Discrete Surfaces in Isotropic Geometry

Helmut Pottmann [a] and Yang Liu [b]

[a] Geometric Modeling and Industrial Geometry, Vienna University of Technology, Wiedner Hauptstr. 8-10, A-1040 Wien
[b] Department of Computer Science, University of Hong Kong
pottmann@geometrie.tuwien.ac.at
yliu@cs.hku.hk
http://www.geometrie.tuwien.ac.at/ig

Abstract. Meshes with planar quadrilateral faces are desirable discrete surface representations for architecture. The present paper introduces new classes of planar quad meshes, which discretize principal curvature lines of surfaces in so-called isotropic 3-space. Like their Euclidean counterparts, these isotropic principal meshes meshes are visually expressing fundamental shape characteristics and they can satisfy the aesthetical requirements in architecture. The close relation between isotropic geometry and Euclidean Laguerre geometry provides a link between the new types of meshes and the known classes of conical meshes and edge offset meshes. The latter discretize Euclidean principal curvature lines and have recently been realized as particularly suited for freeform structures in architecture, since they allow for a supporting beam layout with optimal node properties. We also present a discrete isotropic curvature theory which applies to all types of meshes including triangle meshes. The results are illustrated by discrete isotropic minimal surfaces and meshes computed by a combination of optimization and subdivision.

Key words: discrete differential geometry, surfaces in architecture, isotropic geometry, conical mesh, edge offset mesh, isotropic minimal surface.

1 Introduction

Recently, *discrete differential geometry* enjoys increasing interest in Computer Graphics and Geometric Modeling. This emerging field at the border between differential and discrete geometry provides attractive tools for processing discrete surface representations with methods that may be seen as an extension of the classical theory. This is quite different from numerical differential geometry where approximation by smooth representations is directly or indirectly used as a preprocessing step in order to employ the classical results.

Very recently it turned out that the precise geometric results and relations offered by discrete differential geometry are of high importance in architectural design [7, 10, 11, 13, 19, 21]. This is rooted in the fact that in architectural freeform structures which are based on polyhedral surfaces the mesh representation (i) is seen and greatly influences the aesthetics and (ii) is the basis for the actual construction.

Though meshes are very well understood in geometric computing, their application to architecture is difficult for at least two reasons. One is that triangle meshes, though easy to compute and inherently stable due to their geometry, lead to complications in node manufacturing and supporting beam layout [19, 21]. The other reason is that alternatives, namely quad meshes or hexagonal meshes with planar faces, are harder to deal with: Quad meshes with planar faces discretize so-called conjugate curve networks [24]; their computation in general requires nonlinear optimization [13] and aesthetical requirements may be very hard to achieve. Planarity of faces in hexagonal meshes is easy to realize since only three faces meet in a vertex, but an aesthetic layout of such a mesh on a given freeform surface is an unsolved problem. Building on recently achieved progress [7, 13, 19, 21], the present paper introduces new remarkable classes of planar quad meshes. These are formulated and studied with help of so-called isotropic geometry, which is motivated as follows.

1. Principal curvature lines are sometimes viewed as appropriate curves for guiding the force flow and for visually expressing fundamental shape characteristics. These are reasons why they are preferred for beam layout. The ideal force flow depends on the direction of gravity, whereas Euclidean principal curvature lines are independent of it. Isotropic geometry has a distinguished (isotropic) direction (cf. Section 2.1) and thus it is promising to investigate whether isotropic principal curvature lines provide structural advantages. Before doing so, we need tools to design meshes whose polygons are aligned in isotropic principal direction. This is a main goal of the present paper. An example is shown in Fig. 1. It also illustrates that isotropic principal curvature lines may lead to aesthetically pleasing meshes. Isotropic principal curvature lines form a special conjugate curve network, namely the one which appears as orthogonal network in the top view (projection in isotropic direction).
2. Conical meshes [7, 13] and edge offset meshes [7, 21] have various advantages for architecture, including a supporting beam layout with optimized nodes [7, 13, 21]. These two types of meshes are objects of Laguerre geometry. Isotropic geometry occurs naturally as a model of Laguerre geometry. Hence, it is interesting to see how these meshes appear in the so-called isotropic model of Laguerre geometry. We have here a principle of transfer between meshes. This is useful both for the theoretical understanding and for applications.

1.1 Previous work

Discrete differential geometry. Particularly important in the present context are results on quadrilateral meshes with planar faces (*PQ meshes*). PQ meshes discretize so-called conjugate curve networks on surfaces [24]. They are a basic concept in the integrable system viewpoint of discrete differential geometry [4]. *Circular meshes* [14] and *conical meshes* [13] are special PQ meshes which discretize the network of principal curvature lines. In a circular mesh, each face

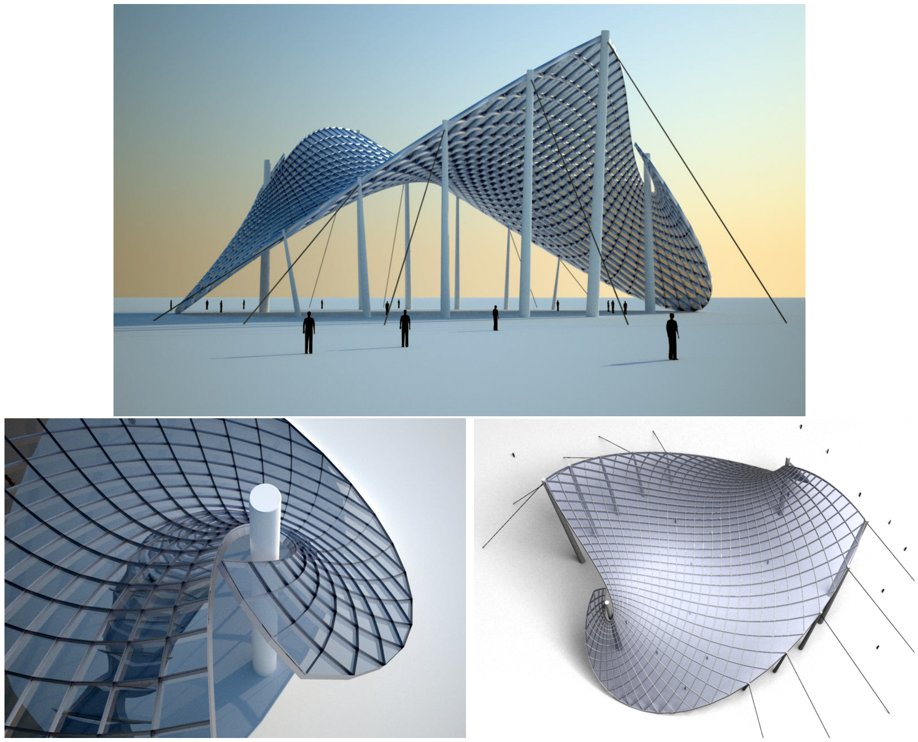

Fig. 1. This architectural design study is based on a quadrilateral mesh with planar faces which discretizes the network of isotropic principal curvature lines on a minimal surface of isotropic geometry.

has a circum-circle; in a conical mesh, the face planes meeting at a vertex are tangent to a cone of revolution. Principal curvature lines are a concept of Lie sphere geometry, and thus circular and conical meshes may be treated in a unified way within Lie geometry [5]. Elementary relations between circular and conical meshes are discussed in [20]. *Mesh parallelism* turned out to be an important tool for studying polyhedral surfaces with a view towards applications in architecture [7, 21]. Due to its high importance for the present paper, mesh parallelism is briefly reviewed in subsection 3.1. PQ meshes which are discrete counterparts to minimal surfaces are the topic of several contributions (e.g. [2, 3, 21, 33]).

Geometry processing. Computational issues concerning the meshes mentioned above are only addressed in a few papers [2, 13, 21]. PQ meshes, in particular circular and conical meshes, may be designed by a combination of subdivision and nonlinear optimization [13]. The same optimization algorithm can be used to approximate a given shape by a circular/conical mesh, if the input mesh is aligned along a network of principal curvature lines. Methods for the design and computation of meshes with offset properties relevant for architecture and for the layout of supporting beams with optimized nodes are based on the concept of parallel meshes [21].

Approximation of smooth surfaces by meshes with planar faces, without a focus on beam layout and offset properties, can be performed via variational shape approximation [9]. Cutler and Whiting [10] modified this method with regard to aesthetics and architectural design. Research on geometry for architecture in general is promoted by the Smart Geometry group (*www.smartgeometry.com*).
Laguerre geometry is the geometry of oriented planes and spheres in Euclidean 3-space [1, 8]. It has been applied in the study of rational curves and surfaces with rational offsets [16, 18] and in the reconstruction of developable surfaces from point clouds [15]. Its recent application in the study of conical meshes [5, 13, 33] and meshes with edge offsets [21] is the most important one for the present paper and will be continued in Section 3.
Isotropic geometry is based on a simple semi-Riemannian metric [22]; cf. Section 2.1. It naturally appears when properties of functions are to be geometrically visualized and interpreted at hand of their graph surfaces [17]. In particular, this holds for the visualization of stress properties in planar elastic systems at hand of their Airy surfaces [30]. An application of isotropic geometry in Image Processing has been given by Koenderink and van Doorn [12].

