Analytic Combinatorics: Complex-analytic Methods and Applications

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Hayman admissible functions

$$y_n = [x^n]y(x) = \frac{1}{2\pi i} \int_{|x|=r} \frac{y(x)}{x^{n+1}}\, dx$$

If $y(x)x^{-n}$ is concentrated at a saddle point ζ_n:

$$y_n \sim \frac{y(\zeta_n)\zeta_n^{-n}}{\sqrt{2\pi\sigma^2(\zeta_n)}}, \qquad (n \to \infty)$$

where

$$\sigma^2(\zeta) = \left[\frac{\partial^2}{\partial v^2} \log y(\zeta e^v)\right]_{v=0} = \frac{\zeta^2 y''(\zeta)}{y(\zeta)} - \left(\frac{\zeta y'(\zeta)}{y(\zeta)}\right)^2 + \frac{\zeta y'(\zeta)}{y(\zeta)}.$$

Definition (Hayman admissible function)

1. $f(x)$ analytic in $|x| < R$, positive in (R_0, R)
2. $\exists \delta : (R_0, R) \to (0, \pi)$ such that in (R_0, R):

$$f\left(re^{i\theta}\right) \sim f(r) \exp\left(i\theta a(r) - \frac{\theta^2}{2} b(r)\right), r \to R,$$

   uniformly for $|\theta| \leq \delta(r)$, where
   $a(r) = rf'(r)/f(r), b(r) = ra'(r)$
3. In (R_0, R)

$$f\left(re^{i\theta}\right) = o\left(\frac{f(r)}{\sqrt{b(r)}}\right), r \to R,$$

   uniformly for $\delta(r) \leq |\theta| \leq \pi$
4. $b(r) \to \infty$ as $r \to R$.

Theorem (Hayman 1956)

Let $f(x)$ be H-admissible ($\in \mathcal{H}_R$). Then, as $r \to R$, we have

$$f_n = \frac{f(r)}{r^n \sqrt{2\pi b(r)}} \left(\exp\left(-\frac{(a(r)-n)^2}{2b(r)} \right) + o(1) \right),$$

uniformly for all integers n.

Examples: e^x, e^{e^x}, $\exp\left(\frac{1}{1-x}\right)$ are admissible.

Corollary

Let $f(z) \in \mathcal{H}_R$, then

$$f_n \sim \frac{f(\zeta_n)}{\sqrt{2\pi b(\zeta_n)}\zeta_n^n}$$

where ζ_n is defined by $a(\zeta_n) = n$.

Remark: For sufficiently large n, ζ_n is unique.

Proof: Let $\delta = \delta(r)$

$$f_n r^n = \frac{1}{2\pi}\int_{-\delta}^{2\pi-\delta} f(re^{i\theta})e^{-in\theta}\,\mathrm{d}\theta$$
$$= \underbrace{\frac{1}{2\pi}\int_{-\delta}^{\delta} f(re^{i\theta})e^{-in\theta}\,\mathrm{d}\theta}_{I_1} + \underbrace{\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} f(re^{i\theta})e^{-in\theta}\,\mathrm{d}\theta}_{I_2}$$

Clearly, $I_2 = o\left(\frac{f(r)}{\sqrt{b(r)}}\right)$.

$$I_1 = \frac{f(r)}{2\pi}\int_{-\delta}^{\delta} \exp\left(i(a(r)-n)\theta - \frac{b(r)}{2}\theta^2\right)(1+o(1))\,\mathrm{d}\theta$$
$$= \frac{f(r)}{2\pi}\left(\int_{-\delta}^{\delta} \exp\left(i(a(r)-n)\theta - \frac{b(r)}{2}\theta^2\right)\,\mathrm{d}\theta + o\left(\frac{1}{\sqrt{b(r)}}\right)\right)$$

Remaining part:

$$\frac{f(r)}{2\pi}\int_{-\delta}^{\delta}\exp\left(i(a(r)-n)\theta-\frac{b(r)}{2}\theta^2\right)\,\mathrm{d}\theta$$

$$\frac{|f(re^{i\delta})|}{f(r)}\sim e^{-b(r)\delta^2/2}=o\left(\frac{1}{\sqrt{b(r)}}\right)\implies b(r)\delta^2\to\infty$$

