surfaces. In particular, pipe surfaces, which possess a constant radius function, to a rational spine curve are always rational.*

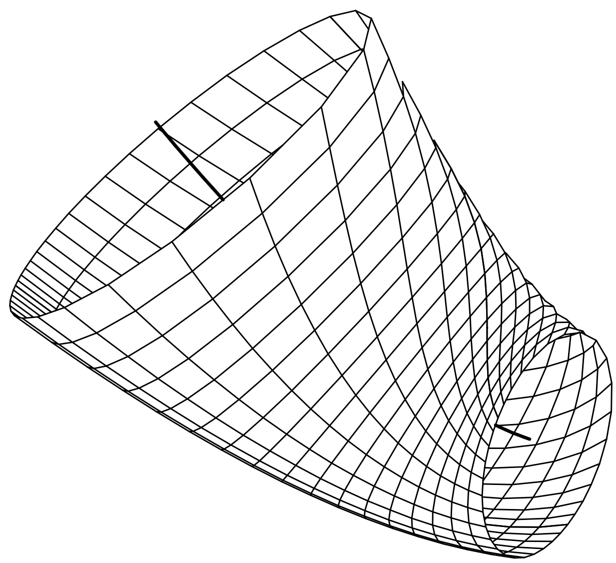

Figure 4.3: Canal surface with cubic spine curve and radius function.

The envelope of a moving cylinder, whose axis runs on a developable surface φ, is the pipe surface around the line of regression of φ and not the offset of φ. More generally, one can show that any moving developable surface has a rational envelope [23]. Only special rational ruled surfaces can be generated by a line under a rational motion, namely those whose unit vectors of the generator lines define a rational curve on the unit sphere.

Corollary 4.4 *Let $\Delta(t)$ be the transforms of a fixed cone of revolution Δ_0 under a rational, one parametric family of Laguerre transformations. It follows that the non developable envelope of $\Delta(t)$ is a rational PN surface.*

To have an example we may specialize these rational transformations to rational motions or rational similarities. We will generalize this result in Section 4.6.

4.5.1 Details of the Parametrization

The parametrization derived in Theorem 4.5 depends on the choice of the coordinate system in $\mathbb{R}^3$. It will turn out that by an appropriate choice of the coordinate system we obtain a representation of lower degrees.

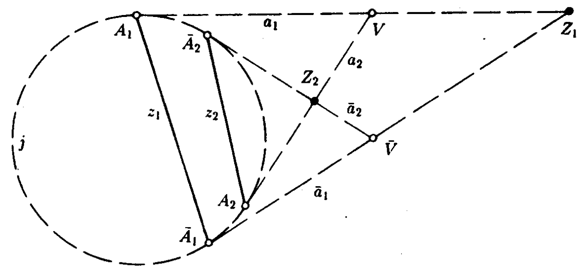

Figure 4.4: Appropriate choice of the coordinate system

We take a closer look to the map Λ^{-1}, and especially to $\bar{\sigma}^{-1} : I^3 \to \Delta$. It is a quadratic map, such that a rational curve $\varphi \subset I^3$ of order n is mapped to a rational curve $q \subset \Delta$ of order $2n$ in general. The parametrization $(y_1, y_2)(t, u)$ in (4.11) leads to a parametrization of the unit normals (Gauss image) of the envelope Φ, which is

$$n(t, u) = \frac{1}{y_1^2 + y_2^2 + 1}(2y_1, 2y_2, y_1^2 + y_2^2 - 1). \tag{4.13}$$

The degree of n reduces, if the numerators and the denominator of the coordinate functions in (4.13) have a common divisor. Then also the degree of the dual parametrization of φ reduces, since the missing coordinate function (4.12) is just a linear combination of the denominator and numerator of the coordinates of n. The parametrization of Φ in normalized plane coordinates is

$$\Lambda^{-1}(y(t, u)) = \frac{1}{y_1^2 + y_2^2 + 1} \begin{pmatrix} -r_i(y_1^2 + y_2^2 + 1) - m_{i,3}(y_1^2 + y_2^2 - 1) - 2m_{i,1}y_1 - 2m_{i,2}y_2 \\ 2y_1 \\ 2y_2 \\ y_1^2 + y_2^2 - 1 \end{pmatrix}. \tag{4.14}$$

First let $M(t) = (M_1, M_2, M_3)(t)$ and assume that the coordinate functions $M_i(t)$ and $R(t)$ are polynomials. Otherwise we assume that the rational functions M_i and R have a common denominator d.

Appropriate Choice of the Coordinate System

Let ω be the ideal plane in $\mathbb{P}^3$, the projective extension of $\mathbb{R}^3$. Let j be the conic in ω defined by $x_1^2 + x_2^2 + x_3^2 = 0$. The bilinear form corresponding to j is the Euclidean scalar product $x \cdot y = x_1y_1 + x_2y_2 + x_3y_3$ which represents the polarity with respect to j. The construction of the frame, illustrated in Figure 4.4, depends essentially on a configuration in ω. Complex lines are represented by dashed lines, real lines by solid lines. Complex points are represented by circles, real points by filled circles.

Let $\tau, \bar{\tau}$ be conjugate complex zeros or a real double zero of the definite polynomial

$$p = M_1^2 + M_2^2 + M_3^2 - R^2 \geq 0,$$

which is the numerator of (4.8). Let $v = M(\tau)$ and $\bar{v} = M(\bar{\tau})$. The vectors v and $\bar{v}$ determine conjugate complex points $V, \bar{V}$ in ω. Let a_i, $\bar{a}_i$ for $i = 1, 2$ be conjugate complex tangent lines of j, passing through V and $\bar{V}$. These lines are tangent to j in points A_i, $\bar{A}_i$. The homogeneous coordinates in ω are

$$\begin{aligned} A_i &= (\alpha_i, \beta_i, \gamma_i) = (-v_1v_3 \mp iv_2\sqrt{\lambda}, -v_2v_3 \pm iv_1\sqrt{\lambda}, v_1^2 + v_2^2), && (4.15) \\ \bar{A}_i &= (\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i) = (-\bar{v}_1\bar{v}_3 \pm i\bar{v}_2\sqrt{\bar{\lambda}}, -\bar{v}_2\bar{v}_3 \mp i\bar{v}_1\sqrt{\bar{\lambda}}, \bar{v}_1^2 + \bar{v}_2^2), && (4.16) \end{aligned}$$

where $\lambda = v \cdot v$ and $\bar{\lambda} = \bar{v} \cdot \bar{v}$. Using the scalar product it follows that the lines a_i and $\bar{a}_i$ are represented by

$$\alpha_i x_1 + \beta_i x_2 + \gamma_i x_3 = 0 \text{ and } \bar{\alpha}_i x_1 + \bar{\beta}_i x_2 + \bar{\gamma}_i x_3 = 0.$$

Further let $Z_i = a_i \cap \bar{a}_i$ and let z_i be the lines connecting $A_i, \bar{A}_i$. It is clear that Z_i and z_i are real. We choose for instance the pair Z_1, z_1 and denote it for simplicity by Z, z. Analogously $A, \bar{A}$ denote the points $A_1, \bar{A}_1$. The point Z and the line z can be represented by the unit vector

$$\zeta = (\zeta_1, \zeta_2, \zeta_3) = \frac{A \times \bar{A}}{||A \times \bar{A}||}. \tag{4.17}$$

The new coordinate system is chosen such that ζ describes the new x_3-axis. Let ξ and η be unit vectors of the new axes x_1, x_2. They can be chosen arbitrarily, but have to satisfy the conditions of an orthonormal frame. This implies $\eta = \zeta \times \xi$, where ξ is for instance

$$\xi_1 = \frac{\zeta_2}{\sqrt{\zeta_1^2 + \zeta_2^2}}, \quad \xi_2 = -\frac{\zeta_1}{\sqrt{\zeta_1^2 + \zeta_2^2}}, \quad \xi_3 = 0.$$

Since this construction depends only on a configuration in ω, the coordinate transformation is only determined up to sign changes of the basis vectors and up to arbitrary translations. The signs will be determined later.

Degree Reductions

Firstly, we rewrite the formulae of the parametrization (4.11) in terms of M_i and R, assumed to be polynomials. Let p_1, p_2 be polynomials satisfying $p_1^2 + p_2^2 = M \cdot M - R^2$. Then the planar rational curve $f(t)$ (see formula (4.10)) is given by

$$f(t) = \frac{1}{R + M_3}(p_1 - M_1, p_2 - M_2). \tag{4.18}$$

Let $\tilde{n}_i(t, u)$ be polynomials and let

$$n(t, u) = \frac{1}{\tilde{n}_0}(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$$

be the rational unit normal vector of the envelope Φ. Using the following substitutions

$$\alpha = M_1^2 + M_2^2,\ \beta = R + M_3,\ \gamma = p_1M_1 - p_2M_2,\ \delta = p_1M_2 + p_2M_1,$$

the polynomials $\tilde{n}_i$ are

$$\begin{aligned}
\tilde{n}_0(t,u) &= u^2(\alpha + M_3\beta + \gamma) + 2u\delta + \alpha + M_3\beta - \gamma, \\
\tilde{n}_1(t,u) &= (-u^2(M_1 + p_1) - 2up_2 + p_1 - M_1)\beta, \\
\tilde{n}_2(t,u) &= (-u^2(M_2 - p_2) - 2up_1 - p_2 - M_2)\beta, \\
\tilde{n}_3(t,u) &= u^2(\alpha - R\beta + \gamma) + 2uh + \alpha - R\beta - \gamma.
\end{aligned} \tag{4.19}$$

Let the frame be chosen as described in Section 4.5.1, up to the signs of the basis vectors. Let $d = (t-\tau)(t-\bar{\tau})$ and let $\pi : x_3 = 0$. We will show that d is a common divisor of the polynomials $\tilde{n}_0, \ldots, \tilde{n}_3$.

The orthogonal projection $\mu : \mathbb{P}^3 \to \pi$ with center Z induces a planar projection $\mu_\omega : \omega \to z$ in the ideal plane. The projection μ maps $v = M(\tau)$ to $\mu(v)$ and $\bar{v} = M(\bar{\tau})$ to $\mu(\bar{v})$. Further μ_ω maps $V, \bar{V}$ to $A, \bar{A} \subset j$. Since $\mu(v)$ and $\mu(\bar{v})$ describe the points A and $\bar{A}$, which are contained in j, it follows that

$$\mu(v) \cdot \mu(v) = \mu(\bar{v}) \cdot \mu(\bar{v}) = 0.$$

This implies that the polynomial $\alpha = \mu(M) \cdot \mu(M)$ has zeros at τ and $\bar{\tau}$, such that d divides α. Since τ and $\bar{\tau}$ are also zeros of $M \cdot M - R^2$ it follows that d divides $M_3^2 - R^2$. We choose the orientation of ζ in (4.17) such that $M_3(\tau) = -R(\tau)$. This guarantees that d divides β.