1.2 Contributions and Overview

The contributions of the present paper are the following ones.

1. We provide an introduction into isotropic geometry, Laguerre geometry and the relations between them (Section 2).
2. Meshes which discretize the network of isotropic principal curvature lines are studied in Section 3. There, we also address optimization algorithms for their design and computation.
3. We introduce discrete isotropic curvatures for meshes with planar faces and provide examples for discrete minimal surfaces of isotropic geometry (Sec. 4).

2 Fundamentals

2.1 Isotropic Geometry

Motions and metric. Isotropic geometry has been developed by Strubecker in the 1940s. The results we need for the present investigations may be found in [29] or in the monograph by Sachs [22]. Isotropic 3-space I^3 is based on the following group G_6 of affine transformations $(x, y, z) \mapsto (x', y', z')$ in $\mathbb{R}^3$,

$$\begin{aligned} x' &= a + x\cos\phi - y\sin\phi, \\ y' &= b + x\sin\phi + y\cos\phi, \\ z' &= c + c_1 x + c_2 y + z, \quad a, b, c, c_1, c_2, \phi \in \mathbb{R}, \end{aligned} \tag{1}$$

which are called *isotropic congruence transformations (i-motions)*. We see that i-motions appear as Euclidean motions (translation vector $(a,b)^T$ and rotation angle ϕ) in the projection onto the xy-plane; the result of this projection $\mathbf{p} = (x,y,z) \mapsto \mathbf{p}' = (x,y,0)$ is called *top view* henceforth. Hence, an isotropic congruence transformation is composed of a Euclidean motion in the xy-plane and an affine shear transformation in z-direction. Many metric properties in isotropic 3-space I^3 (invariants under G_6) are actually Euclidean invariants in the top view. For example, one defines the *i-distance* of two points $\mathbf{x}_j = (x_j, y_j, z_j),\ j = 1,2$, as the Euclidean distance of their top views $\mathbf{x}'_j$,

$$\|\mathbf{x}_1 - \mathbf{x}_2\|_i := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{2}$$

Thus, two points (x,y,z_j) with the same top view (called *parallel points*) have i-distance zero, but they need not agree. Since the i-metric (2) degenerates along z-parallel lines, these lines are called *isotropic lines*. Due to the architectural applications we have in mind, the isotropic z-direction is to be thought as vertical. *Isotropic angles* between straight lines are measured as Euclidean angles in the top view.

Planes, circles and spheres. There are two types of planes in I^3.

(i) *Non-isotropic planes* are not parallel to the z-direction. In these planes we basically have a Euclidean metric: This is not the one we are used to, since we we have to make the usual Euclidean measurements in the top view. An *i-circle (of elliptic type)* in a non-isotropic plane P is an ellipse, whose top view is a Euclidean circle. Such an i-circle with center $\mathbf{m} \in P$ and radius r is the set of all points $\mathbf{x} \in P$ with $\|\mathbf{x} - \mathbf{m}\|_i = r$.

(ii) *Isotropic planes* are parallel to the z-axis. There, I^3 induces an isotropic metric. An *isotropic circle (of parabolic type)* is a parabola with z-parallel axis and thus it lies in an isotropic plane. An i-circle of parabolic type is not the iso-distance set of a fixed point.

There are also two types of *isotropic spheres*. An *i-sphere S of the cylindrical type* is the set of all points $\mathbf{x} \in I^3$ with $\|\mathbf{x} - \mathbf{m}\|_i = r$. Speaking in a Euclidean way, such a sphere is a right circular cylinder with z-parallel rulings; its top view is the Euclidean circle with center $\mathbf{m}'$ and radius r. Any point which is parallel to $\mathbf{m}$ lies on the axis of this cylinder and may also serve as center of S. The more interesting and important type of spheres are the *i-spheres of parabolic type*,

$$z = \frac{A}{2}(x^2 + y^2) + Bx + Cy + D,\ A \neq 0. \tag{3}$$

From a Euclidean perspective, they are paraboloids of revolution with z-parallel axis. The intersections of these i-spheres with planes P are i-circles: If P is not isotropic, then the intersection is an i-circle of elliptic type. If P is isotropic, the intersection curve is an i-circle of parabolic type.

Curvature theory of surfaces. In the present context, isotropic differential geometry of surfaces is of great importance. To avoid degeneracies, we study only those surfaces Φ which are regular and do not possess isotropic (i.e., vertical) tangent planes. Thus, we may write Φ in explicit form,

$$\Phi: \ z = f(x, y). \tag{4}$$

The isotropic version of Gaussian curvature theory uses the isotropic unit sphere,

$$\Sigma : z = \frac{1}{2}(x^2 + y^2) =: s(x, y). \tag{5}$$

To each point $\mathbf{x} = (x, y, f(x, y)) \in \Phi$ we associate as Gaussian image that point $\sigma(\mathbf{x}) = \tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}, s(\tilde{x}, \tilde{y})) \in \Sigma$ whose tangent plane is parallel to the tangent plane of Φ at $\mathbf{x}$. This requires agreement of the gradients $\nabla f(x, y) = (f_x, f_y)$ and $\nabla s(\tilde{x}, \tilde{y}) = (\tilde{x}, \tilde{y})$ and thus yields for the Gaussian mapping σ,

$$\sigma : \ (x, y, f(x, y)) \mapsto (f_x, f_y, \frac{1}{2}(f_x^2 + f_y^2)). \tag{6}$$

The derivative of σ is the *isotropic shape operator*. It is a linear mapping between the parallel and thus identified tangent spaces of Φ at $\mathbf{x}$ and Σ at $\tilde{\mathbf{x}}$. The top view of the shape operator acts on vectors $(t_1, t_2)^T \in \mathbb{R}^2$ and is given by

$$(t_1, t_2)^T \mapsto \nabla^2 f \cdot (t_1, t_2)^T. \tag{7}$$

Its transformation matrix is the Hessian $\nabla^2 f$ of f,

$$\nabla^2 f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

The eigenvectors of the shape operator determine the isotropic principal curvature directions. They are conjugate directions and appear as orthogonal vectors in the top view (eigenvectors of $\nabla^2 f$); the corresponding eigenvalues are called *i-principal curvatures* κ_1, κ_2. Their product equals $\det(\nabla^2 f)$ and is called *isotropic curvature* (or *relative curvature*)

$$K = \kappa_1 \kappa_2 = f_{xx} f_{yy} - f_{xy}^2. \tag{8}$$

Isotropic mean curvature H is given by the Laplacian of f,

$$2H = \kappa_1 + \kappa_2 = \mathrm{trace}(\nabla^2 f) = f_{xx} + f_{yy} = \Delta f. \tag{9}$$

Isotropic minimal surfaces are characterized by $H = 0$ and thus they are graphs of harmonic functions f ($\Delta f = 0$). They may have branching points (see e.g. Fig. 8) and thus the surfaces can have several sheets; we understand a "graph" in this general sense. Isotropic minimal surfaces possess many properties which are analogous to their Euclidean counterparts [22, 31]. We will construct discrete i-minimal surfaces in Section 4.

Finally we point to the integral curves of the field of principal directions. These are called *isotropic principal curvature lines*. They constitute exactly that conjugate curve network on S which appears as orthogonal network in the top view. Discrete counterparts form the content of Section 3.

Metric duality. In projective 3-space we have a principle of duality. To give some examples, points are dual to planes and vice versa, straight lines are dual to straight lines and inclusions are reversed. However, duality cannot be applied to metric quantities of Euclidean geometry. This is different in isotropic geometry, which enjoys a metric duality. It may be realized by the polarity with respect to the i-sphere (5), which maps point $\mathbf{p} = (p_1, p_2, p_3)$ to plane $P: \ z = p_1x + p_2y - p_3$. Points $\mathbf{p}$ and $\mathbf{q} = (q_1, q_2, q_3)$ with i-distance d (from (2), recall $d^2 = (p_1 - q_1)^2 + (p_2 - q_2)^2$) are mapped to planes P and $Q: \ z = q_1x + q_2y - q_3$. The i-angle ϕ of the two planes P, Q equals d (we may even define i-angles in this way) and thus ϕ is simply computed as $\phi^2 = (p_1 - q_1)^2 + (p_2 - q_2)^2$.