Substitute $y=\theta\sqrt{b(r)/2}$:

$$I_1\sim\frac{f(r)}{\pi\sqrt{2b(r)}}\int_{-\delta\sqrt{b(r)/2}}^{\delta\sqrt{b(r)/2}}\exp\left(-y^2+iy\frac{a(r)-n}{\sqrt{b(r)}}\sqrt{2}\right)\,\mathrm{d}y$$

$$\sim\frac{f(r)}{\pi\sqrt{2b(r)}}\underbrace{\int_{-\infty}^{\infty}\exp\left(-y^2+iy\frac{a(r)-n}{\sqrt{b(r)}}\sqrt{2}\right)\,\mathrm{d}y}_{\sqrt{\pi}\exp\left(-(a(r)-n)^2/(2b(r))\right)}\qquad\square$$

Consequences:

- Admissible functions have at least exponential growth, their maximal modulus is on the positive real line.
- "Distribution" of partial sums asymptotically Gaussian, i.e.,

$$\sum_{n \le a(r)+\omega\sqrt{2b(r)}} a_n r^n \sim \frac{f(r)}{\sqrt{\pi}} \int_{-\infty}^{\omega} e^{-t^2}\,\mathrm{d}t.$$

- Growth of derivatives: $f^{(k)}(r) \sim f(r)(a(r)/r)^k$, as $r \to R$.

Theorem (Closure Properties, Hayman 1956)

1. *$p(x)$ real polynomial, $e^{p(x)} = \sum a_n x^n$, almost all a_n positive, then $e^{p(x)} \in \mathcal{H}$*
2. $f_1(x), f_2(x) \in \mathcal{H} \quad \Longrightarrow \quad e^{f_i(x)} \in \mathcal{H}$ *and* $f_1(x) f_2(x) \in \mathcal{H}$
3. *$f(x) \in \mathcal{H}_R$, $p(x)$ polynomial with positive leading coefficient and $p(R) > 0 \implies f(x)p(x) \in \mathcal{H}_R$*
4. *$f(x) \in \mathcal{H}$ and $g(x) = O\left(f^{1-\delta}\right) \implies f(x) + g(x) \in \mathcal{H}$*

Remark: Maple packages by Salvy et al. (Algolib)

Example (Stirling's formula)

Find asymptotic of $n!$:

$$\frac{1}{n!} = [z^n]f(z) = [z^n]e^z$$

$e^z \in \mathcal{H}$,

$$a(r) = \frac{rf'(r)}{f(r)} = r, \qquad b(r) = ra'(r) = r.$$

Saddle point equation: $a(\zeta_n) = n$.
This implies $\zeta_n = n$ and thus

$$f_n = \frac{1}{n!} \sim \frac{1}{\sqrt{2\pi b(\zeta_n)}} \frac{f(\zeta_n)}{\zeta_n^n} = \frac{1}{\sqrt{2\pi n}} \frac{e^n}{n^n}.$$

Example (Bell numbers)

$e^z \in \mathcal{H}$, likewise $e^z - 1 \in \mathcal{H}$ and thus $f(z) = e^{e^z-1} \in \mathcal{H}$.
This function corresponds to $\mathtt{Set}(\mathtt{Set}(\mathcal{Z}) \setminus \{\varepsilon\})$.