Maybe after a substitution of p_2 by $-p_2$ we achieve that the real polynomial d divides the complex polynomial $(M_1 + \mathrm{i}M_2)(p_1 + \mathrm{i}p_2)$, such that d is a factor of its real and imaginary parts, γ and δ, respectively. We summarize that all polynomials n_i have the common divisor d, which implies that the degree of the Cartesian coordinates $n_i = \tilde{n}_i/\tilde{n}_0$ of the unit normal n reduces. With this considerations we might give an estimation of the polynomial degrees of the dual parameterization of an envelope.

Proposition 4.1 *Let Φ be envelope of a real rational one parameter set of cones of revolution with rational radius function. Let k be the degree of the polynomial center curves $m_i(t)$ and the polynomial radius functions $r_i(t)$. Then there exists a unit normal vector $n(t,u)$ of Φ, which is of degree $2k-2$ in t. The resulting dual parametrization of Φ is in general of degree $3k-2$ in t and 2 in u.*

For canal surfaces with polynomial center curve and polynomial radius function we obtain a better estimation of the degrees of the parametrization, namely $3k-4$. This is also the degree of a point representation (see [32]).

4.6 Envelopes of Developable PN Surfaces

Let Δ be a rational developable PN surface, given by normalized plane coordinates

$$(e_0, \ldots, e_3)(t) = \left(h, \frac{2ac}{a^2+b^2+c^2}, \frac{2bc}{a^2+b^2+c^2}, \frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right), \tag{4.20}$$

where a, b and c are polynomials in $\mathbb{R}[t]$ and h is an arbitrary rational function in $\mathbb{R}(t)$. Consider a rational one parameter family of Laguerre transformations in $\mathbb{R}^3$. They shall be represented in the cyclographic model $\mathbb{R}^4$ by pe similarities $\alpha(u)$ (see Section 3.2). Let these mappings $\alpha(u)$ be extended to projective collineations in $\mathbb{P}^4$, which map the absolute quadric Ω of the cyclographic model onto itself. The oriented tangent planes $e(t)$ of Δ are mapped by ζ^* to γ–hyperplanes $E(t)$. The dual transformation $\alpha^*(u)$ to $\alpha(u)$ maps the set of γ–hyperplanes H onto itself. Using homogeneous coordinates, $\alpha^*(u)$ can be represented by

$$G(t,u) = \alpha^*(u)(E(t)) = (e_0, e_1, e_2, e_3, 1)(t) \cdot \left[\begin{array}{cc} 1, & 0 \\ a(u), & A(u) \end{array} \right], \tag{4.21}$$

where $a(u)$ denotes the translational part of $\alpha^*(u)$, a vector in $\mathbb{R}^4$. The matrix $A(u)$ is a pe orthogonal 4×4–matrix for each $u \in \mathbb{R}$ with rational entries. Multiplying by the common denominator of the rational entries, we obtain a matrix with polynomial entries. Compare the representation of pe similarities in Formula (3.8).

This implies that $G(t,u)$ is a rational parametrized set of γ–hyperplanes in the cyclographic model $\mathbb{R}^4$. If $G(t,u)$ is really 2–parametric, we apply the cyclographic map η^* to $G(t,u)$ and obtain a two parametric set of oriented tangent hyperplanes

$$e(t,u) = G(t,u) \cap \mathbb{R}^3.$$

Let $e(t,u)$ be a regular dual parametrization of a surface Φ, which means that e and the first partial derivative planes e_t, e_u are linearly independent and intersect in a unique point.

Proposition 4.2 *Let Φ be the envelope of a rational developable PN surface Δ in $\mathbb{R}^3$ under a rational one parameter family of Laguerre transformations. If Φ is non developable, then it is a rational PN surface.*

Example: Let a parabolic cylinder Δ be given by its tangent planes

$$\left[-\frac{p}{2}\frac{1+t^2}{2t}, \frac{2t}{1+t^2}, -\frac{1-t^2}{1+t^2}, 0 \right].$$

For simplicity we choose the one parameter family of Laguerre transformations to be a one parameter family of motions $\beta(u)$. Let X and X' be coordinates of the original and the transformed plane. Then, $\beta(u)$ shall be

$$X' = X \cdot \left[\begin{array}{cccc} 1+u^2 & 0 & 0 & 0 \\ \frac{1}{4}u(1+u^2) & 1+u^2 & 0 & 0 \\ 0 & 0 & 1-u^2 & -2u \\ 0 & 0 & 2u & 1-u^2 \end{array} \right].$$

A point representation of the envelope Φ in homogeneous coordinates is

$$\left[\begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[\begin{array}{c} -4t^2(t^2-1)(u^2+1) \\ (2t^4 - ut^2 - 12t^2 + 2)(t^2-1)(u^2+1) \\ 2t(4(u^2-1)(t^2-1)^2 + t^2u(1+u^2)) \\ t(16u(t^2-1)^2 + t^2(1-u^4)) \end{array} \right].$$

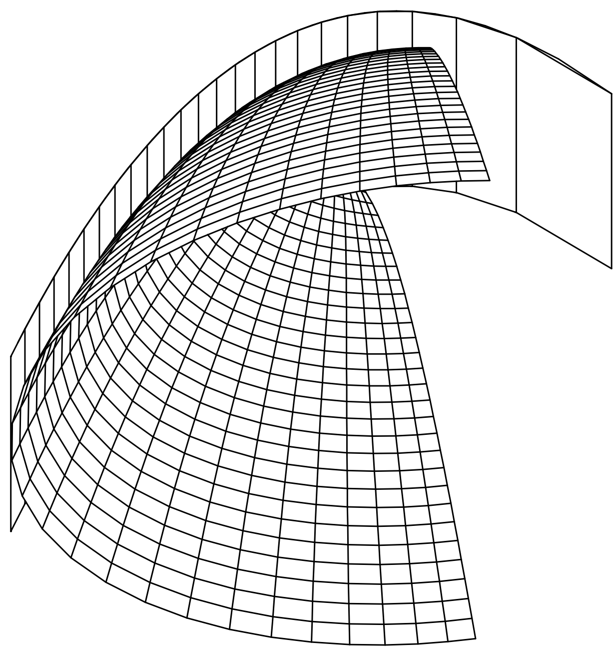

Figure 4.5: Envelope of a parabolic cylinder

Figure 4.5 shows rational parameter lines of a section of Φ. It is plotted for values $t \in [1.5, 4]$ and $u \in [-1.2, 1.2]$.

4.7 Hypercyclides

Let π be a hyperplane in $\mathbb{R}^4$ and $\varphi \subset \pi$ a quadric in it. The cyclographic image $\eta(\varphi)$ of φ is called a *hypercyclide*. A brief discussion of these surfaces is given in [3]. To generate $\eta(\varphi)$ one forms the γ–hypersurface $\Gamma(\varphi)$, which is the envelope of the two parameter set of γ–hyperplanes, passing through the tangent planes of φ. Then, the hypercyclide $\eta(\varphi)$ is the intersection $\Gamma(\varphi) \cap \mathbb{R}^3$.

Let Ω be the absolute quadric in $\mathbb{P}^4$ defined by $x_0 = 0$, $x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$.(See Section 3.2). The hypersurface $\Gamma(\varphi)$ can be interpreted as envelope of common tangent hyperplanes of the pencil of dual hyperquadrics $\lambda\varphi + \mu\Omega$ in $\mathbb{P}^4$. The quadrics φ and Ω are considered as sets of tangent hyperplanes, which implies that they are singular hypersurfaces in this pencil. This generation of $\eta(\varphi)$ indicates that hypercyclides are of algebraic class ≤ 4, but they are in general not of order 4.

Remark: The notation 'hypercyclide' goes back to W. Blaschke [3] and is just a translation of the german expression 'Hyperzyklide'. These surfaces should not be mixed up with 'supercyclides', which are introduced by M.J. Pratt [44]. Supercyclides are special conic surfaces of type C (see Chapter 1), which possess two 1–parameter families of quadratic tangent cones. A supercyclide admits a twofold generation as conic surface. Additionally, the planes containing the conics of each family form two pencils. They are also called double Blutel conic surfaces. Their algebraic degree is ≤ 4, which was proved by W. Degen, [11]. Mainly they are complex projective transforms of Dupin cyclides.

It can be proved that the intersection surface of a pencil of hyperquadrics in $\mathbb{P}^4$ is a quartic del Pezzo surface, which is rationally parametrizable, see for instance [1]. But a del Pezzo surface is dual to the hypersurface $\Gamma(\varphi)$, which implies that the two parameter family of tangent hyperplanes of $\Gamma(\varphi)$ can be rationally parametrized. This proves the following result.

Proposition 4.3 *The cyclographic images $\eta(\varphi)$ of two dimensional quadrics $\varphi \subset \mathbb{R}^4$ are rational surfaces, so–called hypercyclides.*

Hypercyclides are natural generalizations of Dupin cyclides and contain interesting surfaces such as the offsets of quadrics. Those occur as cyclographic images $\eta(\varphi)$ if $\pi \supset \varphi$ is parallel to $\mathbb{R}^3$. The cyclographic image of a quadric in any Euclidean hyperplane π is related to the offset of a quadric by a Laguerre transformation (compare last but one paragraph in Section 3.2).

To obtain rational parametrizations of hypercyclides one can follow ideas of classical geometry, which provides parametrizations of a del Pezzo surface. Choose five base points in $\mathbb{P}^2$ and consider the four parametric linear system of cubics, passing through these points. This leads to a parametrization of a del Pezzo surface, see [1].

We will construct rational PN parametrizations of cyclographic images $\eta(\varphi)$ of quadrics φ by representing $\eta(\varphi)$ as envelope of cones of revolution. But firstly, we assume that φ is not a quadratic cone but a regular quadric. Secondly, we assume that φ is contained in a Euclidean hyperplane π. As we mentioned above, there exists pe similarity in $\mathbb{R}^4$, which maps π to $x_4 = 0$. This implies that the cyclographic image $\eta(\varphi)$ is related to the offsets of φ by a Laguerre transformation. So it is sufficient to study offset surfaces of quadrics. Another proof of the rationality of the offsets of regular quadrics can be found in [27]. By the way, the offsets of conics in the plane are only rational for circles and parabolas.