Parallel points, i.e. points with the same top view, in the duality correspond to parallel planes. A surface $\Phi : z = f(x, y)$, seen as set of contact elements (points plus tangent planes) corresponds to a surface Φ^*, parameterized by

$$x^* = f_x(x, y), \ y^* = f_y(x, y), \ z^* = xf_x + yf_y - f. \tag{10}$$

Contact elements along i-principal curvature lines of Φ and Φ^* correspond in the duality. Note that Φ^* may have singularities which correspond to parabolic surface points of Φ ($K = 0$). This is reflected in the following relations between the isotropic curvature measures of dual surface pairs [32],

$$H^* = H/K, \ K^* = 1/K. \tag{11}$$

Thus, the dual surface to an i-minimal surface is also i-minimal. For further properties of the metric duality, see [22]. We will use duality in Section 4 to obtain curvatures at mesh vertices from curvatures attached to faces. In particular, duality will allow us to derive curvatures in triangle meshes from those in hexagonal meshes.

Already this very brief introduction into isotropic geometry reveals that it is actually simpler than Euclidean geometry. Isotropic counterparts to nonlinear problems of Euclidean geometry may be linear and thus computationally less demanding than their Euclidean version. For example, equation (9) shows that the PDEs characterizing minimal surfaces or surfaces of constant mean curvature are linear in isotropic geometry; the corresponding equations of Euclidean geometry are nonlinear. Since we use these tools mainly for aesthetical shape generation, it may be sufficient to resort to the isotropic version.

In order to present the relation between isotropic geometry and Laguerre geometry, we have to discuss sphere transformations.

Isotropic sphere transformations. In I^3, there exists a counterpart to Euclidean Möbius geometry (recall that planes and spheres form the set of so-called Möbius spheres and Möbius transformations act bijectively on this set). In I^3, one puts *i*-spheres of parabolic type and non-isotropic planes into the same class $\mathcal{S}$ of *isotropic Möbius spheres*; they are given by (3), including $A = 0$. Together with an appropriate extension $\mathcal{P}$ of the point set of I^3 (one adds $\mathbb{R}$ as set of "ideal points"), there exists a group M^i_{10} of so-called *isotropic Möbius (i-M) transformations* which act bijectively in $\mathcal{P}$ and in $\mathcal{S}$. The group of i-M-transformations also acts bijectively on the set of *i-M-circles*. An i-M-circle is defined as intersection of two i-M-spheres and may be an i-circle of elliptic or parabolic type or a non-isotropic line. The 10-dimensional group M^i_{10} is isomorphic to the group of Euclidean Laguerre transformations (see subsection 2.2). The top view of an i-M-transformation is a planar Euclidean Möbius transformation. The basic i-M-transformations are inversions with respect to i-spheres. The inversion (reflection) at an i-M-sphere $S : z = A(x^2 + y^2)/2 + ... =: s(x, y)$ is given by $(x, y, z) \mapsto (x, y, 2s(x, y) - z)$. The top view remains unchanged and in z-direction we have a reflection at the corresponding point of S. An inversion κ with respect to an i-sphere S of cylindrical type appears in the top view as ordinary inversion with respect to the circle S', the top view of S. Also in 3D, corresponding points $\mathbf{x}, \kappa(\mathbf{x})$ lie collinear with a fixed center $\mathbf{c}$ which must be contained in the axis of S.

2.2 Laguerre Geometry

Laguerre geometry is the geometry of oriented planes and oriented spheres in Euclidean E^3 [1, 8]. We may write an or. plane P in Hesse normal form $\mathbf{n}^T \cdot \mathbf{x} + h = 0$, where the unit normal vector $\mathbf{n}$ defines the orientation; $\mathbf{n}^T \cdot \mathbf{x} + h$ is the signed distance of the point $\mathbf{x}$ to P. An oriented sphere S, with center $\mathbf{m}$ and signed radius r, is tangent to an oriented plane P if the signed distance of $\mathbf{m}$ to P equals r, i.e., $\mathbf{n}^T \cdot \mathbf{m} + h = r$. Points are viewed as or. spheres with radius zero.

A *Laguerre transformation (L-transformation)* is a mapping which is bijective on the sets of or. planes and or. spheres, respectively, and keeps plane/sphere tangency. Viewing or. spheres S as points $\mathbf{S} := (\mathbf{m}, r) \in \mathbb{R}^4$, an L-transformation is seen in $\mathbb{R}^4$ as a special affine map $\mathbf{S}' = \mathbf{a} + R \cdot \mathbf{S}$, where R describes a linear map which preserves the inner product $\langle \mathbf{x}, \mathbf{y} \rangle := x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$. With the diagonal matrix $D := diag(1, 1, 1, -1)$ we have $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \cdot D \cdot \mathbf{y}$, and the condition on R reads $R^T \cdot D \cdot R = D$. A simple example of an L-transformation is the offsetting operation (given by the identity matrix R and $\mathbf{a} = (0, 0, 0, d)$), which adds a constant d to the radius of each sphere and in particular maps a point to a sphere of radius d.

Let us return to the standard model of Laguerre geometry in E^3. A pencil of parallel or. planes has the same normal vector $\mathbf{n}$ (image point on the "Gaussian" sphere S^2). An L-transformation keeps the parallelity of or. planes and induces a Möbius transformation of the Gaussian sphere S^2. The or. planes which are tangent to two or. spheres envelope an or. right circular cone, and thus or. cones (including limit cases) are also objects of Laguerre geometry.

The isotropic model of Laguerre geometry. There is the following remarkable relation between Laguerre geometry and isotropic Möbius geometry. We may use $(\mathbf{n}, h)$ as coordinates of an or. plane P. However, these four coordinates are not independent due to $\|\mathbf{n}\| = 1$. Thus, one replaces $\mathbf{n} = (n_1, n_2, n_3)$ (point of S^2) by its image point $(n_1/(n_3 + 1), n_2/(n_3 + 1), 0)$ under the stereographic projection of S^2 from $(0, 0, -1)$ onto $z = 0$. It is then convenient (and can be explained in a more geometric way, see e.g. [18]) to view the point

$$P^i := \frac{1}{n_3 + 1}(n_1, n_2, h) \tag{12}$$

as a kind of dual image of the oriented plane $P : n_1x+n_2y+n_3z+h = 0$. P^i should be seen as a point of isotropic space I^3, whose point set has been extended to $\mathcal{P}$ (see above). Thus, one also speaks of the *isotropic model of Laguerre geometry.* Parallel or. planes P, Q appear in the isotropic model as parallel points P^i, Q^i. The or. tangent planes of an or. sphere S are seen as points of an isotropic Möbius sphere in the isotropic model. The common tangent planes of two or. spheres (= or. cone) correspond to the common points of two i-M-spheres (= i-M-circle). A non-developable surface Φ viewed as set of or. tangent planes is mapped to a surface Φ^i in the isotropic model. Tangent planes along a Euclidean principal curvature line of Φ are mapped to points of an isotropic principal curvature line of Φ^i. Underlying all these facts is the essential result that an L-transformation corresponds to an i-M-transformation in the isotropic model. Hence, the two groups are isomorphic.

These classical relations will be important to realize the close connection between certain known meshes of Euclidean geometry and the meshes of isotropic geometry which form the content of the present paper. For more results and details on Laguerre geometry, see [1, 8, 18].

3 Principal meshes in isotropic geometry

3.1 Meshes with planar faces

Here, we first introduce some essential methods and results on mesh parallelism [7, 21] and then apply them in the context of isotropic geometry.

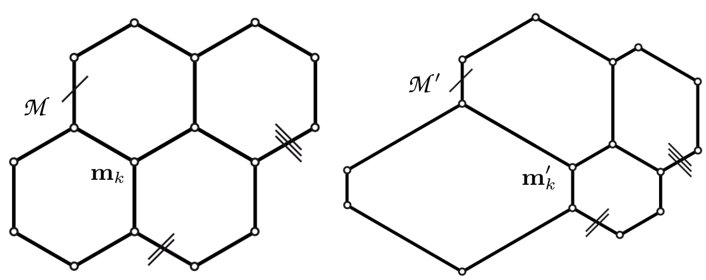

Fig. 2. Meshes $\mathcal{M}, \mathcal{M}'$ with planar faces are parallel if they are combinatorially equivalent and corresponding edges are parallel.