We get

$$a(r) = \frac{rf'(r)}{f(r)} = \frac{re^r f(r)}{f(r)} = re^r, \quad b(r) = ra'(r) = r(1+r)e^r,$$

Saddle point equation: $\zeta_n e^{\zeta_n} = n$.
Taking logarithms and then using $\zeta_n = \log n - \log \zeta_n$ gives

$$\zeta_n \sim \log n - \log\log n + o(1).$$

Thus

$$B_n = n![z^n]f(z) \sim \frac{n!}{\sqrt{2\pi\zeta_n^2 e^{\zeta_n}}} \cdot \frac{e^{e^{\zeta_n}-1}}{\zeta_n^n} \sim \frac{n!}{e\sqrt{2\pi n \log n}} \cdot \frac{e^{n/\log n}}{\zeta_n^n}$$

Example (Involutions)

An involution is a permutation with no cycles of length 3 or more. Thus

$$f(z) = e^{z+\frac{z^2}{2}} = \sum_{n\geq 0} f_n \frac{z^n}{n!} \in \mathcal{H}$$

We compute

$$a(r) = r + r^2, \quad b(r) = r + 2r^2, \quad \zeta_n^2 + \zeta_n = n$$

and so $\zeta_n = \sqrt{n} - \frac{1}{2} + \frac{1}{8\sqrt{n}} + O(n^{-3/2})$.

This implies

$$f(\zeta_n) = e^{\zeta_n + \frac{1}{2}\zeta_n^2} = e^{\frac{1}{2}(\zeta_n + n)} = \exp\left(\frac{\sqrt{n}}{2} - \frac{1}{4} + O(n^{-1/2})\right)$$

and $b(\zeta_n) = \zeta_n + 2\zeta_n^2 \sim 2\zeta_n^2 \sim 2n$.

Example (Involutions – cont'd)

$$\begin{aligned}
\zeta_n^n &= \left(\sqrt{n} - \frac{1}{2} + \frac{1}{8\sqrt{n}} + O(n^{-3/2})\right)^n \\
&= n^{n/2}\left(1 - \frac{1}{2\sqrt{n}} + \frac{1}{8n} + O(n^{-2})\right)^n \\
&= n^{n/2}\exp\left(n\left[\left(-\frac{1}{2\sqrt{n}} + \frac{1}{8n}\right) - \frac{1}{2}\left(-\frac{1}{2\sqrt{n}} + \frac{1}{8n}\right)^2 + O\left(\frac{1}{n^{3/2}}\right)\right]\right) \\
&= n^{n/2}\exp\left(-\frac{\sqrt{n}}{2} + O\left(n^{-1/2}\right)\right) \sim n^{n/2}e^{-\sqrt{n}/2}
\end{aligned}$$

Thus

$$\frac{f_n}{n!} \sim \frac{f(\zeta_n)}{\sqrt{2\pi b(\zeta_n)}\zeta_n^n} \sim \frac{\exp\left(\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right)}{2n^{n/2}\sqrt{\pi n}}$$

Multivariate Hayman admissibility

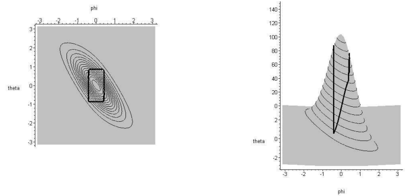
Notations

$\mathbf{x} = (x_1, \ldots, x_d)$, $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_d^{n_d}$
For a function $y(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}^d$:
$\mathbf{a}(\mathbf{x}) = (a_j(\mathbf{x}))_{j=1,\ldots,d}$ vector of the logarithmic (partial) derivatives, i.e.,

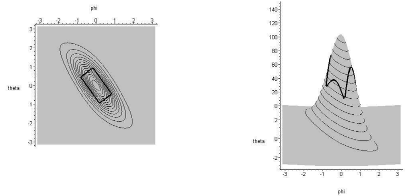
$$a_j(\mathbf{x}) = \frac{x_j y_{x_j}(\mathbf{x})}{y(\mathbf{x})},$$

$B(\mathbf{x}) = (B_{jk}(\mathbf{x}))_{j,k=1,\ldots,d}$ matrix of the second logarithmic (partial) derivatives of $y(\mathbf{x})$, i.e.,

$$B_{jk}(\mathbf{x}) = \frac{x_j x_k y_{x_j x_k}(\mathbf{x}) + \delta_{jk} x_j y_{x_j}(\mathbf{x})}{y(\mathbf{x})} - \frac{x_j x_k y_{x_j}(\mathbf{x}) y_{x_k}(\mathbf{x})}{y(\mathbf{x})^2},$$



