UE Discrete Mathematics

Exercises for Jan 9, 2024

111) Let $(R, +, \cdot)$ be an integral domain. Show that $x \in R$ is a unit if and only if it is a divisor of every $a \in R$.

112) Let $(R, +, \cdot)$ be a Euclidean ring and let its Euclidean function be denoted by n. Show that $n(x) = n(1)$ for all units x of R.
Prove moreover that, if $x, y \in R$ and y is a unit, then $n(xy) = n(x)$.

113) List all irreducible polynomials up to degree 3 over $\mathbb{Z}_3$.

114) Decompose $x^5 + x^4 + 1$ into irreducible factors over $\mathbb{Z}_2$.

115) Let K be a field and $p(x) \in K[x]$ a polynomial of degree m. Prove that $p(x)$ cannot have more than m zeros in K (counted with multiplicities).

Hint: Use the fact that $K[x]$ is a factorial ring.

116) Consider the ring $R[[x]]$ of formal power series with coefficients in some integral domain R. Set $I = \left\{\sum_{n\geq 0} a_n z^n \mid \forall i \in \mathbb{N} : a_i \in R \text{ and } a_0 = 0\right\}$. Show that I is an ideal of $R[[x]]$.

117) Let R be a (not necessarily commutative) ring and $A \subseteq R$. Furthermore, let $\mathcal{I}(A)$ denote the set of all ideals of R that contain A as a subset. Prove: $J := \bigcap_{I\in\mathcal{I}(A)} I$ is the smallest ideal of R with $A \subseteq J$.

118) Let $\varphi : R_1 \to R_2$ be a ring homomorphism and I be an ideal of R_2. Prove that $\varphi^{-1}(I) := \{x \in R_1 \mid \varphi(x) \in I\}$ is an ideal of R_1.

119) Let R be a ring and I one of its ideals. Define the relation $\sim_I$ on R by $a \sim_I b :\Leftrightarrow a - b \in I$. Show that $\sim_I$ is an equivalence relation. Let $[x]$ denote the equivalence class of x with respect to $\sim_I$. Prove that for all $x \in R$ we have $[x] = x + I$.

120) Let R be a ring and I be an ideal of R. Then $(R/I, +)$ is the quotient group of $(R, +)$ over $(I, +)$. Define a multiplication on R/I by

$$(a + I) \cdot (b + I) := (ab) + I.$$

Prove that this operation is well-defined, *i.e.* that

$$\left.\begin{array}{rr} & a + I = c + I \\ \text{and} & b + I = d + I \end{array}\right\} \Longrightarrow (ab) + I = (cd) + I.$$

Furthermore, show that $(R/I, +, \cdot)$ is a ring.