Mesh parallelism. A mesh $\mathcal{M}$ is represented by its vertices, concatenated in $(\mathbf{m}_1, \ldots, \mathbf{m}_N) \in \mathbb{R}^{3N}$ and the combinatorics, i.e., edges and faces. If $\mathcal{M}', \mathcal{M}''$ have the same combinatorics, a linear combination $\lambda'\mathcal{M}' + \lambda''\mathcal{M}''$ is defined vertex-wise; this operation corresponds to the linear combination of vectors in $\mathbb{R}^{3N}$. Meshes $\mathcal{M}, \mathcal{M}'$ are *parallel*, if they have the same combinatorics and corresponding edges are parallel (see Fig. 2). We use this definition only if the *faces of $\mathcal{M}$ (and hence of $\mathcal{M}'$) are planar.* Clearly, corresponding faces of $\mathcal{M}$ and $\mathcal{M}'$

lie in parallel planes. The set of meshes parallel to $\mathcal{M}$ is denoted by $\mathcal{P}(\mathcal{M})$. This space does not only contain 'nice' meshes; we may see various undesirable effects such as unevenly distributed face sizes, sharp regression edges or overlapping regions in meshes of this space, but it is theoretically and practically important to use it. Even visually unpleasant meshes of $\mathcal{P}(\mathcal{M})$ are helpful in the computation of optimized supporting beam layouts (cf. [21]).

Since triangles with parallel edges are scaled copies of each other, two parallel triangle meshes are scaled copies of each other. This degeneracy does not matter since we do not consider triangle meshes anyway (except at the end of Sec. 4 where a different type of parallelism is used).

Suppose $\mathcal{M}', \mathcal{M}'' \in \mathcal{P}(\mathcal{M})$. Then, for each edge $\mathbf{m}_i\mathbf{m}_j$, the vectors $\mathbf{m}'_i - \mathbf{m}'_j$, $\mathbf{m}''_i - \mathbf{m}''_j$ are multiples of $\mathbf{m}_i - \mathbf{m}_j$. It follows that any expression $(\lambda'\mathbf{m}'_i + \lambda''\mathbf{m}''_i) - (\lambda'\mathbf{m}'_j + \lambda''\mathbf{m}''_j)$ is a multiple of $\mathbf{m}_i - \mathbf{m}_j$. This shows that the linear combination $\lambda'\mathcal{M}' + \lambda''\mathcal{M}''$ is also parallel to $\mathcal{M}$, and thus $\mathcal{P}(\mathcal{M})$ is a linear subspace of $\mathbb{R}^{3N}$. The zero vector of $\mathcal{P}(\mathcal{M})$ is the mesh $\mathbf{o} = 0 \cdot \mathcal{M}$, all of whose vertices coincide with the origin of the coordinate system.

Meshes with offsets. A mesh $\mathcal{M}' \in \mathcal{P}(\mathcal{M})$ at constant distance from $\mathcal{M}$ is called an *offset* of $\mathcal{M}$. Different ways to define the precise meaning of a constant offset distance, $\mathrm{dist}(\mathcal{M}, \mathcal{M}') = d$, lead to different kinds of offsets:

(i) *vertex offsets*: $\|\mathbf{m}_i - \mathbf{m}'_i\| = d$, independently of the vertex $\mathbf{m}_i$.
(ii) *edge offsets*: The distance of corresponding parallel edges (actually, lines which carry those edges) does not depend on the edge and equals d.
(iii) *face offsets*: The distance of faces (actually, planes which carry faces) is independent of the face and equals d.

To characterize offset pairs of parallel meshes $\mathcal{M}, \mathcal{M}'$, one defines a further parallel mesh, the *Gauss image* $\mathcal{S} = (\mathcal{M}' - \mathcal{M})/d$, which satisfies $\mathrm{dist}(\mathcal{S}, \mathbf{o}) = 1$. Then, the following result is easy to show [21]:

(i) For the vertex offsets, the vertices of the Gauss image $\mathcal{S}$ are contained in the unit sphere S^2. If $\mathcal{S}$ is a quad mesh and no edges degenerate, then $\mathcal{M}$ has a vertex offset if and only if it is *circular*, i.e., each face has a circum-circle.
(ii) For edge offsets, the edges of the Gauss image $\mathcal{S}$ are tangent to S^2. These *edge offset meshes* are studied in [21] and allow for a realization as architectural designs with beams of constant height and the cleanest possible nodes.
(iii) For the face offsets, the faces of the Gauss image $\mathcal{S}$ are tangent to S^2. A mesh has a face offset if and only if it is *conical*, i.e., the faces around each vertex are tangent to a cone of revolution.

Hence, if $\mathcal{M}$ has the v-offset (e-offset, f-offset, resp.) property, any parallel mesh including $\mathcal{S}$ has the same offset property.

Relative Gauss mapping and r-principal meshes. The condition that parallel meshes $\mathcal{M}, \mathcal{M}'$ are at constant distance (in the notation from above, $\mathrm{dist}(\mathcal{M}', \mathcal{M}) = d$) is too rigid for certain applications in architecture, in particular for the layout of fair planar quad meshes, see [21]. One therefore considers the following generalization: We let $\mathcal{M}' = \mathcal{M} + d\mathcal{S}$, where $\mathcal{S} \in \mathcal{P}(\mathcal{M})$ is a mesh which approximates a certain convex surface Σ, then called the "relative unit sphere". The mesh $\mathcal{S}$, which is parallel to $\mathcal{M}$, is called *r-Gauss image* of $\mathcal{M}$. The vector $\mathbf{s}_i$ of $\mathcal{S}$ corresponding to a vertex $\mathbf{m}_i$ of $\mathcal{M}$ is the *r-normal vector*. Then the r-offset at r-distance d has vertices $\mathbf{m}_i + d\mathbf{s}_i$. The correspondence $\sigma : \mathcal{M} \to \mathcal{S}$ between the meshes $\mathcal{M}$ and $\mathcal{S}$ is called *r-Gauss mapping*. We use here the terminology from *relative differential geometry* [28], which generalizes Euclidean differential geometry of curves and surfaces by choosing a different "unit sphere" Σ instead of the standard Euclidean one. Below, we will choose a paraboloid Σ.

Definition 1. *If $\mathcal{M}$ and $\mathcal{S}$ are parallel quad meshes such that $\mathcal{S}$ approximates the r-unit sphere Σ, we call $\mathcal{M}$ an* r-principal mesh.

Here, "approximation" needs to be specified. We have seen three important examples above: $\mathcal{S}$ may have its vertices on Σ ($\mathcal{S}$ inscribed to Σ), it may have its face planes tangent to Σ ($\mathcal{S}$ circumscribed to Σ) or its edges may be tangent to Σ (here, $\mathcal{S}$ is called "midscribed" to Σ [27]). An r-principal mesh can be considered as discrete counterpart to the network of relative principal curvature lines of a smooth surface. One reason for this is the following: The relative normals $\mathbf{v}_i + t\mathbf{s}_i$, $\mathbf{s}_i \in \mathcal{S}$, $\mathbf{v}_i \in \mathcal{M}$, along each mesh row or column of $\mathcal{M}$ form a discrete model of a developable surface (since normals at consecutive vertices are coplanar; see [21]).

Mesh parallelism in isotropic space. Without explicitly mentioning it, we have described isotropic surface theory in terms of relative differential geometry with the paraboloid Σ of (5) as r-unit sphere. The presented concepts are ideally suited to study principal meshes in I^3 (*i-principal meshes*). Before doing this, we address some consequences of metric duality in I^3 onto mesh parallelism and principal meshes in I^3.

Consider a mesh $\mathcal{M}$ and a parallel mesh $\mathcal{S}$ approximating the isotropic sphere Σ. Corresponding *face planes are parallel* and thus we call $\mathcal{M}$ and $\mathcal{S}$ *f-parallel*.

Let us now apply metric duality, namely the polarity with respect to Σ, to an f-parallel mesh pair $(\mathcal{M}, \mathcal{S})$. Corresponding vertices $\mathbf{m}_i^*, \mathbf{s}_i^*$ of the dual meshes $\mathcal{M}^*, \mathcal{S}^*$ are the poles of corresponding (parallel) face planes M_i, S_i of $\mathcal{M}, \mathcal{S}$. Thus, corresponding vertices $\mathbf{m}_i^*, \mathbf{s}_i^*$ are parallel points, i.e., their top views agree; we call $\mathcal{M}^*, \mathcal{S}^*$ *v-parallel*. The v-parallel pair $(\mathcal{M}^*, \mathcal{S}^*)$ has identical top views and the combinatorics is dual to that of $(\mathcal{M}, \mathcal{S})$. However, corresponding face planes of $(\mathcal{M}^*, \mathcal{S}^*)$ are no longer parallel. Clearly, the meshes which are v-parallel to $\mathcal{M}^*$ form a linear space.

Both $\mathcal{S}$ and $\mathcal{S}^*$ approximate Σ; if one mesh is inscribed, the other is circumscribed, whereas being midscribed remains unchanged under duality. By Def. 1 we can say: *If $\mathcal{M}$ and $\mathcal{S}$ are f-parallel quad meshes and $\mathcal{S}$ approximates the i-unit sphere Σ, then $\mathcal{M}$ is an i-principal mesh.*

In the smooth setting, metric duality preserves i-principal curvature lines. Thus we can also use the dual approach: *An i-principal mesh $\mathcal{M}$ is v-parallel to a mesh $\mathcal{S}$ approximating Σ.* Now viewing I^3 as isotropic model of Laguerre geometry, we may perform the change to the standard model in E^3. This is a geometric transformation τ which maps $\mathcal{M} \subset I^3$ to a mesh $\mathcal{N} = \tau(\mathcal{M}) \subset E^3$.

