UE Discrete Mathematics

Exercises for Dec 5, 2023

81) An involution is a permutation π such that $\pi \circ \pi = \mathrm{id}_M$ where $M = \{1, 2, \ldots, n\}$. Let $\mathcal{I}$ be the set of involutions. Give a specification of $\mathcal{I}$ as a combinatorial construction and use this to determine the exponential generating function $I(z)$ of $\mathcal{I}$.

82) Let $\mathcal{T}$ be the class of rooted and labelled trees, i.e. the n vertices of a tree of size n are labelled with the labels $1, 2, \ldots, n$. Use the theory of combinatorial constructions to determine a functional equation for the exponential generating function of $\mathcal{T}$. Finally, apply the following theorem to prove that the number of trees in $\mathcal{T}$ which have n vertices is equal to n^{n-1}. (You are not asked to prove the theorem.)

Theorem. *Let* $\Phi(w) = \sum_{n\geq 0} \phi_n w^n$ *with* $\phi_0 \neq 0$. *If* $z = w/\Phi(w)$, *then* $[z^n]w = \frac{1}{n}[w^{n-1}]\Phi(w)^n$.

83) Let P be the set of all divisors of 12. Determine the Möbius function of $(P, |)$ using the definition of the Möbius function and compare your result with the one from the last example in the lecture.

84) Let $(P, \leq)$ be the poset defined by $P = \{0, 1, 2, 3, 4\}$ and $0 \leq 1 \leq 4,\ 0 \leq 2 \leq 4,\ 0 \leq 3 \leq 4$. Compute all values $\mu(x, y)$ for $x, y \in P$.

85) Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two locally finite posets with 0-element and $(P, \leq)$ be defined by $P = P_1 \times P_2$ and for $(a, x), (b, y) \in P$:

$$(a, x) \leq (b, y) :\Longleftrightarrow a \leq_1 b \wedge x \leq_2 y.$$

Show that $(P, \leq)$ is a locally finite poset with 0-element.

86) Draw the Hasse diagram of $(2^{\{1,2,3\}}, \supseteq)$ and redo the proof of the principle of inclusion and exclusion for the special case of three sets $A_1, A_2, A_3 \subseteq M$. Carry out every step in detail.

87) Let p, q, r be three distinct prime numbers and $m = pqr$. How many of the numbers $1, 2, \ldots, m$ are relatively prime to m? (Two numbers x and y are called relatively prime if their greatest common divisor is 1.)

88) Given an alphabet of size four, how many ways are there to create a password of 20 characters, if it is required that every letter of the alphabet must occur?

89) Let a_n denote the number of permutations $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ that have no fixed point. Use the inclusion-exclusion principle to show that

$$a_n = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

90) Let $\mathcal{A}_n$ denote the set of the permutations considered in Exercise 89 and set $\mathcal{A} = \{\varepsilon\} \cup \bigcup_{n\geq 1} \mathcal{A}_n$, where ε is the zero-sized combinatorial object and $\pi \in \mathcal{A}$ has size n if and only if $\pi \in \mathcal{A}_n$. Specify $\mathcal{A}$ as a combinatorial construction, use that specification to determine its generating function and rederive the formula from Exercise 89