Theorem 4.6 *All regular quadrics are PN surfaces.*

Proof: Let φ be a regular quadric possessing real points. Firstly, we note that quadrics of revolution are canal surfaces. From Corollary 4.3 it follows that they are rational PN surfaces. Secondly, if φ is a ruled surface, we can apply Corollary 4.2.

Otherwise let $\mathbb{R}^3$ be embedded in $\mathbb{R}^4$ and let φ be contained in the hyperplane $x_4 = 0$. We form the pencil of dual hyperquadrics $\lambda\varphi + \mu\Omega$ in $\mathbb{R}^4$ and form the γ–hypersurface $\Gamma(\varphi)$. All singular quadrics in this pencil possess the same cyclographic image $\Gamma(\varphi) \cap \mathbb{R}^3$. Let π be parallel to $\mathbb{R}^3$. Then we obtain the offset surfaces of φ by projecting $\Gamma(\varphi) \cap \pi$ onto $\mathbb{R}^3$.

We will prove that the pencil $\lambda\varphi + \mu\Omega$ contains a ruled singular quadric ψ. Since φ possesses real points it follows that ψ possesses a real cyclographic image $\eta(\psi)$. This implies that ψ possesses a family of hyperbolic generator lines $g(t)$. But note that $g(t)$ needs not to be hyperbolic for all $t \in \mathbb{R}$, but in an interval $\subset \mathbb{R}$. Three cases have to be distinguished.

Ellipsoid: Let φ be an ellipsoid in $x_4 = 0$ given by

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1, x_4 = 0,$$

and $a > b > c$. Then the ruled quadric ψ is given by

$$\frac{x_1^2}{a^2 - b^2} - \frac{x_3^2}{b^2 - c^2} + \frac{x_4^2}{b^2} = 1, x_2 = 0.$$

Two sheet hyperboloid: Let φ be a two sheet hyperboloid in $x_4 = 0$ given by

$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1, x_4 = 0,$$

and $b > c$. Then the ruled quadric ψ is given by

$$\frac{x_1^2}{a^2 + b^2} + \frac{x_3^2}{b^2 - c^2} - \frac{x_4^2}{b^2} = 1, x_2 = 0.$$

Elliptic paraboloid: Let φ be an elliptic paraboloid in $x_4 = 0$ given by

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 2x_3 = 0, x_4 = 0,$$

and $a > b$. Then the ruled quadric ψ is given by

$$-\frac{1}{a^2 - b^2} x_2^2 + \frac{1}{a^2} x_4^2 = 2x_3 - a, x_1 = 0,$$

and is a hyperbolic paraboloid.

We mentioned that $\eta(\varphi) = \eta(\psi)$. Since ψ is a ruled quadric with generators $g(t)$, $\eta(\psi)$ is the envelope of a one parameter family of cones of revolution $\Delta(t) = \eta(g(t))$. The cones $\Delta(t)$ can be defined by two rational families of cycles $c_1(t)$ and $c_2(t)$. These cycles can be chosen to be the cyclographic images of two conics on ψ. Since the generator lines induce a projective map between these two conics, they can be rationally parametrized by a common parameter t. We can choose $c_1(t)$ to be the conic $\psi \cap \mathbb{R}^3$, where $\mathbb{R}^3$ is considered to be the hyperplane $x_4 = 0$. Further $c_2(t)$ is the cyclographic image of $\psi \cap (x_4 = \text{const.})$, (see the following example). This proves that $\eta(\varphi) = \eta(\psi)$ is an envelope of a real rational one parameter family of cones of revolution $\Delta(t)$ with rational radius function. With Theorem 4.5, all regular quadrics are PN surfaces. $\square$

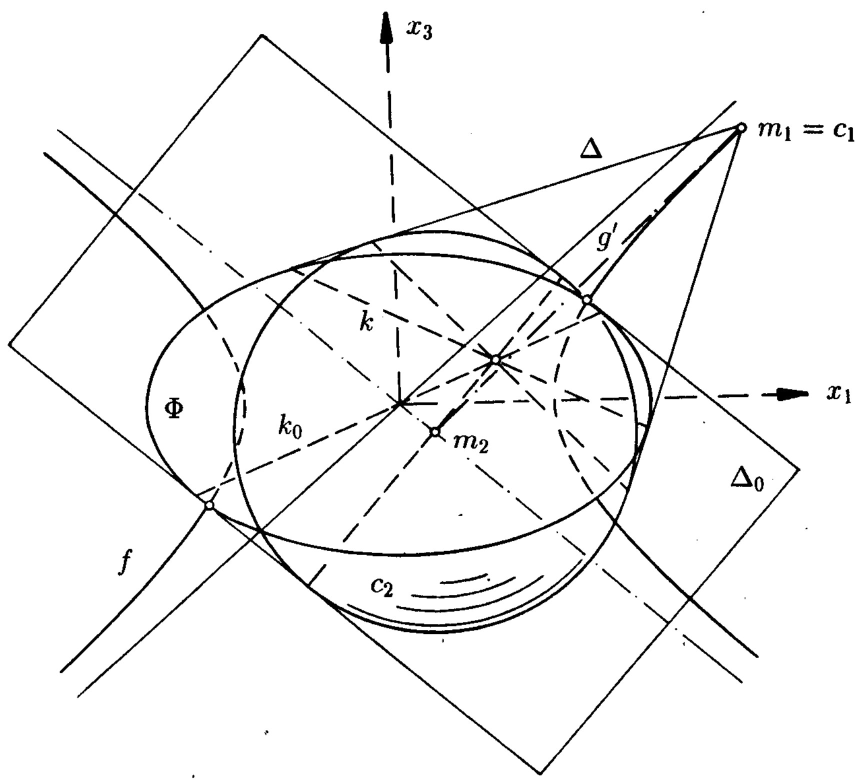

Figure 4.6: Ellipsoid as envelope of cones of revolution

From above formulas we see that ψ is contained in a hyperplane parallel to the x_4–axis which implies that the top views g' of all generator lines g are contained in a fixed plane. These lines g' are tangent lines of a conic f, which is called *focal conic* of φ.

If φ is an ellipsoid in $\mathbb{R}^3$, f is a hyperbola in the plane $x_2 = 0$. If φ is a two sheet hyperboloid, f is an ellipse in the plane $x_2 = 0$. At least, if φ is an elliptic paraboloid, f is a parabola in the plane $x_1 = 0$. The focal conic f intersects φ orthogonally at its umbilic points. The points of f are the vertices and the tangent lines of f are the axes of cones of revolution $\Delta(t)$, which envelope φ. But only points in the exterior of φ correspond to real cones $\Delta(t)$. In the following example the parametrization of an ellipsoid and its offset surfaces is discussed in detail.

Example: Let Φ be an ellipsoid

$$\Phi : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1,$$

where x_i are Cartesian coordinates in $\mathbb{R}^3$ and $a \geq b \geq c > 0$ are constants. First we assume

$a > b > c$, which implies that one focal conic is a hyperbola f defined by

$$f : \frac{x_1^2}{a^2 - b^2} - \frac{x_3^2}{b^2 - c^2} = 1, x_2 = 0.$$

If $a = b$ or $b = c$, f degenerates to a segment of a line and Φ is an ellipsoid of revolution. We discuss this later. To obtain low degree representations, we apply a rotation of the frame with axis x_2. The new x_3 axis shall be parallel to the normal to Φ at one umbilic point. We use the substitutions

$$\alpha = \sqrt{b^2 - c^2}, \beta = \sqrt{a^2 - c^2}, \gamma = \sqrt{a^2 - b^2}.$$

With respect to the new coordinate system a rational parametrization of an appropriate segment of f is

$$f = m_1(t) = \frac{1}{b(t^4 - 1)}\Big(\alpha\gamma(t^4 + 1), 0, ac(t^4 + 1) + 2b^2t^2)\Big),$$

which represents one arc in the exterior of Φ, connected in the projective extension, see Figure 4.6. This arc of f defines the vertices of the cones $\Delta(t)$, which are the cycles $c_1(t)$ in accordance with Theorem 4.5. Let Δ_0 be a cylinder of revolution, tangent to Φ along k_0 (see Fig. 4.6). The intersection points of the tangent lines of f with the axis of Δ_0 are the centers

$$m_2(t) = \frac{(t^2 - 1)}{b(t^2 + 1)}(\alpha\gamma, 0, ac - b^2)$$

of the cycles $c_2(t)$ with constant radius $r_2 = b$. We transfer these two families of cycles c_1 and c_2 by the map Λ to isotropic 3–space. According to formulae (4.7) and (4.8) the centers $n(t)$ and radius function $\rho(t)$ of the circles are

$$\begin{aligned}
(n_1, n_2) &= \Big(\frac{-\alpha\gamma t^2}{act^2 + b^2}, 0\Big),\\
\rho^2 &= \frac{b^2t^2}{(act^2 + b^2)^2}\Big(ac(t^4 + 1) + t^2(a^2 + c^2)\Big).
\end{aligned}$$

A decomposition of $\rho^2(t)$ leads to

$$(\rho_1, \rho_2) = \frac{bt}{(act^2 + b^2)}\Big(\sqrt{ac}(t^2 + 1), (a - c)t\Big),$$

with $\rho_1^2 + \rho_2^2 = \rho^2$. With formula (4.11) one constructs a rational parametrization of Φ as set of tangent planes, in general of degrees 6 and 2 in t and u. It is not difficult to see that ellipsoids of revolution Φ can be treated in the same way. For instance, let $a = b$. The not normalized homogeneous plane coordinates of the one–sided offset Φ_d at distance d are

$$\begin{aligned}
\varphi_0 &= -(1 + u^2)\Big(a(ct^4 + 2at^2 + c) + d(at^4 + 2ct^2 + a)\Big),\\
\varphi_1 &= -2t\Big(\sqrt{ac}(t^2 + 1)(u^2 - 1) + 2tu(a - c)\Big),\\
\varphi_2 &= 2t\Big(-2\sqrt{ac}(t^2 + 1)u + t(u^2 - 1)(a - c)\Big),\\
\varphi_3 &= a(1 + u^2)(t^4 - 1).
\end{aligned}$$

Setting $d = 0$ parametrizes the ellipsoid Φ itself. The conversion to a point representation is done by intersecting the planes $\varphi_0 x_0 + \ldots + \varphi_3 x_3 = 0$ with the derivative planes with respect to t and u. This leads to a point representation of Φ_d

$$\begin{aligned}
x_0 &= -(1+u^2)(ct^4+2at^2+c)(at^4+2ct^2+a),\\
x_1 &= -2t(d(ct^4+2at^2+c)+a(at^4+2ct^2+a))\\
&\quad ((u^2-1)\sqrt{ac}(t^2+1)+2ut(a-c)),\\
x_2 &= 2t(d(ct^4+2at^2+c)+a(at^4+2ct^2+a))\\
&\quad (t(a-c)(u^2-1)-2\sqrt{ac}u(t^2+1)),\\
x_3 &= -(1+u^2)(t^4-1)(ad(ct^4+2at^2+c)+c^2(at^4+2ct^2+a)).
\end{aligned}$$

Again, $d = 0$ parametrizes the ellipsoid Φ itself. The parametrization of Φ is of degrees 4 and 2. The t–lines on Φ are rational quartics, containing an isolated double point. The t–lines on Φ_d are rational curves of degree 8.