Corresponding parallel vertices of $\mathcal{M}$ and $\mathcal{S}$ are transformed into corresponding parallel face planes of $\mathcal{N}$ and $\mathcal{T} = \tau(\mathcal{S})$. Since $\mathcal{T}$ is an approximation of the Euclidean sphere $\tau(\Sigma)$ and f-parallel to $\mathcal{N}$, the mesh $\mathcal{N}$ is a Euclidean principal mesh. More precise versions of this general result will be given below.

3.2 Conical and circular meshes in isotropic geometry

In this section we describe two main classes of principal meshes in I^3 and derive properties which are fundamental for their design and computation. These meshes contain aesthetically pleasing ones, which will be demonstrated in section 3.3. We even expect structural advantages, which is a topic of ongoing research.

An *i-conical mesh* $\mathcal{M} \in I^3$ is parallel to a quad mesh $\mathcal{S}$ whose planar faces are tangent to an isotropic sphere Σ of parabolic type such as (5). All tangent planes of Σ which pass through a vertex $\mathbf{s}_i \in \mathcal{S}$ envelope an *isotropic cone of revolution* Δ_i and thus the face planes of $\mathcal{S}$ which meet at $\mathbf{s}_i$ are tangent to Δ_i. By parallelity of corresponding faces in $\mathcal{S}$ and $\mathcal{M}$ also the faces of $\mathcal{M}$ which meet at the corresponding vertex $\mathbf{v}_i \in \mathcal{M}$ are tangent to an isotropic cone of revolution Γ_i. The cone Γ_i arises from Δ_i by the translation $\mathbf{s}_i \mapsto \mathbf{v}_i$.

Let us now apply duality to the f-parallel pair $(\mathcal{M}, \mathcal{S})$. As discussed above, we obtain a v-parallel pair $(\mathcal{M}^*, \mathcal{S}^*)$, where $\mathcal{S}^*$ has its vertices $\mathbf{s}_i^*$ on Σ; $\mathbf{s}_i^*$ are the points of tangency between the face planes of $\mathcal{S}$ and Σ. Hence, any quad Q^* in $\mathcal{S}^*$ has an isotropic circum-circle (intersection of Q^*'s plane and Σ). This means that the top view of each face Q^*, which agrees with the top view of the corresponding face of $\mathcal{M}^*$, has a Euclidean circum-circle. Thus also all faces in $\mathcal{M}^*$ have an isotropic circum-circle and we have to call $\mathcal{M}^* \in I^3$ an *i-circular mesh*. We see that in I^3 conical and circular meshes are dual to each other. Recall that regularity of surfaces is not preserved under duality. Thus duality is practically only useful if $\mathcal{S}$ is regular (without over-foldings), which implies that $\mathcal{M}$ and $\mathcal{M}^*$ approximate surfaces without parabolic surface points.

Let us now show that any i-circular mesh $\mathcal{M}$ is not only v-parallel to an inscribed mesh $\mathcal{S}$ of Σ, but also f-parallel to another inscribed mesh $\mathcal{S}_1$ of Σ. We pick an arbitrary vertex $\mathbf{v}_i \in \mathcal{M}$ and define a point $\mathbf{s}_i \in \Sigma$ as corresponding vertex of $\mathcal{S}_1$. By parallelism of corresponding edges we can now uniquely construct $\mathcal{S}_1$. As in Euclidean geometry, there are infinitely many parallel meshes $\mathcal{S}_1$ of $\mathcal{M}$ which are inscribed to Σ. Conversely, any mesh $\mathcal{M}$ which is parallel to an inscribed mesh $\mathcal{S}_1$ of Σ is circular: If we denote the isotropic angles at the vertices of a quad $Q \in \mathcal{S}_1$ by $\alpha_1, \ldots, \alpha_4$ (these are the Euclidean angles in the top view, Fig. 3, left), being circular means that the sum of opposite angles equals π,

$$\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 = \pi. \tag{13}$$

Since the angles of the quad $F \in \mathcal{M}$ parallel to Q are the same, also F has an isotropic circum-circle and thus $\mathcal{M}$ is i-circular. Let us summarize our results:

Theorem 1. *I-circular meshes discretize the network of isotropic principal curvature lines. An i-circular mesh $\mathcal{M}$ is v-parallel to an inscribed PQ mesh $\mathcal{S}$ of* *the i-unit sphere Σ. Moreover, it is f-parallel to infinitely many inscribed PQ meshes $\mathcal{S}_1$ of Σ. Each quad in $\mathcal{M}$, $\mathcal{S}$ and $\mathcal{S}_1$ has an isotropic circum-circle, i.e., the top views of these meshes are Euclidean circle patterns (Fig. 3, left).*

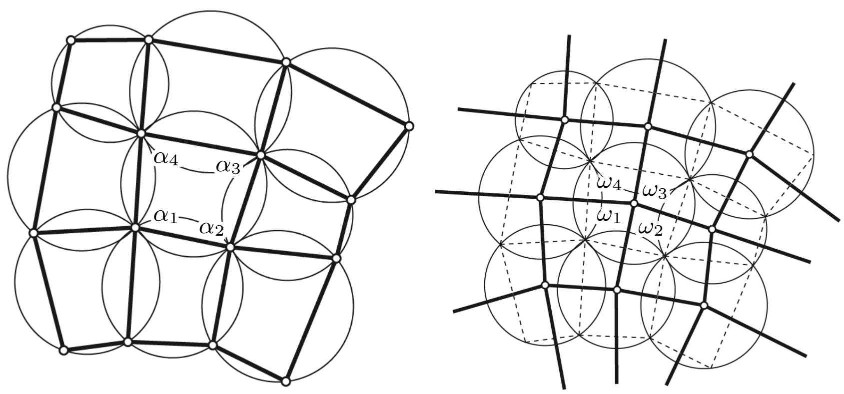

Fig. 3. The top view of an *i*-circular mesh is a Euclidean circle pattern (left). The centers of the circles define the dual mesh, which appears as top view of an *i*-conical mesh (right).

By duality between i-circular and i-conical meshes and between inscribed and circumscribed meshes of Σ we see that any conical mesh M is v-parallel to infinitely many circumscribed meshes $\mathcal{S}_1$ of Σ. Moreover, we note that the pole of a plane Q with respect to Σ has as top view the center of the circle which appears as top view of the i-circle $Q \cap \Sigma$. This proves the following result:

Theorem 2. *I-conical meshes discretize the network of isotropic principal curvature lines. They are dual to i-circular meshes. An i-conical mesh $\mathcal{M}$ is f-parallel to a circumscribed mesh $\mathcal{S}$ of the i-unit sphere Σ. It is also v-parallel to infinitely many circumscribed meshes $\mathcal{S}_1$ of Σ. The top view $\mathcal{M}'$ of $\mathcal{M}$ is a quad mesh, whose vertices are the centers of circles in a pattern and thus $\mathcal{M}'$ is a special Voronoi diagram (Fig. 3, right).*

By duality, an *i*-conical mesh has at each vertex the angle balance condition

$$\omega_1 + \omega_3 = \omega_2 + \omega_4 = \pi. \tag{14}$$

These are isotropic angles and may be measured as Euclidean angles in the top view (cf. Fig. 3, right). (14) also follows from the fact that the vertices of $\mathcal{M}'$ are centers of circles in a pattern.

Remark 1. I-circular meshes are closely related to conical meshes of Euclidean geometry. The change τ from the isotropic model of Laguerre geometry to the standard model maps an i-circular mesh $\mathcal{M}$ to a conical mesh $\mathcal{N} = \tau(\mathcal{M})$ in

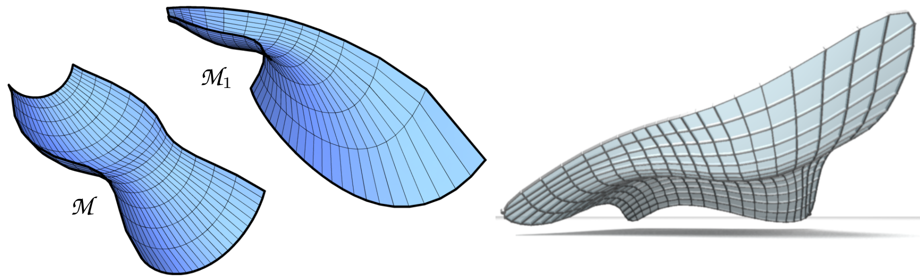

Fig. 4. The i-circular mesh $\mathcal{M}$ (left) has been obtained by combined Catmull-Clark subdivision and optimization. The image $\mathcal{M}_1$ of $\mathcal{M}$ under an isotropic Möbius transformation is again an i-circular mesh. The mesh on the right hand side has been obtained by subdivision and optimization towards an i-conical mesh.

