UE Discrete Mathematics

Exercises for Nov 21, 2023

61) Prove that for all complex numbers x and all $k \in \mathbb{N}$ we have

$$\binom{-x}{k} = (-1)^k \binom{x+k-1}{k}.$$

62) Prove the following identity:

$$x^n = \sum_{k=0}^{n} S_{n,k} (x)_k \qquad (n \geq 0).$$

63) Let A, B be two finite sets with $|A| = n$ and $|B| = k$. How many injective mappings $f : A \to B$ are there? Furthermore, show that the number of surjective mappings $f : A \to B$ equals $k! S_{n,k}$.

64) The n-th Bell number equals the number of set partitions of $\{1, 2, \ldots, n\}$. We set $B_0 := 1$. Prove the following identities:

$$B_n = \sum_{k=0}^{n} S_{n,k} \qquad \text{and} \qquad B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

65) Prove that the squares of the Fibonacci number satisfy the recurrence relation $a_{n+3} - 2a_{n+2} - 2a_{n+1} + a_n = 0$ and solve this recurrence relation with the correct initial conditions.

66) Let a_n denote the number of fat subsets of $\{1, 2, \ldots, n\}$ where a set A is called *fat* if $A = \emptyset$ or $\forall k \in A : k \geq |A|$. Prove that $a_n = F_{n+2}$ (as usual $(F_n)_{n \geq 0}$ denotes the sequence of the Fibonacci numbers) and show that this implies

$$F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k}.$$

67) Solve the following recurrence using generating functions: $a_{n+1} = 3a_n - 2$ for $n \geq 0$, $a_0 = 2$.

68) Solve the following recurrence using generating functions: $a_{n+1} = a_n + (n+1)^2$ for $n \geq 0$, $a_0 = 1$.

69) Solve the following recurrence using generating functions: $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$ with $a_0 = 1$, $a_1 = -2$.

70) Use generating functions to find a closed form expressions for the sum $\sum_{k=0}^{n} (k^2 + 3k + 2)$.