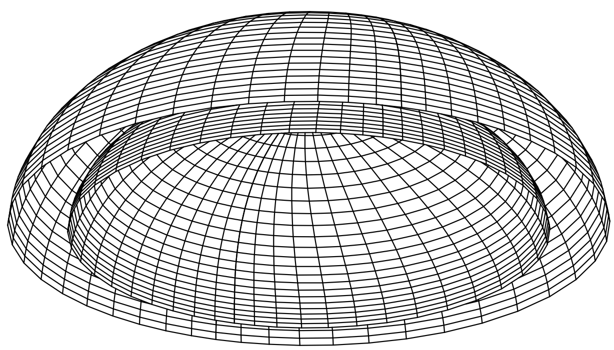

Figure 4.7: Ellipsoid of revolution and outside offset

Let Φ be a general triaxial ellipsoid. The point representation as PN surface is rather long, but we give a representation as set of tangent planes. Substituting $\alpha = \sqrt{b^2-c^2}$,$\gamma = \sqrt{a^2-b^2}$ and $\mu = \sqrt{ac}$ we obtain

$$\begin{aligned}
\varphi_0 &= -b\Big[(u^2+1)\Big(a^2c^2t^2(t^4-1)+b^2ac(3t^4+1)+2b^2(a^2+c^2)t^2\Big)\\
&\quad +2b\mu\alpha\gamma(u^2-1)t(t^2+1)+4b(a-c)\alpha\gamma ut^2\Big],\\
\varphi_1 &= -2t(act^2+b^2)\Big(b\mu(u^2-1)(t^2+1)+2(a-c)but+\alpha\gamma(u^2+1)t\Big),\\
\varphi_2 &= 2tb(act^2+b^2)\Big((a-c)t(u^2-1)-2\mu u(t^2+1)\Big),\\
\varphi_3 &= (u^2+1)\Big(acb^2t^2(t^4-1)-b^4(t^4+1)+2t^4(b^2c^2+a^2b^2-a^2c^2)\Big)+\\
&\quad 2b\mu\alpha\gamma(u^2-1)t^3(t^2+1)+4b(a-c)\alpha\gamma ut^4.
\end{aligned}$$

Figure 4.8 shows rational parameter lines of an ellipsoid and an inside offset. All t–lines pass through two umbilic points of Φ, such that the parametrization is singular there, (see the black dots at the figure).

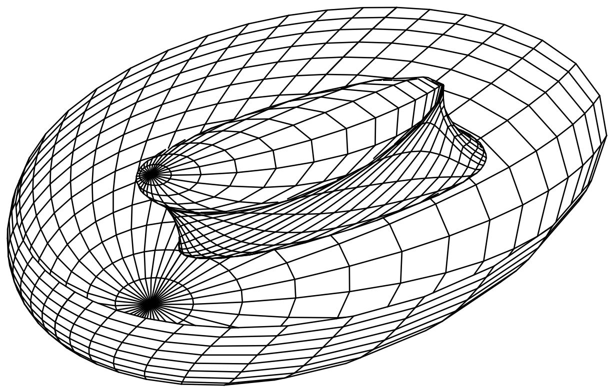

Figure 4.8: Ellipsoid and inside offset

Chapter 5

Generalizations in n–Space

We will show that the most of the results obtained before in $\mathbb{P}^3$ and $\mathbb{R}^3$ can be generalized to $\mathbb{P}^n$ and $\mathbb{R}^n$, respectively in an appropriate way. This is not a complete description of a generalization of earlier ideas but a short overview. We only pick out some details and start with an analogue to Section 1.3.

5.1 Rational One Parameter Families of Quadrics

Let $Q(t)$ be a one parameter family of quadrics in $\mathbb{P}^n$. We call the set $Q(t)$ rational, if a representation exists, such that the defining equations of $Q(t)$ possess rational coefficients. Using rational coordinate transformations we can assume that $Q(t)$ is given by its real projective normal form

$$Q(t) : \beta_0(t)x_0^2 + \ldots + \beta_p(t)x_p^2 - \beta_{p+1}(t)x_{p+1}^2 - \ldots - \beta_r(t)x_r^2 = 0, \tag{5.1}$$

where $\beta_i(t)$ are assumed to be polynomials. Further, let $\beta_i(t) \geq 0$, for all $t \in \mathbb{R}$. Some assumptions are necessary, equivalent to those given in Section 1.3.

1. The quadrics $Q(t)$ possesses regular real points for all real t, such that not only the singular set consists of real points.

2. There are at least three polynomials $\beta_i(t)$, which are not identically zero. This implies that $Q(t)$ does not contain hyperplanes.

3. The quadrics $Q(t)$ are of fixed type for all real numbers, except the zeros of $\beta_i(t)$. That means that $Q(t)$ has constant rank and a constant maximum dimension of subspaces contained in $Q(t)$.

Since $Q(t)$ possesses regular real points, there are at least three polynomials $\beta_i(t)$, which are not identically zero. Then, there exists a plane intersection $c(t)$ of $Q(t)$, which is an irreducible conic containing real points for almost all t. By an appropriate renumbering, let this plane be $x_3 = 0, \ldots, x_n = 0$. The conic $c(t)$ is given by

$$c(t) : \beta_0(t)x_0^2 + \beta_1(t)x_1^2 + \beta_2(t)x_2^2 = 0, \tag{5.2}$$

Theorem 1.1 says that there are polynomials $(y_0, y_1, y_2)(t) \neq (0, 0, 0)$, which satisfies equation (5.2) identically. This leads to an equivalent statement to Theorem 1.1.

Corollary 5.1 *Let $Q(t)$ be a real rational one parameter family of quadrics in $\mathbb{P}^n$. Assume that $Q(t)$ possesses real conic sections, which means that $Q(t)$ does not contain hyperplanes. Then there exists a rational curve $f(t)$ such that for all $t \in \mathbb{R}$ the curve point $f(t)$ is contained in the corresponding quadric $Q(t)$.*

Applying a stereographic projection (see Section 1.4) to each quadric $Q(t)$ results in a rational parametrization $g(t, u)$ of the family $Q(t)$, with $u = (u_1, \ldots, u_{n-1})$.

Proposition 5.1 *Let $Q(t)$ be a real rational one parameter family of quadrics, satisfying above assumptions. Then there exists a rational parametrization $g(t, u)$. For all $t \in \mathbb{R}$ the point $g(t, u)$ is contained in the quadric $Q(t)$. For a fixed t_0, $g(t_0, u)$ is an affine parametrization of $Q(t)$.*

This construction will be applied to some n–dimensional problems. One could study analogues to conic surfaces (Chapter 1) and envelopes of quadratic cones (Chapter 2).

5.2 PN Hypersurfaces

Definition: Let Φ be a rational hypersurface in $\mathbb{R}^n$. Let $u = (u_1, \ldots, u_{n-1})$ be a vector in $\mathbb{R}^{n-1}$, considered as parameter space. Let $x(u) = (x_1, \ldots, x_n)(u)$ be a rational parametrization of Φ. The parametrization $x(u)$ is called a *PN parametrization*, if and only if the unit normal vectors $n(u)$ of Φ are rational. Further, Φ is called a *PN hypersurface*, if and only if it possesses a PN parametrization.

All offset hypersurfaces Φ_d to a PN surface Φ at an arbitrary distance d, which are parametrized by $x_d = x + dn$ are rational. As in $\mathbb{R}^3$ it is good to use the dual parametrization. Let $e(u) = (e_0, \ldots, e_n)(u)$ be homogeneous coordinate functions of hyperplanes in $\mathbb{R}^n$. If Φ be a PN hypersurface, it admits a dual parametrization of the form

$$(e_0, e_1, \ldots, e_{n-1}, e_n) = \left(g, 2a_1cf, 2a_2cf, \ldots, 2a_{n-1}cf, (a_1^2 + a_2^2 + \ldots a_{n-1}^2 - c^2)f\right), \quad (5.3)$$

where a_i, c, g, f are polynomials in $\mathbb{R}[u]$, and can be assumed to have no common factor. The not normalized surface normal is represented by $(e_1, \ldots, e_n)$. The normalized coordinate vector of $e(u)$ is

$$e(u) = \left(h, \frac{2a_1c}{a_1^2 + \ldots + a_{n-1}^2 + c^2}, \ldots, \frac{2a_{n-1}c}{a_1^2 + \ldots + a_{n-1}^2 + c^2}, \frac{a_1^2 + \ldots + a_{n-1}^2 - c^2}{a_1^2 + \ldots + a_{n-1}^2 + c^2}\right), \quad (5.4)$$

where h is an arbitrary rational function in $\mathbb{R}(u)$ and a_i, c are polynomials in $\mathbb{R}[u]$ without a common factor. Immediately we see that substituting h by $h + d$ in (5.4) parametrizes the offset surface Φ_d at distance d, which is again a PN surface. If the tangent planes $e(u)$ and the first partial derivatives e_{u_j} are linearly independent, the conversion to a point representation $p(u,v)$ is done by intersecting

$$(e \cap e_{u_1} \cap \ldots \cap e_{u_{n-1}})(u) = p(u).$$

Otherwise one obtains more dimensional analogues to developable PN surfaces.

5.3 Envelopes of Hypercones

First we look for an analogue of a cone of revolution. Further, we give a n–dimensional version of Theorem 4.5.

Definition: Let c_1 and c_2 be two cycles in Euclidean Laguerre space $\mathbb{R}^n$. The set of common oriented tangent hyperplanes of c_1 and c_2 define a quadratic hypercone Δ, which shall be called *sphere–hypercone*.