E^3. This follows immediately from the fact that the i-circum-circles of $\mathcal{M}$'s faces are mapped to the tangent cones of the face planes meeting at $\mathcal{N}$'s vertices. Any isotropic Möbius transformation maps an i-circular mesh $\mathcal{M}$ (seen as set of vertices) to another i-circular mesh $\mathcal{M}_1$ (Fig. 4). The transformation under τ yields the known invariance of Euclidean conical meshes (viewed as sets of oriented face planes) under Laguerre transformations.

3.3 Optimization algorithms

The computation of i-circular and i-conical meshes can be performed with algorithms from Liu et al. [13]. We just have to use conditions (13) and (14), respectively, as constraints. A simple way for designing such meshes is to alternate between subdivision and optimization (Figs. 4 and 5). It should be noted that optimization, even if this seems to be the case if we read the description of algorithms above, does not magically convert any given mesh into an i-circular or i-conical mesh: Only initial meshes which do not deviate too much from one with the desired properties will behave in a controlled and expected way. Since it is easier to define a coarse mesh which has nearly planar faces and further desired properties (e.g. being close to a discretely i-orthogonal mesh), the combination with subdivision usually works quite well. Although planarity of faces and other constraints are in general destroyed by any linear subdivision method, this can be 'repaired' in the optimization phase.

3.4 Isothermic meshes and their dual counterparts in I^3

A further remarkable type of i-principal meshes $\mathcal{M}$ are those which are v-parallel to a planar quad mesh $\mathcal{S}$ all of whose edges touch the i-sphere Σ. Each face plane of $\mathcal{S}$ intersects Σ in an isotropic circle which is tangent to the edges of that face. If we transform Σ into a Euclidean sphere with an appropriate projective map , $\mathcal{S}$ is mapped to a Koebe polyhedron [34]. Thus, we may call $\mathcal{S}$, which is midscribed to Σ, an *isotropic Koebe polyhedron*. The top view of the isotropic

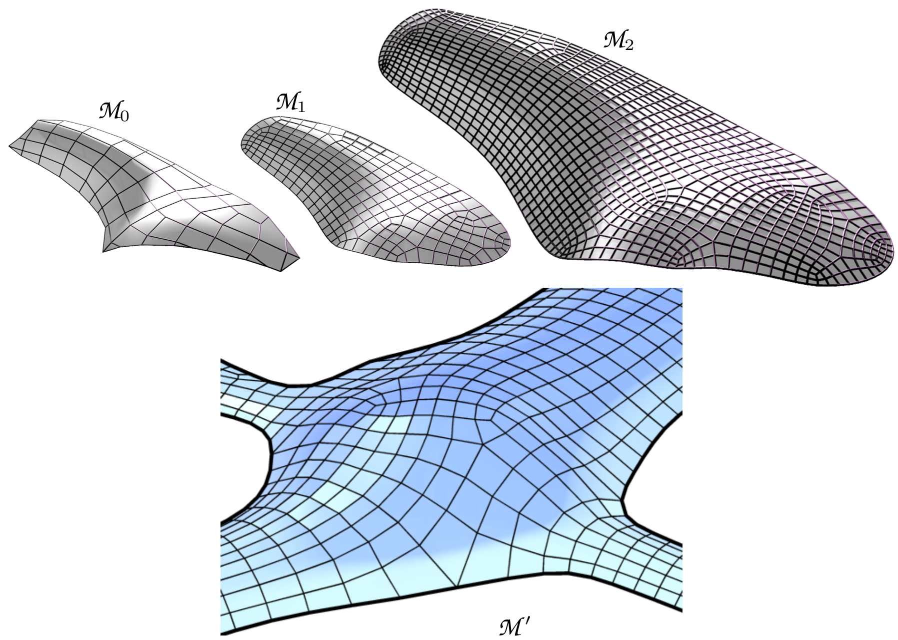

Fig. 5. Meshes $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_2$ (enlarged) have been generated by repeated subdivision and i-conical optimization. The same holds for the i-conical mesh $\mathcal{M}'$, but here the side conditions that $\mathcal{M}'$ approximates a given design plus its boundary resulted in a loss of smoothness.

Koebe polyhedron is a quad mesh with the following properties: (i) each face has an inscribed circle, and (ii) per edge there is only one point of tangency with the inscribed circles of the adjacent faces. This shows the existence of an orthogonal circle packing (Fig. 6). The arising orthogonal circle patterns have been first discussed by Schramm [26]; their computation from given combinatorics amounts to the minimization of a convex function [6]. We call $\mathcal{M}$ an *i-isothermic mesh*; it is an isotropic counterpart of the S-isothermic meshes by Bobenko at al. [2]. As in the Euclidean case, *these meshes are not suitable for approximating arbitrary surfaces.* They discretize isothermic surfaces of I^3. In section 4 we will investigate a particularly beautiful class of i-isothermic meshes, namely discrete versions of i-minimal surfaces.

Theorem 3. *I-isothermic meshes discretize the network of isotropic principal curvature lines on isothermic surfaces in* I^3*. An i-isothermic mesh is v-parallel to an isotropic Koebe polyhedron and has a top view mesh whose faces have inscribed circles which belong to a Schramm circle pattern (Fig. 6).*

Remark 2. If we view an i-isothermic mesh $\mathcal{M}$ as set of i-circles inscribed to its faces, an i-Möbius transformation maps $\mathcal{M}$ to another i-isothermic mesh. Moreover, if we map the set of inscribed i-circles via the transfer τ to the standard model of Laguerre geometry in E^3, we obtain a set of right circular cones Γ_j,

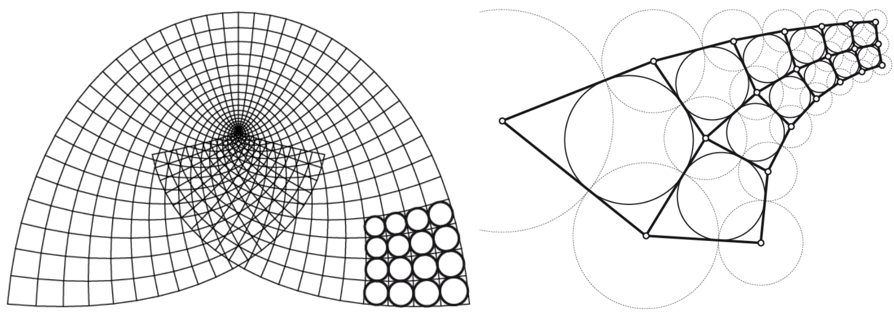

Fig. 6. The top view of an i-isothermic mesh is a quad mesh whose faces have inscribed circles. These circles are part of a circle packing (left). Quadruples of points of tangency are co-circular; these orthogonal circles form a second circle packing (right).

whose vertices $\mathbf{g}_j$ form a quad mesh $\mathcal{N}$ with planar faces and the edge offset property (for direct constructions of the meshes $\mathcal{N}$, see [21]). The cone Γ_j with vertex $\mathbf{g}_j$ contains the 4 edges emanating from $\mathbf{v}_j$. The invariance of i-isothermicity under i-Möbius transformations proves the invariance of edge offset quad meshes $\mathcal{N}$ under Laguerre transformations if $\mathcal{N}$ is viewed as set of vertex cones Γ_j.

4 Isotropic curvatures in meshes with planar faces

The aesthetics of meshes is essential for artistic applications such as architecture. Aesthetics is closely related to the curvature behavior and thus it is important to have an adapted discrete curvature theory at our disposal. This is the topic of the present section. We will illustrate the results with discrete minimal surfaces, i.e., surfaces with vanishing discrete i-mean curvature.

We consider a mesh $\mathcal{M} \in I^3$ with planar faces, not necessarily a quad mesh, and an f-parallel mesh $\mathcal{S}$ which approximates Σ. Thus $\mathcal{S}$ may be viewed as *i-Gauss image* $\sigma(\mathcal{M})$ of $\mathcal{M}$. For such a situation, an adapted discrete curvature theory has been developed recently [21]. We briefly describe this theory, using the simplifications which arise in I^3, and later present a dual counterpart which is more specific to isotropic geometry.