The typical shape of $|y(re^{i\varphi}, se^{i\theta})|$ (for fixed r and s)



$B(\mathbf{r})$ positive definite
$\mathbf{v}_1(\mathbf{r}), \ldots, \mathbf{v}_d(\mathbf{r})$ orthonormal basis of eigenvectors

Definition (Multivariate Hayman admissible function)

A function

$$y(\mathbf{x}) = \sum_{n_1,\dots,n_d \geq 0} y_{n_1\cdots n_d} x_1^{n_1} \cdots x_d^{n_d},$$

with real coefficients $y_{n_1\cdots n_d}$ is called **H-admissible in $\mathcal{R} \subset \mathbb{R}^d$** if it is entire and positive in $\mathcal{R}$ (for some fixed $R_0 > 0$) and satisfies

(I) $B(\mathbf{r})$ is positive definite and there exists a function $\boldsymbol{\delta} : \mathbb{R}^d \to [-\pi, \pi]^d$ such that

$$y\left(\mathbf{r}e^{i\theta}\right) \sim y(\mathbf{r}) \exp\left(i\boldsymbol{\theta}\mathbf{a}(\mathbf{r})^t - \frac{\boldsymbol{\theta}B(\mathbf{r})\boldsymbol{\theta}^t}{2}\right), \text{ as } \mathbf{r} \to \infty \text{ in } \mathcal{R},$$

uniformly for

$$\boldsymbol{\theta} \in \Delta(\mathbf{r}) = \{\sum_{j=1}^{d} \mu_j \mathbf{v}_j(\mathbf{r}) \text{ with } |\mu_j| \leq \delta_j(\mathbf{r}), \text{ for } j = 1, \dots, d\}.$$

Definition (Mv Hayman admissible function – cont'd)

(II) The asymptotic relation

$$y\left(\mathbf{r}e^{i\boldsymbol{\theta}}\right) = o\left(\frac{y(\mathbf{r})}{\sqrt{\det B(\mathbf{r})}}\right), \text{ as } \mathbf{r} \to \infty \text{ in } \mathcal{R},$$

holds uniformly for $\boldsymbol{\theta} \notin \Delta(\mathbf{r})$.

(III) The eigenvalues $\lambda_1(\mathbf{r}), \ldots, \lambda_d(\mathbf{r})$ of $B(\mathbf{r})$ satisfy

$$\lambda_i(\mathbf{r}) \to \infty, \text{ as } \mathbf{r} \to \infty \text{ in } \mathcal{R}, \text{ for all } i = 1, \ldots, d.$$

(IV) We have $B_{ii}(\mathbf{r}) = o\left(a_i(\mathbf{r})^2\right)$, as $\mathbf{r} \to \infty$ in $\mathcal{R}$.

(V) For $\mathbf{r}$ sufficiently large and $\boldsymbol{\theta} \in [-\pi, \pi]^d \setminus \{\mathbf{0}\}$:

$$|y(\mathbf{r}e^{i\boldsymbol{\theta}})| < y(\mathbf{r}).$$

Remark: (I)–(III) imply $\|B(\mathbf{r})\| = o(\|\mathbf{a}\,(\mathbf{r})\|^2)$, as $\mathbf{r} \to \infty$ in $\mathcal{R}$.

Theorem

Let $y(\mathbf{x})$ be H-admissible. Then as $\mathbf{r} \to \infty$ we have

$$y_{\mathbf{n}} \sim \frac{y(\mathbf{r})}{\mathbf{r}^{\mathbf{n}}(2\pi)^{d/2}\sqrt{\det B(\mathbf{r})}} \exp\left(-\frac{1}{2}(\mathbf{a}(\mathbf{r})-\mathbf{n})B(\mathbf{r})^{-1}(\mathbf{a}(\mathbf{r})-\mathbf{n})^t\right),$$

uniformly for all $\mathbf{n} \in \mathbb{Z}^d$.