Let two rational families $c_1(t)$ and $c_2(t)$ be given. If the common oriented tangent hyperplanes are real, we call $\Delta(t)$ a *real rational family with rational radius function*, analogously to Section 4.5.

This prepares a generalization of Theorem 4.5. We also give the proof here, although it is nearly a copy of the 3-dimensional case.

Theorem 5.1 *Let Φ be the envelope of a real rational one–parameter family of sphere–hypercones $\Delta(t)$ with rational radius function, degenerate cases are allowed. If the Gaussian image of Φ is at least two dimensional, then Φ is a PN surface in $\mathbb{R}^n$.*

Proof: The set $\Delta(t)$ is determined by two rational sets of cycles $c_i(t)$ with centers $m_i(t)$ and radii $r_i(t),\ i = 1, 2$. We map them by Λ to the isotropic model I^n and by Lemma 3.1 we obtain two isotropic Möbius spheres

$$\begin{aligned} \Psi_1(t) : 2y_n + (y_1^2 + \ldots + y_{n-1}^2)(r_1 + m_{1,n}) + 2y_1 m_{1,1} + \ldots 2y_{n-1} m_{1,n-1} + r_1 - m_{1,n} = 0, \\ \Psi_2(t) : 2y_n + (y_1^2 + \ldots + y_{n-1}^2)(r_2 + m_{2,n}) + 2y_1 m_{2,1} + \ldots 2y_{n-1} m_{2,n-1} + r_2 - m_{2,n} = 0. \end{aligned} \tag{5.5}$$

We required that $\Delta(t)$ possesses real oriented tangent hyperplanes for all t. This implies that the intersection $\psi_1 \cap \psi_2$ is a quadric $d(t)$ which possesses real points. Its projection $d'(t)$ onto $y_n = 0$ is

$$d'(t) :\ (y_1^2 + \ldots y_{n-1}^2)(R + M_n) + 2y_1 M_1 + \ldots + 2y_{n-1} M_{n-1} + R - M_n = 0, \tag{5.6}$$

where $M_j := m_{1,j} - m_{2,j}$ and $R := r_1 - r_2$. If $R + M_n \neq 0$, $d'(t)$ is a Euclidean sphere of dimension $n-2$ in the hyperplane $\pi : y_n = 0$. Note that $d'(t)$ is the stereographic projection of the Gaussian image of Φ. If Φ is the envelope of a rational one parameter family of hyperplanes $e(t)$, its Gaussian image is 1–dimensional and not necessarily a rational curve. This curve is only rational, if $e(t)$ possesses rational unit normals.

We will construct a rational parametrization of the one parameter family $d'(t)$ in the hyperplane π, considered as parameter space. Section 5.1 says that a rational parametrization exists and gives an algorithm to compute it. But here we will proceed as in the 3–dimensional case. The center of the sphere $d'(t)$ is

$$(p_1, \ldots, p_{n-1})(t) = \frac{-1}{R + M_n}(M_1, \ldots, M_{n-1}). \tag{5.7}$$

We define the vector $M = (M_1, \ldots, M_{n-1}, M_n)$. The radius function $\rho(t)$ is not rational, however we have

$$\rho^2(t) = \frac{1}{(R + M_n)^2(t)}(M \cdot M - R^2)(t). \tag{5.8}$$

Next, two rational functions $\rho_1(t)$ and $\rho_2(t)$ are constructed, satisfying

$$\rho_1^2(t) + \rho_2^2(t) = \rho^2(t). \tag{5.9}$$

This leads to a curve

$$f(t) = (p_1 + \rho_1, p_2 + \rho_2, p_3, \ldots, p_{n-1}), \tag{5.10}$$

which satisfies $f(t) \subset d'(t)$ for all $t \in \mathbb{R}$. Now we can apply stereographic projection to parametrize each sphere $d'(t)$, as described in Section 5.1. Another way, which shall be described here, is to reflect $f(t)$ at all hyperplanes passing through the center of $d'(t)$. Let $u = (u_1, \ldots, u_{n-2})$, where u_i are affine real parameters. The normals of these hyperplanes can be parametrized by $c(u) = (u_1, \ldots, u_{n-2}, 1)$. This leads to a rational parametrization of the top view $d'(t)$ of $d(t)$

$$(z_1, \ldots, z_{n-1})(t, u) = f(t) + 2\frac{(p(t) - f(t)) \cdot c(u)}{c(u) \cdot c(u)}c(u). \tag{5.11}$$

Further, by inserting $(z_1, \ldots, z_{n-1})$ into (5.5), we obtain

$$2z_n(t, u) = -r_i(z_1^2 + \ldots z_{n-1}^2 + 1) - m_{i,n}(z_1^2 + \ldots z_{n-1}^2 - 1) - 2m_{i,1}z_1 - \ldots 2m_{i,n-1}z_{n-1}. \tag{5.12}$$

Thus, we have found a rational parameterization $z(t, u)$ of the envelope in the isotropic model. Applying the map Λ^{-1} to the paramerization $z(t, u_1, \ldots, u_{n-2})$ results in a PN parametrization of the envelope of the family $\Delta(t)$. $\square$

5.4 Ruled $k + 1$–Manifolds

Let $\mathbb{R}^n$ be Euclidean n–space and let $\mathbb{P}^n$ be the projective extension. Further let ω be the hyperplane at infinity. We also interpret $\mathbb{R}^n$ as vector space. Let $l(t)$ for $t \in \mathbb{R}$ be a smooth curve in $\mathbb{R}^n$. Let $g_j(t)$ for $t \in \mathbb{R}$ be vector functions in $\mathbb{R}^n$ and $j = 1, \ldots, k$. Assume that $G_k(t) = l(t) + \mathrm{span}(g_1, \ldots, g_k)(t)$ is a subspace of dimension k for (almost) all $t \in \mathbb{R}$.

Definition: The parametrized $k+1$–dimensional manifold

$$\Phi : x(t, u_1, \ldots, u_k) = l(t) + \sum_{j=1}^{k} u_j g_j(t) \qquad (5.13)$$

is called a *ruled k+1–manifold* in $\mathbb{R}^n$. The curve $l(t)$ is called *directrix*, the subspaces $G(t)$ are called *generating subspaces* of Φ. The $k+1$–manifold Φ is regular, if

$$\mathrm{rk}(\dot{l} + \sum_{1}^{k} u_j \dot{g}_j, g_1, \ldots, g_k)(t) = k + 1.$$

We will prove that the offset hypersurfaces of unirational ruled $k+1$–manifolds are unirational. For that we collect some basic geometric ideas concerning ruled $k+1$–manifolds. A detailed description, mainly from the differential geometric viewpoint, can be found in [14], [15].

Two vector spaces shall be defined. The first one is

$$A(t) = \mathrm{span}(g_1, \ldots, g_k, \dot{g}_1, \ldots, \dot{g}_k)(t), \qquad (5.14)$$

with dimension $k+m$ for a fixed t, and $0 \leq m \leq k$. For a fixed t the vector space $A(t)$ contains all tangent vector spaces, which are tangent to $\Phi \cap \omega$. Such tangent spaces are called *asymptotic spaces* and $\cup A(t)$ is called *asymptotic bundle.*

The second vector space is

$$T(t) = \mathrm{span}(\dot{l}, g_1, \ldots, g_k, \dot{g}_1, \ldots, \dot{g}_k)(t). \qquad (5.15)$$

For its dimension we have $k+m \leq \dim(T(t)) \leq k+m+1$ for all fixed t. For a fixed t, the vector space $T(t)$ contains all tangent vector spaces in regular points of the generating subspace $G_k(t)$. Mainly there are two cases to distinguish.

Case 1: $\dim T(t) = k + m$.
It follows from the definition of the vector spaces, that $\dot{l}(t) \in A(t)$. We look for singular points in each generating space G_k. A point $s = l + \sum_1^k u_j g_j$ is singular, if

$$\dot{s} = \dot{l} + \sum_{1}^{k} u_j \dot{g}_j = \sum_{1}^{k} \lambda_j g_j.$$

This leads to a linear system for each t. Since $\dot{l} \in A$ and $\dim A = k+m$, there is a solution of dimension $k-m$. But note that $k-m < k$, because otherwise all points in the generating space would be singular. The subspace of singular points in G_k shall be denoted by K_{k-m}.

To have an example, consider developable surfaces Φ, different from cylinders in $\mathbb{R}^3$. Here we have $m = k = 1$, such that only one singular point s exists in each generating line g. If s is a fixed point, Φ is a cone. The singular point s can be obtained by intersecting $g \cap \dot{g}$. The same is true in the general case.

Lemma 5.1 $K_{k-m} = G_k \cap \dot{G}_k$.

Proof: For each fixed t let N_j be normal vectors of G_k. Let s be a singular point in G_k. We choose s to be the directrix curve of Φ. Let E_k and $\dot{E}_k$ be the corresponding vector spaces to G_k and $\dot{G}_k$, that means $G_k = s + E_k$. The subspaces G_k and $\dot{G}_k$ can be defined by a linear system of equations

$$\begin{aligned} G_k &: (x-s)\cdot N_j = 0, j = 1,\ldots,n-k, && (5.16)\\ \dot{G}_k &: (x-s)\cdot \dot{N}_j - \dot{s}\cdot N_j = 0, j = 1,\ldots,n-k, && (5.17)\end{aligned}$$

where N_j are normal vectors of E_k. Since $\dot{s} \in E_k$ it follows that s is a solution. The dimension of $E_k \cap \dot{E}_k$ is $k-m$ and it follows

$$s + E_k \cap \dot{E}_k = K_{k-m}.\square$$

Consider a line $h \in G_k$, which passes through a singular point, but $h \notin K_{k-m}$. The line h can be parametrized by

$$h : x = s + \lambda \sum_1^k \alpha_j g_j,$$

where the constants α_j define the direction of the line h. The tangent space in any regular point x of h is

$$\begin{aligned}\tau(x) &= s + \lambda \sum_1^k \alpha_j g_j + \mu_0(\dot{s} + \lambda \sum_1^k \alpha_j \dot{g}_j) + \mu_1 g_1 + \ldots + \mu_k g_k \\ &= s + \mathrm{span}(g_1, \ldots, g_k, \sum_1^k \alpha_j \dot{g}_j).\end{aligned}$$

The tangent space $\tau(x)$ only depends on the direction defined by α_j and is fixed for all $x \in h \setminus s$.