I-curvatures at faces. Let F be a face of $\mathcal{M}$ with vertices $\mathbf{v}_0, \ldots, \mathbf{v}_{k-1}$. The corresponding parallel face $S = \sigma(F)$ of $\mathcal{S}$ shall have vertices $\mathbf{s}_0, \ldots, \mathbf{s}_{k-1}$. The isotropic area of a domain can be measured as usual area in the top view. Let $\mathbf{v}'_i, \mathbf{s}'_i \in \mathbb{R}^2$ denote the top views of the involved vertices. The signed areas of the top views of F and S (isotropic areas of F and S) can be computed as

$$\text{area}(F') = \frac{1}{2} \sum_{j=0}^{k-1} \det(\mathbf{v}'_j, \mathbf{v}'_{j+1}), \ \text{area}(S') = \frac{1}{2} \sum_{j=0}^{k-1} \det(\mathbf{s}'_j, \mathbf{s}'_{j+1}), \qquad (15)$$

with indices modulo k. We also need the so-called *mixed area* of the two polygons F' and S', which is a well known concept in convex geometry [25],

$$\operatorname{area}(F', S') := \frac{1}{4} \sum_{j=0}^{k-1} [\det(\mathbf{v}'_j, \mathbf{s}'_{j+1}) + \det(\mathbf{s}'_j, \mathbf{v}'_{j+1})]. \tag{16}$$

I-curvature K and i-mean curvature H of mesh $\mathcal{M}$ at face F are then defined as

$$K(F) = \frac{\operatorname{area}(S')}{\operatorname{area}(F')}, \quad H(F) = \frac{\operatorname{area}(F', S')}{\operatorname{area}(F')}. \tag{17}$$

It has been shown in [21] that these definitions of curvatures in many aspects behave as their counterparts in the smooth setting. An example is the following: If F, S or any linear combination of them is convex, one can prove $H(F)^2 - K(F) \geq 0$ and thus compute principal curvatures $\kappa_1(F), \kappa_2(F)$ which satisfy $H(F) = [\kappa_1(F) + \kappa_2(F)]/2$ and $K(F) = \kappa_1(F)\kappa_2(F)$. A further example is the curvature behavior of the offset meshes $\mathcal{M}^d = \mathcal{M} + d\mathcal{S}$,

$$K(F^d) = \frac{K(F)}{1 + 2dH(F) + d^2K(F)}, \; H(F^d) = \frac{H(F) + dK(F)}{1 + 2dH(F) + d^2K(F)}. \tag{18}$$

This includes the following discrete isotropic version of a known result on surfaces of constant mean curvature in Euclidean geometry: If $\mathcal{M}$ has constant isotropic mean curvature $H \neq 0$, then its offset at distance $d = -1/H$ has constant i-mean curvature $-H$ and the offset at distance $d = -1/(2H)$ has constant i-Gaussian curvature $K = 4H^2$.

Remark 3. We have adopted common sign conventions in isotropic geometry which differ from the Euclidean ones at some places. This is seen in the last remark on offsets, in the sign of H, and in the fact that we defined the derivative (not the negative derivative) of the i-Gauss mapping as shape operator.

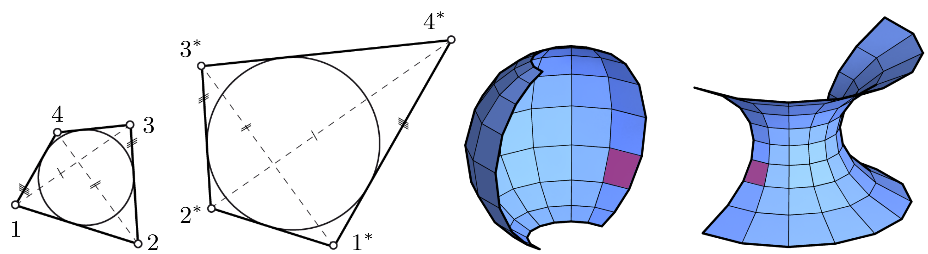

Fig. 7. Left: Two parallel quads have vanishing mixed area if they possess opposite orientation and parallel diagonals. If one of the two quads possesses an inscribed circle, so does the other. Right: Christoffel dual meshes are parallel meshes where all pairs of corresponding faces have vanishing mixed area.

I-minimal surfaces and Christoffel duality. Discrete minimal surfaces in I^3 are meshes with $H = 0$, i.e. with

$$\text{area}(F, S) = 0, \tag{19}$$

for all parallel face pairs (F, S), which is equivalent to $\text{area}(F', S') = 0$. According to (9), they discretize graph surfaces of harmonic functions. A simple characterization of parallel quads $F = 1234$ and $S = 1^*2^*3^*4^*$ with vanishing mixed area has been presented in [21],

$$\text{area}(F, S) = 0 \iff 13 \text{ parallel } 2^*4^*,\ 24 \text{ parallel } 1^*3^*. \tag{20}$$

This is illustrated in Fig. 7, left. Note that the two quads differ in their orientation. If F has an inscribed circle, so does S (Fig. 7, left). It may seem easy now to construct isotropic minimal surfaces. However, starting from a mesh $\mathcal{S}$ which approximates Σ, it will in general not be possible to derive a minimal mesh $\mathcal{M}$ by a face-wise construction from the faces $S \in \mathcal{S}$, since the construction of the four faces around a vertex does not close. If the construction is possible in a consistent way, one calls the pair $(\mathcal{M}, \mathcal{S})$ a *Christoffel-dual pair* (Fig. 7, right). Christoffel duality should not be mixed up with metric duality.

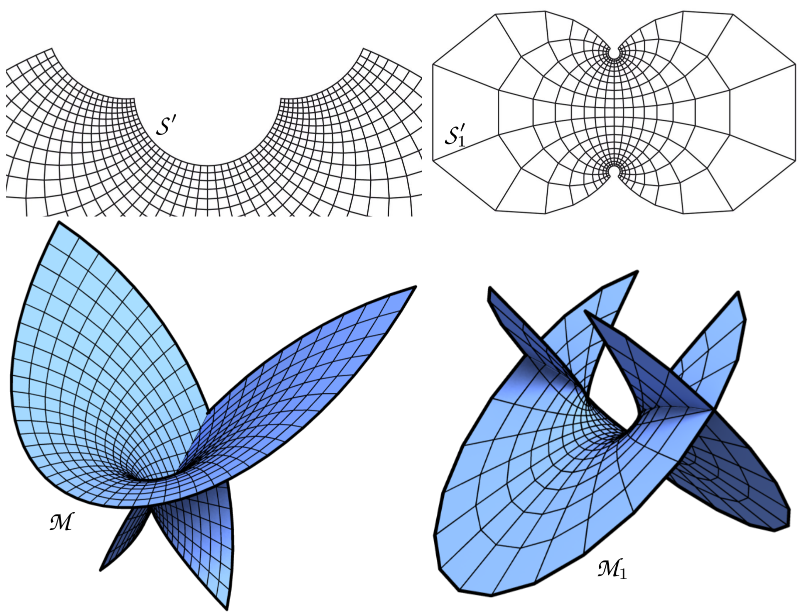

Fig. 8. Discrete i-minimal surfaces represented by i-isothermic meshes: isotropic Enneper surface $\mathcal{M}$ and isotropic counterpart $\mathcal{M}_1$ to Bonnet's minimal surface; the top views of the corresponding isotropic Koebe meshes are $\mathcal{S}', \mathcal{S}'_1$. In meshes $\mathcal{S}, \mathcal{M}, \mathcal{S}_1, \mathcal{M}_1$ all row and column polygons are planar.

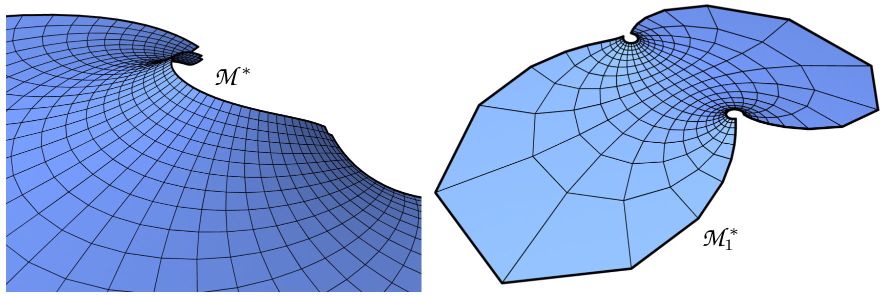

Fig. 9. Discrete i-minimal surfaces which are dual to the surfaces from Fig. 8.

We have already presented meshes to which Christoffel duality can be applied. These are the i-isothermic meshes, in particular isotropic Koebe meshes. *The Christoffel dual of an isotropic Koebe mesh is a discrete i-minimal surface, and a special i-isothermic mesh.* This is the isotropic counterpart to the discrete Euclidean minimal surfaces of Bobenko et al. [2].