Proof (sketch): Let $\mathcal{E} = \left\{\sum_j \mu_j \mathbf{v}_j \mid |\mu_j| \le \delta_j\right\}$.
Then we have $y_{\mathbf{n}}\mathbf{r}^{\mathbf{n}} = I_1 + I_2$ with

$$I_1 = \frac{1}{(2\pi)^d} \underset{\mathcal{E}}{\int \cdots \int} \frac{y\left(\mathbf{r}e^{i\boldsymbol{\theta}}\right)}{e^{i\mathbf{n}\boldsymbol{\theta}^t}}\, d\theta_1 \cdots d\theta_d$$

and

$$I_2 = \frac{1}{(2\pi)^d} \underset{[-\pi,\pi]^d \setminus \mathcal{E}}{\int \cdots \int} \frac{y\left(\mathbf{r}e^{i\boldsymbol{\theta}}\right)}{e^{i\mathbf{n}\boldsymbol{\theta}^t}}\, d\theta_1 \cdots d\theta_d = o\left(\frac{y(\mathbf{r})}{\sqrt{\det B(\mathbf{r})}}\right)$$

Corollary

Let $y(x)$ be an H-admissible function. If $n_1, \ldots, n_d \to \infty$ in such a way that all coordinates of the solution $\rho_{\mathbf{n}}$ of $\mathbf{a}(\rho_{\mathbf{n}}) = \mathbf{n}$ tend to infinity as well, then we have

$$y_{\mathbf{n}} \sim \frac{y(\rho_{\mathbf{n}})}{\rho_{\mathbf{n}}^{\mathbf{n}} \sqrt{(2\pi)^d \det B(\rho_{\mathbf{n}})}},$$

where ρ_n is uniquely defined for sufficiently large $\mathbf{n}$, i.e., $\min_j n_j > N_0$ for some $N_0 > 0$.

A Class of H-admissible Functions

Theorem

Let $P(\mathbf{z}) = \sum_{\mathbf{m} \in M} b_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$ *be a polynomial in* $\mathbf{z}$ *with real coefficients and*

$$y(\mathbf{z}) = \sum_{\mathbf{n} \in \mathbf{n}^d} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} = e^{P(\mathbf{z})}$$

Then the following conditions are equivalent

1. $\forall \boldsymbol{\vartheta} \in [-\pi, \pi]^d \setminus \mathbf{0}$ *we have:*

$$\left| y(\mathbf{r}e^{i\boldsymbol{\vartheta}}) \right| < y(\mathbf{r}), \text{ for sufficiently large } \mathbf{r} (\text{in } \mathcal{R})$$

2. $y(\mathbf{z})$ *is H-admissible in* $\mathcal{R}$*.*

Theorem

Let $P(z_1, \ldots, z_d) = \sum_{j=1}^{L} a_j z_1^{k_{1j}} \cdots z_d^{k_{dj}}$ *be a polynomial in* d *variables and with positive coefficients* $a_j > 0$
$\mathbf{K}_j := (k_{1j}, \ldots, k_{dj})$*: exponent vector of the* j*th monomial*

Then $e^{P(z_1,\ldots,z_d)}$ *is admissible if and only if the span of the* $\mathbf{K}_j := (k_{1j}, \ldots, k_{dj})$ *over* $\mathbb{Z}$ *equals* $\mathbb{Z}^d$.
Equivalently, this means that

$$\mathbf{K}_j \boldsymbol{\theta}^{\mathrm{T}} \equiv 0 \mod 2\pi, \; j = 1 \ldots, L,$$

has only the trivial solution $\boldsymbol{\theta} \equiv \mathbf{0} \mod 2\pi$.