Lemma 5.2 *Let $h \in G_k$ be a fixed line, but $h \notin K_{k-m}$, which passes through one singular point s. All regular points $x \neq s$ of h possess a constant tangent $k+1$–space.*

Case 2: $\dim T = k + m + 1$. There exists a vector a in T, such that $\mathrm{span}(A, a) = T$ and $a \perp A$. Further we are looking for points r, satisfying

$$\dot{r} \in \mathrm{span}(a, g_1, \ldots, g_k).$$

In detail, the condition is

$$\dot{r} = \dot{l} + \sum_1^k u_j \dot{g}_j = \lambda a + \mu_j g_j, j = 1, \ldots, k.$$

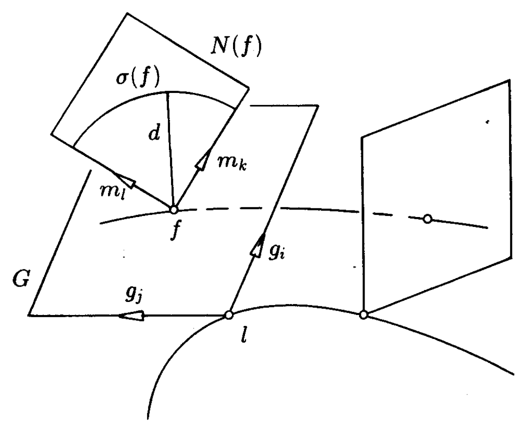

Figure 5.1: Geometric properties of ruled $k+1$–manifolds

This is again a linear system for each fixed t. The solution leads to a subspace of G_k, which shall be denoted by Z_{k-m}, and usually is called *central space*, see [14], [15].

Let y be an arbitrary point in Z_{k-m}. Its derivative vector is

$$\dot{y} \in \mathrm{span}(a, g_1, \ldots, g_k) = \tau.$$

We see that all points of the central $Z_{k-m}(t)$ space possess a constant tangent space $\tau(t)$. For example, cylinders in $\mathbb{R}^3$ are obtained for $k=1$ and $m=0$. A non developable ruled surface in $\mathbb{R}^3$ is obtained for $k=1$ and $m=1$, thus Z is zero dimensional and $Z(t)$ is usually called *striction curve*.

5.5 Offset Hypersurfaces of Ruled $k+1$–Manifolds

Assume that $\Phi \subset \mathbb{R}^n$ is a rationally parametrized ruled $k+1$–manifold, that means $l(t)$ and $g_j(t)$, $j = 1 \ldots, k$ are rational vector functions in t. In the literature Φ is called unirational. Further $g_j(t)$ are assumed to be linearly independent for almost all $t \in \mathbb{R}$.

The offset hypersurface Φ_d to Φ shall be defined to be the envelope of the $k+1$–parametric set of hyperspheres $S(t, u_i)$ with constant radius d, centered at points of Φ. Let

$$p(t, u_i) = l(t) + \sum_{i=1}^{k} u_i g_i(t)$$

be a regular point on Φ. The tangent vector space

$$\tau(p) = \mathrm{span}(\dot{l}(t) + \sum_{i=1}^{k} u_i \dot{g}_i(t), g_1(t), \ldots, g_k(t)). \tag{5.18}$$

at a regular point $p \in \Phi$ is $k+1$–dimensional, and the normal vector space $N(p)$ is $n-(k+1)$–dimensional. Let $S(p) : (x-p)\cdot(x-p) - d^2 = 0$ be a hypersphere of radius d, centered at p and let

$$\begin{array}{lll} S_t & : (x-p)\cdot p_t & = 0, \\ S_{u_i} & : (x-p)\cdot p_{u_i} & = 0, \ \text{for } i = 1,\dots,k, \end{array}$$

be the partial derivatives of $S(p)$ with respect to t and u_i. The later ones are just hyperplanes passing through p, which are perpendicular to p_t and $g_i = p_{u_i}$, respectively. The envelope Φ_d consists of spheres

$$\sigma(p) = S(p) \cap S_t(p) \cap S_{u_1}(p) \cap \ldots \cap S_{u_k}(p) \tag{5.19}$$

with radius d in each normal space $p + N(p)$ of Φ. This is a generalization of (2.14). So, we can obtain the sphere $\sigma(p)$ as intersection $S(p) \cap (p + N(p))$.

We will prove that the offset hypersurface Φ_d to a rationally parametrizable ruled $k+1$–manifold Φ is a PN hypersurface, if Φ is not a hypersurface with constant tangent hyperplanes $\tau(t)$ along the generating subspaces $G_{n-2}(t)$. The construction shall be done in three steps. Firstly, we will construct a rational curve $f(t)$ and a rational unit normal vector $M_1(t)$, which satisfies $M_1(t) \perp \tau(f(t))$. Secondly, we will find a rational unit normal $M(p)$ at all regular points p of Φ. Thirdly, we give a parametrization of the spheres $\sigma(p)$.

Step 1: Let p be a regular point of Φ. A vector z is a normal vector of Φ at p if

$$\begin{aligned} p_t \cdot z = (\dot{l}(t) + \sum_1^k u_i \dot{g}_i(t)) \cdot z &= 0, && (5.20) \\ g_i(t) \cdot z &= 0, \ \text{for } i = 1,\dots,k. && (5.21) \end{aligned}$$

The solutions of (5.21) form the subspace $G_k^\perp$, spanned by b_i, and can be written as

$$y(t) = \lambda_1 b_1(t) + \ldots \lambda_{n-k} b_{n-k}(t), \quad \lambda_i \in \mathbb{R}. \tag{5.22}$$

We note that $b_i \cdot g_j = 0$, for all possible i and j. Some technical details are necessary. The vectors g_i are renumbered such that $(g_1,\dots,g_k,\dot{g}_1,\dots,\dot{g}_m)$ is a linearly independent set.

This implies that $G_k^\perp \not\subset \dot{g}_j^\perp$, but $G_k^\perp \cap \dot{g}_j^\perp$ is a hyperplane in $G_k^\perp$, for all $j = 1,\dots,m$. It is necessary to distinguish two cases. If $\dim T(t) = k+m$, thus $\dot{l} \in A(t)$, we form the set

$$W_1 = G_k^\perp \setminus \{G_k^\perp \cap \dot{g}_1^\perp, \dots, G_k^\perp \cap \dot{g}_m^\perp\}.$$

If $\dim T(t) = k+m+1$, thus $\dot{l} \notin A(t)$, we form the set

$$W_2 = G_k^\perp \setminus \{G_k^\perp \cap \dot{g}_1^\perp, \dots, G_k^\perp \cap \dot{g}_m^\perp, G_k^\perp \cap \dot{l}^\perp\}.$$

We choose the vector $b_1 \in W_1$ or W_2, respectively. This implies that $b_1 \cdot \dot{g}_j \neq 0$ for $j = 1,\dots,m$ in Case 1 and in Case 2 we additionally have $b_1 \cdot \dot{l} \neq 0$. Then, we insert (5.22) into (5.20) and obtain a condition for λ_1. This leads to the representation of the normal vector z

$$z = -[\sum_{i=2}^{n-k} \lambda_i b_i \cdot p_t] b_1 + \sum_{i=2}^{n-k} \lambda_i (b_1 \cdot p_t) b_i. \tag{5.23}$$

Since p_t is linear in u_i, the solution z is linear in u_i. Inserting $p_t = \dot{l} + \sum u_i \dot{g}_i$ into (5.23) leads to

$$z(t, u_i) = \bar{u}_0 c_0(t) + \bar{u}_1 c_1(t) + \ldots + \bar{u}_m c_m(t). \tag{5.24}$$

where $\bar{u}_i$ are homogeneous parameters, obtained by $u_i = \bar{u}_i / \bar{u}_0$. The vectors c_i are

$$\begin{aligned} c_0 &= -[(\sum_{i=2}^{n-k} \lambda_i b_i) \dot{l}] b_1 + (b_1 \cdot \dot{l}) \sum_{i=2}^{n-k} \lambda_i b_i, \\ c_j &= -[(\sum_{i=2}^{n-k} \lambda_i b_i) \dot{g}_j] b_1 + (b_1 \cdot \dot{g}_j) \sum_{i=2}^{n-k} \lambda_i b_i, \text{ for } j = 1, \ldots, m. \end{aligned}$$

They are rational in t and depend additionally on parameters $\lambda_1, \ldots, \lambda_{n-k}$. Is it possible to choose the λ_i such that the vectors $c_i(t)$ are linearly independent? We have to discuss two cases.

Case 1: Firstly, let Φ be a ruled $k+1$–manifold of the type $\dim T(t) = k + m$, that means $\dot{l} \in A(t)$. We can choose $l \in K_{k-m}$ such that $\dot{l} \in \mathrm{span}(g_1, \ldots, g_k)$. In this case we obtain $c_0 = 0 \in \mathbb{R}^n$ and further

$$z = \bar{u}_1 c_1(t) + \ldots + \bar{u}_m c_m(t).$$

Since we know that $b_1 \cdot \dot{g}_j \neq 0$ for $j = 1, \ldots, m$, we can choose the parameters $\lambda_1, \ldots, \lambda_m$ such that $(c_1, \ldots, c_m)$ are linearly independent.

Since we exclude manifolds with only singular points, the case $m = 0$ does not appear. If $m = 1$, the normal z and also the normal space $p + N$ only depends on t but not on the parameters u_i. So, N is constant for all regular points of a fixed generating space G_k. This case is treated separately and the dimension k will be important.

Case 2: Secondly, let Φ be a ruled $k+1$–manifold of the type $\dim T(t) = k + m + 1$, that means $\dot{l} \notin A(t)$. Similar to Case 1 it is possible to choose $m + 1$ parameters λ_i in a way such that the representation (5.24) contains already $m + 1$ linearly independent vectors $c_0, \cdot, c_m$. Here, the case $m = 0$ is allowed and represents the cylinders with directrix curve l. If $m = 0$, the normal vector space N is independent of u_i. We will discuss this later and again the dimension k will be important.