Figures 8 and 9 show some examples which have been generated as follows: Apply an inversion to the planar regular square grid (viewed as set of circles inscribed to the grid cells) to get a circle packing and a Koebe mesh $\mathcal{S}'$ in the plane. This is a discrete version of two orthogonal pencils of circles. Then lift this mesh to the paraboloid Σ such that one obtains an i-Koebe mesh $\mathcal{S}$; the edges of $\mathcal{S}$ have to touch Σ. The Christoffel dual $\mathcal{M}$ of $\mathcal{S}$ is a discrete isotropic Enneper surface, whose smooth analogue has been studied by Strubecker [31]. Just lifting the square grid to Σ and then computing the Christoffel dual yields a discrete right hyperbolic paraboloid, the simplest i-minimal surface. Computing an i-Koebe mesh $\mathcal{S}_r$ with rotational symmetry and dualizing it, one obtains a discrete rotational i-minimal surface. Its smooth counterpart is the so-called logarithmoid, obtained by rotating the curve $z = \ln x$ of the xz-plane around the z-axis (for a hexagonal version, see Fig. 10, left). Finally, we may apply an isotropic Möbius transformation to $\mathcal{S}_r$ and obtain an i-Koebe mesh $\mathcal{S}_1$ whose top view $\mathcal{S}_1'$ discretizes two orthogonal pencils of circles through two points. The resulting i-minimal surface $\mathcal{M}_1$ is a discrete isotropic counterpart to Bonnet's minimal surface. The i-minimal meshes of this paragraph are exactly those which possess *planar discrete principal curvature lines* (row and column polygons).

Surfaces $\mathcal{M}^*$ and $\mathcal{M}_1^*$ which correspond to $\mathcal{M}$ and $\mathcal{M}_1$, resp., via metric duality are illustrated in Fig. 9. These surfaces have the property that the planes of all faces along the same row or column of the mesh pass through a fixed point (which may be at infinity).

The concept is not limited to quad meshes. We may also construct hexagonal meshes which are discrete versions of minimal surfaces, starting from hexagonal Koebe polyhedra. In the example of Fig. 10, left, the realization of $\text{area}(F, S) = 0$ is simple due to rotational symmetry. It allows us to split each hexagon by the symmetry axis into two quads and compute the Christoffel dual. Fig. 10 also shows rotational hexagonal meshes whose discrete curvature measures H or K are constant; their construction is based on formula (17) and the exploitation of symmetry.

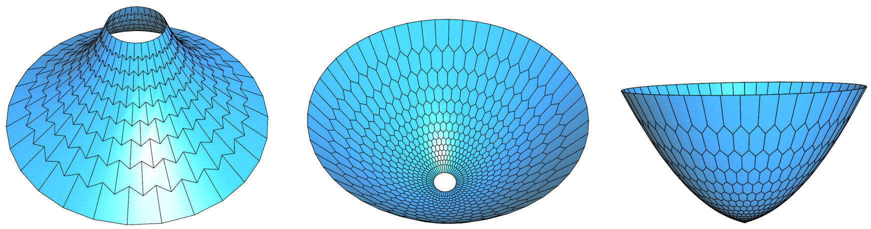

Fig. 10. Hexagonal meshes with planar faces and rotational symmetry, representing an i-minimal surface (left), a surface with constant i-mean curvature H (middle) and a surface with constant i-curvature K (right).

Remark 4. Application of an i-M-inversion κ to the inscribed circles of the faces in an i-minimal mesh $\mathcal{M}$ results in a collection of inscribed circles of the faces of another mesh $\kappa(\mathcal{M})$. The latter is no longer minimal, but can be shown to discretize the graph $z = f(x, y)$ of a biharmonic function ($\Delta^2 f = 0$), also known as an Airy surface [30]. By mapping $\mathcal{M}$ or $\kappa(\mathcal{M})$ back to the standard model of Laguerre geometry in the way explained in Remark 2, one obtains discrete versions of so-called Laguerre minimal surfaces (of the spherical type), which have been investigated by Blaschke [1]. Since the presented i-minimal meshes $\mathcal{M}$ are i-isothermic, the resulting Laguerre-minimal meshes $\mathcal{N}$ possess the edge offset property (cf. Remark 2). There are further types of Laguerre-minimal edge offset meshes, all of which can be constructed from Koebe meshes. These remarkable discrete surfaces will be the subject of a forthcoming publication.

I-curvatures at vertices. Metric duality in I^3 allows us to define discrete curvatures at vertices as well. Dual to the points of a domain F in a plane P is a set F^* of planes through a point P^*. By duality, the *isotropic measure (density)* $D(F^*)$ *of the plane set* F^* in the sense of integral geometry [23] equals the isotropic area of F (area(F') of the top view F'). Let $P_0, \ldots, P_{k-1}$ be the face planes of a mesh $\mathcal{M}$ around vertex $\mathbf{v}$. Each plane P_j can be written as $z = p_{j,1}x + p_{j,2}y + p_{j,3}$ and we set $\mathbf{p}_j := (p_{j,1}, p_{j,2})$. Then, the measure $D(\mathbf{v})$ of the "vertex planes around $\mathbf{v}$" (which are dual to the points in the face $\mathbf{v}^*$ of the dual mesh $\mathcal{M}^*$) is computed as the areas in (15),

$$D(\mathbf{v}) = \frac{1}{2} \sum_{j=0}^{k-1} \det(\mathbf{p}_j, \mathbf{p}_{j+1}). \tag{21}$$

The rest follows easily: We consider a v-parallel mesh $\mathcal{S}$ of $\mathcal{M}$ which approximates Σ. Let $\mathbf{s} \in \mathcal{S}$ be the vertex corresponding to $\mathbf{v} \in \mathcal{M}$ (i.e., $\mathbf{s}' = \mathbf{v}'$). The planes through $\mathbf{s}$ possess a density measure $D(\mathbf{s}) = (1/2)\sum \det(\mathbf{q}_j, \mathbf{q}_{j+1})$ and analogous to (16) we define the *mixed measure* $D(\mathbf{v}, \mathbf{s}) = (1/4)\sum[\det(\mathbf{p}_j, \mathbf{q}_{j+1}) + \det(\mathbf{q}_j, \mathbf{p}_{j+1})]$. In view of equations (11) and (17) *curvatures of* $\mathcal{M}$ *at vertex* $\mathbf{v}$ *are defined by*

$$K(\mathbf{v}) = \frac{D(\mathbf{v})}{D(\mathbf{s})}, \quad H(\mathbf{v}) = \frac{D(\mathbf{v}, \mathbf{s})}{D(\mathbf{s})}. \tag{22}$$

Discrete face curvatures do not make much sense for triangle meshes $\mathcal{M}$ since f-parallel triangle meshes are similar and thus one cannot find an appropriate Gaussian image $\mathcal{S}$ unless $\mathcal{M}$ itself approximates Σ. However, vertex curvatures work for triangle meshes as well (see Fig. 11). In fact, here it is particularly easy to get $\mathcal{S}$: We project the vertices of $\mathcal{M}$ in isotropic z-direction onto Σ. The relation between face curvatures, vertex curvatures and curvatures associated with edges of principal meshes will be described in a separate publication.

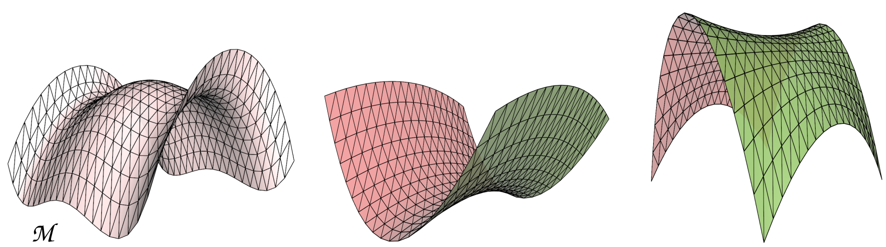

Fig. 11. Diagram surfaces for vertex curvatures H (middle) and K (right), computed with formula (22) for the triangle mesh $\mathcal{M}$, whose vertices lie on a given analytic surface $z = f(x, y)$. The curvatures match the true values with high accuracy.

Conclusion and future research. The main contribution of the present paper is the geometric discussion of remarkable new types of meshes. For applications, i-minimal surfaces are certainly interesting. They deserve a much more detailed investigation and the derivation of tools such as the computation of a discrete i-minimal surface through a prescribed closed boundary.

Discrete differential geometry in its interplay with architecture is a wide area for future work. We only address a few topics related to isotropic geometry: Due to their optimal node properties, we need to know more about the possible shapes of quadrilateral edge offsets meshes and the related i-isothermic meshes. A careful investigation of smooth and discrete Laguerre-minimal surfaces and their suitability for architecture is also missing. Discrete surfaces in I^3 with constant H or K are also of interest.

Probably the most important research task towards the implementation of freeform geometry in architecture is to find new ways of approximating a given freeform shape by meshes with properties that are desirable for architecture (for a result obtained by currently available methods, see Fig. 5, bottom).

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