In both cases we can assume that the rational vectors $c_i(t)$ form an orthogonal set, but clearly not normalized. We will construct a normal vector $M_1(t)$ or $M_2(t)$ for the Cases 1 or 2, respectively, which possess rational length. In both cases, the squared length of z is

$$\begin{aligned} &\text{Case 1:} \quad z \cdot z = \textstyle\sum_{i=1}^{m} c_i \cdot c_i \bar{u}_i^2, \\ &\text{Case 2:} \quad z \cdot z = \textstyle\sum_{i=0}^{m} c_i \cdot c_i \bar{u}_i^2. \end{aligned}$$

This defines a family of not necessarily regular quadrics

$$\begin{array}{ll} \text{Case 1:} & Q_1(t) : \sum_{i=1}^{m} c_i \cdot c_i \bar{u}_i^2 - w^2 = 0, \\ \text{Case 2:} & Q_2(t) : \sum_{i=0}^{m} c_i \cdot c_i \bar{u}_i^2 - w^2 = 0. \end{array} \tag{5.25}$$

Proposition 5.1 says that there are polynomials $w(t)$ and $\bar{u}_i(t)$ which satisfy $Q_1(t)$ or $Q_2(t)$, respectively, identically, if the quadric possesses real points and conic sections. The reality condition is always satisfied. The quadrics $Q_1(t)$ or $Q_2(t)$ contain conics, except in Case 1 for $m = 1$ or in Case 2 for $m = 0$, which will be studied separately. Otherwise it follows that

$$\begin{aligned} \text{Case 1: } M_1(t) &= c_1(t)\frac{\bar{u}_1}{w}(t) + \ldots c_m(t)\frac{\bar{u}_m}{w}(t) \\ \text{Case 2: } M_2(t) &= c_0(t) + c_1(t)\frac{\bar{u}_1}{\bar{u}_0}(t) \ldots c_m(t)\frac{\bar{u}_m}{\bar{u}_0}(t), \end{aligned}$$

are rational normal vectors. In Case 1, $M_1(t)$ is a unit normal. In Case 2, $M_2(t)$ has the length $\sqrt{M_2 \cdot M_2} = w/\bar{u}_0$ and $M_2(t)$ can be normalized. Inserting the rational parameters $\bar{u}_1(t)/w(t), \ldots, \bar{u}_m(t)/\bar{u}_0(t)$ into the parameter representation of Φ determines a curve $f(t)$ which possesses a rational normal vector field $M_1(t)$ or $M_2(t)$, respectively, with rational length. We summarize the construction of Step 1.

Corollary 5.2 *Let Φ be a rational parametrizable ruled $k+1$–manifold. There exists a rational curve $f(t)$ which possesses a rational unit normal $M_1(t)$ or $M_2(t)$, respectively, except in Case 1 for $m = 1$ or in Case 2 for $m = 0$.*

Step 2: We will construct a unit normal $M(t, u_i)$ for all points of Φ, except in Case 1 for $m = 1$ or in Case 2 for $m = 0$. This can be done by stereographic projection, applied to $Q_1(t)$ or $Q_2(t)$. But we have to consider $Q_1(t)$ or $Q_2(t)$ to be a hyperquadric in $\mathbb{P}^{k+1}$. We note that $Q_1(t)$ always contains singular points, even for $m = k$, whereas $Q_2(t)$ is regular for $m = k$. This proves the following.

Corollary 5.3 *Let Φ be a rationally parametrizable ruled $k+1$–manifold, excluding $m = 1$ in Case 1 and $m = 0$ in Case 2. For all regular points $p(t, u_i)$ there exists a rational unit normal $M(t, u_i)$.*

Step 3: If the normal vector space $N(p)$ of Φ is 1–dimensional we are already done. Otherwise we have to parametrize the spheres $\sigma(p)$, centered at p with radius d, which are contained in the normal space $p + N(p)$, see formula (5.19). Additionally, each sphere $\sigma(p)$ possesses the rational point $p + dM$. We apply a stereographic projection with center $p + dM$ and obtain a rational parametrization of $\sigma(p)$.

Another method is to reflect the point $p + dM$ at all diameter hyperplanes of $\sigma(p)$, considered as hypersphere in the normal space $p + N(p)$. We use local coordinates in $N(p)$. Let $c(v) = (v_1, \ldots, v_{n-(k+2)}, 1)$ be the normals of these diameter hyperplanes. A rational parametrization of the offset hypersurface Φ_d of Φ is

$$\Phi_d : y(t, u_1, \ldots, u_k, v_1, \ldots, v_{n-(k+2)}) = (p + dM)(t, u) - 2d\frac{M(t, u) \cdot c(v)}{c(v) \cdot c(v)}c(v) \tag{5.26}$$

Now we study the two exclusions, Case 1 for $m = 1$ and Case 2 for $m = 0$. First we note that the normal vector space only depends on t and not on the parameters u_i, such that we have $N(p) = N(t)$. All regular points of a fixed generating space $G(t)$ possess the same normal space and also the same tangent space.

If $\dim N(t) = 1$, the length $\sqrt{M \cdot M}$ of the normal vector $M(t)$ is in general not rational. So we conclude that the offset hypersurface of these ruled hypersurfaces Φ are in general not rational parametrizable.

If $\dim N(t) \geq 2$ a general normal vector $M(t)$ at $p \in \Phi$ is a linear combination

$$M(t) = d_1(t)v_1 + \ldots + d_{n-k-1}(t)v_{n-k-1},$$

where we can assume that $(d_1(t), \ldots, d_{n-k-1}(t))$ is a rational orthogonal basis of $N(t)$. The squared length of $M(t)$ is

$$M(t) \cdot M(t) = \sum_{i=1}^{n-k-1} d_i \cdot d_i v_i^2. \tag{5.27}$$

This defines the quadric

$$Q(t) : \sum_{i=1}^{n-k-1} d_i \cdot d_i v_i^2 = w^2 \tag{5.28}$$

in $\mathbb{P}^{n-k-1}$ with homogeneous coordinates v_i, w. Since $n - k - 1 \geq 2$, $Q(t)$ contains conic sections and real points. Proposition 5.1 says that there exist polynomials $v_i(t)$ and $w(t)$, satisfying $Q(t)$ identically.

Since $N(t)$ is constant for all regular points of $G(t)$, it follows that there is a rational unit normal vector $M(t)$ for all regular points of Φ. What remains is equivalent to Step 3 from above. We apply a stereographic projection to the quadric $Q(t)$, given by formula (5.28). This leads to a rational parametrization of the spheres $\sigma(p)$. Together with above constructions this proves the following.

Theorem 5.2 *Let Φ be a ruled $k+1$–manifold different from a hypersurface with constant tangent hyperplanes along its generators $G(t)$. Then, the offset hypersurface Φ_d to Φ at distance d is a PN hypersurface.*

To illustrate the method just described we discuss two examples.

Example: Let Φ be a developable 2–manifold in 4–space, parametrized by

$$\Phi : p(t, u) = c(t) + u\dot{c}(t)$$

Let c be a polynomial (rational) space curve, free of inflection points. The tangent plane is given by

$$\tau : p + \lambda(\dot{c} + u\ddot{c}) + \mu\dot{c}.$$

The normal vectors of Φ are solutions of the linear homogeneous system

$$\begin{aligned} m_j \cdot (\dot{c} + u\ddot{c}) &= 0, \\ m_j \cdot \dot{c} &= 0. \end{aligned}$$

This implies that the normal vector space $N(t)$ does not depend on u. Since c is required to be free of inflections, $N(t)$ is 2–dimensional. Let $m_1(t)$ and $m_2(t)$ be an orthogonal basis of $N(t)$. A general normal vector m of Φ is a linear combination of

$$m(t) = m_1(t)v_1 + m_2(t)v_2.$$

For the squared length of m we obtain a quadratic polynomial in v_i

$$||m||^2 = \alpha_1(t)v_1^2 + \alpha_2(t)v_2^2,$$

whose coefficients α_i are polynomial (rational) functions in t. We form the quadric

$$Q(t) : \alpha_1(t)v_1^2 + \alpha_2(t)v_2^2 = w^2.$$

Proposition 5.1 says that there exist polynomials $v_1(t)$, $v_2(t)$ and $w(t)$ which satisfy $Q(t)$ identically. It follows that

$$m(t) = m_1(t)\frac{v_1}{w}(t) + m_2(t)\frac{v_2}{w}(t)$$

is a unit normal vector field of Φ. To obtain a parametrization of the offset hypersurface Φ_d at distance d, it remains to parametrize circles $\sigma(p)$ of radius d in the normal spaces $p + N(t)$. Let $m_1(t)$ be an arbitrary rational normal vector of Φ, linearly independent of m for almost all t. Then, Φ_d can be parametrized by

$$\Phi : q(t, u, \lambda) = p(t, u) + dm(t) - 2d\frac{m(t) \cdot (\lambda m_1(t) - m(t))}{(\lambda m_1(t) - m(t)) \cdot (\lambda m_1(t) - m(t))}(\lambda m_1(t) - m(t)).$$

Example: Let Φ be a 2–manifold in 4–space, parametrized by

$$\Phi : p(t, u) = c(t) + ug(t),$$

where c is a rational space curve, g a rational vector field. The vectors $\dot{c}, g, \dot{g}$ shall be linearly independent, which implies that Φ is non developable. The normal vectors of Φ are solutions of the linear homogeneous system

$$\begin{aligned} m_j \cdot (\dot{c} + u\dot{g}) &= 0, \\ m_j \cdot g &= 0. \end{aligned} \tag{5.29}$$

The normal vector space $N(t, u)$ depends on u, which is the main difference to the previous example. Let m be a normal vector of Φ and with (5.29) it is a linear combination

$$m(t, u) = m_1(t) + um_2(t).$$

The squared length is a quadratic function in u. Setting $u = u_2/u_1$ it follows

$$||m||^2 = \alpha_1(t)u_1^2 + \alpha_2(t)u_2^2,$$

where α_i are rational in t. Proposition 5.1 says that there exist polynomials $u_1(t)$, $u_2(t)$ and $w(t)$ such that

$$\alpha_1(t)u_1(t)^2 + \alpha_2(t)u_2(t)^2 = w(t)^2$$

is identically satisfied. It follows that

$$m(t) = m_1(t)\frac{u_1}{w}(t) + m_2(t)\frac{u_2}{w}(t)$$

is a unit normal vector field of Φ along a curve $f(t)$. Applying a stereographic projection to the quadric, defined by the squared length, we obtain unit normals $m(t, \tilde{u})$ in each point of Φ. We note that this new parameter $\tilde{u}$ depends on t and on the stereographic projection.

To obtain a parametrization of the offset hypersurface Φ at distance d, it remains to parametrize circles $\sigma(p)$ of radius d in the normal spaces $p + N(t, \tilde{u})$. Let $n(t, \tilde{u})$ be an arbitrary normal vector of Φ, where $(m(t, \tilde{u}), n(t, \tilde{u}))$ is a basis of the normal vector space $N(t, \tilde{u})$. Then, Φ_d can be parametrized by

$$\Phi_d : q(t, \tilde{u}, \lambda) = p(t, \tilde{u}) + dm - 2d\frac{m \cdot (\lambda n - m)}{(\lambda n - m) \cdot (\lambda n - m)}(\lambda n - m).$$

Bibliography

[1] Beauville, A., (1983), *Complex Algebraic Surfaces*, London Mathematical Society, Lecture Note Series 68, Cambridge University Press.

[2] Benz, W., (1992), *Geometrische Transformationen*, BI-Wiss.-Verlag, Mannheim.

[3] Blaschke, W., (1910), Untersuchungen über die Geometrie der Speere in der Euklidischen Ebene, Monatshefte für Mathematik und Physik 21, 3–60.

[4] Blaschke, W., (1929), *Vorlesungen über Differentialgeometrie III*, Springer, Berlin.

[5] Blutel, E., (1890), Recherches sur les surfaces qui sont en meme temps lieux de coniques et envellopes de cones du second degre, Ann. sci. ecole norm. super, (3) 7, 155–216.

[6] Boehm, W. (1990), On cyclides in geometric modeling, Comput. Aided Geom. Design 7, 243–255.

[7] Boehm, W. and Paluszny, M., (1996), Cyclides and Their Generalizations as Joining Surfaces.

[8] T.E. Cecil, T.E., (1992), *Lie Sphere Geometry*, Springer, Berlin, Heidelberg, New York.

[9] Degen, W., (1964), Zur projektiven Differentialgeometrie der Flächen, die von einer einparametrigen Schar von Kegelschnitten erzeugt werden I, Mathematische Annalen 155, 1–34.

[10] Degen, W., (1967), Zur projektiven Differentialgeometrie der Flächen, die von einer einparametrigen Schar von Kegelschnitten erzeugt werden II, Mathematische Annalen 170, 1–36.

[11] Degen, W., (1986), Die zweifach Blutelschen Kegelschnittflächen, manuscripta mathematika 55, 9–38.

[12] Degen, W., (1994), Nets with plane silhouettes, The Mathematics of Surfaces V, 117–133.

[13] DeRose, T.D. (1991), Rational Bézier curves and surfaces on projective domains, in: Farin, G., ed., *NURBS for Curve and Surface Design*, SIAM, Philadelphia, 35–45.

[14] Frank, H. and Giering, O., (1976), Verallgemeinerte Regelflächen, Math. Zeitschrift 150, 261–271.

[15] Frank, H. and Giering, O., (1978), Regelflächen mit Zentralräumen, Sitzungsberichte der mathem.–naturw. Kl., Abt. II, Band 187, Heft 4–7, 139–163.

[16] Farouki, R.T. (1992), Pythagorean–hodograph curves in practical use, in: Barnhill, R.E., ed., *Geometry Processing for Design and Manufacturing*, SIAM, Philadelphia, 3–33.

[17] Farouki, R.T. (1994), The conformal map $z \mapsto z^2$ of the hodograph plane, Comput. Aided Geom. Design 11, 363–390.

[18] Farouki, R.T. and Neff, C.A. (1995), Hermite interpolation by Pythagorean hodograph quintics, Math. Comp. 64, 1589–1609.

[19] Farouki, R.T. and Sakkalis, T. (1990), Pythagorean hodographs, IBM J. Res. Develop. 34, 736–752.

[20] Farouki, R.T. and Sakkalis, T. (1994), Pythagorean–hodograph space curves, Advances in Comp. Math. 2, 41–66.

[21] Farouki, R.T. et al (1994), Offset curves in layered manufacturing, ASME PED-Vol. 68-2, Manufacturing Science and Engineering Book No. G0930B, 557–568.

[22] Fiorot, J.C. and Gensane, T. (1994), Characterizations of the set of rational parametric curves with rational offsets, in: Laurent, P.J., LeMéhauté, A. and Schumaker, L.L., eds., *Curves and Surfaces in Geometric Design*, A K Peters, Wellesley, MA, 153–160.

[23] Jüttler, B. and Wagner, M., (1996), Computer aided design with spatial rational B–spline motions, ASME J. of Mechanical Design 118, 193–201.

[24] Lü, W. and Pottmann, H., (1996), Pipe surfaces with rational spine curve are rational, Comput. Aided Geom. Design 13, 621–628.

[25] Lü, W. (1994), Rationality of the offsets to algebraic curves and surfaces, Applied Mathematics 9 (Ser. B), 265–278.

[26] Lü, W. (1995), Offset–rational parametric plane curves, Computer Aided Geometric Design 12, 601–616.

[27] Lü, W. (1996), Rational parameterization of quadrics and their offsets, Computing 57, 135–147.

[28] Mäurer, Ch. (1996), Rational Curves and Surfaces on Dupin Ring Cyclides, Preprint Nr. 1806, Fachbereich Mathematik, TH Darmstadt.

[29] Mäurer, Ch. (1997), Rationale Bézier –Kurven und Bézier –Flächenstücke auf Dupinschen Zykliden, Dissertation, TH Darmstadt.

[30] Martin, R.R. (1982), Principal patches for computational geometry, PhD Thesis, Cambridge University.

[31] Peternell, M. and Pottmann, H. (1996), Designing rational surfaces with rational offsets, in: Fontanella, F., Jetter, K. and Laurent, P.J., eds., *Advanced Topics in Multivariate Approximation*, World Scientific, 275–286.

[32] Peternell, M. and Pottmann, H. (1997), Computing Rational Parametrizations of Canal Surfaces, J. Symbolic Computation **23**, 255–266.

[33] Peternell, M. and Pottmann, H., (1997), A Laguerre Geometric Approach to Rational Offsets, Comput. Aided Geom. Design, to appear.

[34] Pottmann, H. (1994), Applications of the dual Bézier representation of rational curves and surfaces, in: Laurent, P.J., LeMéhauté, A. and Schumaker, L.L., eds., *Curves and Surfaces in Geometric Design*, A K Peters, Wellesley, MA, 377–384.

[35] Pottmann, H. (1995), Rational curves and surfaces with rational offsets, Comput. Aided Geom. Design 12, 175–192.

[36] Pottmann, H. (1995), Curve design with rational Pythagorean–hodograph curves, Advances in Comp. Math. 3, 147–170.

[37] Pottmann, H. and Farin, G. (1995), Developable rational Bézier and B–spline surfaces, Comput. Aided Geom. Design 12, 513–531.

[38] Pottmann, H., Lü, W. and Ravani, B. (1996), Rational ruled surfaces and their offsets, Graphical Models and Image Processing 58, 544-552.

[39] Pottmann, H. and Peternell, M., (1997), Applications of Laguerre Geometry in CAGD, Comput. Aided Geom. Design, to appear.

[40] Pottmann, H. and Wagner, M., (1997), Principal Surfaces, in: T.N.T. Goodman (ed.), *The Mathematics of Surfaces VII*, Information Geometers Ltd., to appear.

[41] Powell, M.J.D. and Sabin, M.A., (1977), Piecewise quadratic approximation on triangles, ACM Transactions on Mathematical Software **3**, 317–325.

[42] Pratt, M.J., (1990), Cyclides in Computer Aided Geometric Design, Comput. Aided Geom. Design 7, 221–242.

[43] Pratt, M.J., (1995), Cyclides in Computer Aided Geometric Design, Comput. Aided Geom. Design 12, 131–152.

[44] Pratt, M.J., (1997), Quartic supercyclides I: Basic theory, Comput. Aided Geom. Design 14, 671–692.

[45] Shafarevich, I.R., (1977), *Basic Algebraic Geometry I*, Springer, Berlin, Heidelberg.

[46] Srivinas, Y.L. and Dutta, D., (1994), An intuitive procedure for constructing geometrically complex objects using cyclides, Computer Aided Design 26, 327–335.

[47] Srivinas, Y.L. and Dutta, D., (1995), Rational parametric representation of parabolic cyclide: Formulation and applications, Comput. Aided Geom. Design 12, 551–566.

[48] Thomsen, G., (1932), Über Kegelschnitte in Raum, Mathematische Annalen 108, 260–295.

[49] Wunderlich, W., (1948), Spiegelung am elliptischen Paraboloid, Monatshefte f. Math. 52, 13–37.

Lebenslauf

Am 20. August 1964 wurde ich als Sohn des Beamten Kurt Peternell und seiner Frau Renate Peternell, geb. Wagner, in Timelkam / OÖ. geboren. Von 1970 bis 1974 besuchte ich die Volksschule in Timelkam, später das Bundesgymnasium in Vöcklabruck, wo ich am 18. Juni 1982 maturierte.

Im Jahr 1982 immatrikulierte ich an der Technischen Universität Wien und inskribierte Vermessungswesen. Gleichzeitig besuchte ich diverse Vorlesungen über Philosophie und Politikwissenschaft an der Universität Wien. Im Studienjahr 83/84 besuchte ich einen Lehrgang für Vermessungstechnik an der HTBLVA Wien III, den ich am 29. Juni 1984 abschloß. Von Oktober 1984 bis Mai 1985 leistete ich dann meinen Präsenzdienst in Wien ab. In den Jahren 1985 bis 1987 besuchte ich ein Kolleg für Bautechnik an der HTBLVA Wien III, wo ich am 6. Oktober 1987 mit Auszeichnung maturierte.

Ab Oktober 1987 inskribierte ich an der TU Wien das Lehramtsstudium für Darstellende Geometrie und Mathematik. Dieses Studium schloß ich am 11. Jänner 1995 ab und am 26. Jänner 1995 wurde mir der akademische Grad des Magister der Naturwissenschaften verliehen.

In den Jahren 1991 bis 1993 war ich technischer Angestellter in einem Architekturbüro in Wien. Ab März 1994 arbeitete ich an einem zweijährigen Forschungsprojekt des Fonds zur Förderung der wissenschaftlichen Forschung (FWF) über 'Funktionelle B–Spline Formen in der CAD/CAM Technik' unter der Leitung von Prof. H. Pottmann. Seit Jänner 1996 bin ich als Universitätsassistent am Institut für Geometrie der TU Wien beschäftigt.

Seit 26. November 1990 bin ich mit Ingrid Peternell, geb. Eder, verheiratet. Wir haben drei Kinder, Michael, geb. 1983, Iris, geb. 1993 und Nora, geb. 1996.

Wien, Oktober 1997 Martin Peternell