Closure Properties

- If $y(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$, then $e^{y(\mathbf{x})} \in \mathcal{H}_{\mathcal{R}}$.
- If $y_1(\mathbf{x}), y_2(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$ and $\det(B_1 + B_2) \leq C \min\left(\det B_1, \det B_2\right)$ and B_1 and B_2 have same eigenvectors, then $y_1(\mathbf{x})y_2(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$.
- If $y(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$, $p(\mathbf{x})$ polynomial with positive coefficients, then $y(\mathbf{x})p(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$.
- $y(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$, $f(\mathbf{x})$ analytic, real if $\mathbf{x} \in \mathbb{R}^d$ and
$$\max_{x_i = r_i, i=1,\ldots,d} |f(\mathbf{x})| = O\left(y(\mathbf{r})^{1-\delta}\right), \text{ as } \mathbf{r} \to \infty.$$
Then $y(\mathbf{x}) + f(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$.
- If $y(x) \in \mathcal{H}$, then $e^{uy(x)} \in \mathcal{H}_{\mathcal{R}}$ with suitable $\mathcal{R}$.
- If $y(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$, then $\exp(y(\mathbf{x}_1)y(\mathbf{x}_2)) \in \mathcal{H}_{\tilde{\mathcal{R}}}$ with suitable $\tilde{\mathcal{R}}$.

Example (Stirling numbers of the second kind)

gen. function $y(z,u) = e^{u(e^z-1)}$

Example (Permutations with bounded cycle length)

cycle length $\leq \ell \implies$ gen. function

$$y(z,u) = \exp\left(u\sum_{i=1}^{\ell}\frac{z^i}{i}\right)$$

exponent is polynomial $\to$ check conditions of theorem

Example (Partitions of a set of partitions)

partitions of the set of subsets of a given partition

$$y(z,u) = \exp\left(u\left(e^{e^z-1}-1\right)\right)$$

Example (Set partitions with bounded block size)

$$y(z,u) = \exp\left(u\sum_{i=1}^{\ell}\frac{z^i}{i!}\right)$$

Example (Coverings of complete bipartite graphs with complete bipartite graphs)

$$y(z,u) = \exp\left((e^z-1)(e^u-1)\right)$$

Example (Partitions of sets with coloured elements (out of d colours))

$S \subseteq \mathbb{Z}^d$, S finite, in each block the vector of the number of elements per colour must be an element of S.

$$y(\mathbf{z}) = \exp\left(\sum_{\mathbf{n}\in S}\frac{\mathbf{z}^{\mathbf{n}}}{n_1!\cdots n_d!}\right)$$

Rule of thumb

- Small singularities:
  - $f(z)$ grows subexponentially when z approaches the singularity
  - Use singularity analysis
- Large singularities:
  - $f(z)$ grows at least exponentially when z approaches the singularity
  - Use saddle point method

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Multivariate asymptotics and limiting distributions

Introductory remarks

$X\ldots$ random variable on probability space Ω.
Assumption: $X(\Omega) \subseteq \mathbb{N}$; $p_k := \mathbf{P}\{X = k\}$

distribution function: $F(x) = \mathbf{P}\{X \le x\} = \sum_{k \le x} p_k$.

probability generating function: $p(u) = \sum_{k \ge 0} p_k u^k$;
analytic for $|u| < 1$ since $p(1) = 1$.
$p_k = \frac{1}{k!} p^{(k)}(0)$.

Moments: $\mathbf{E}X = \sum_{k \ge 0} k p_k$, generally: $\mathbf{E}h(X) = \sum_{k \ge 0} h(k) p_k$;
for $h(x) = e^{itx}$: characteristic function $\phi_X(t) = \mathbf{E}e^{itX}$.

In particular:

$$\mu_r = \mathbf{E}X^r = \left(u \frac{\partial}{\partial u}\right)^r p(u) \mid_{u=1};$$

$$\mu = \mu_1 = p'(1), \quad \sigma^2 = \mu_2 - \mu^2 = p''(1) + p'(1) - p'(1)^2.$$

In combinatorics: $\mathcal{A}$ combinatorial structure;
probability spaces $(\mathcal{A}_n, P_n)$, where
P_n is a probability distribution on $\mathcal{A}_n$,

$X_n(a) = \chi(a)$ for some random $a \in \mathcal{A}_n$ (recall: $\chi : \mathcal{A} \to \mathbb{N}$, or $\mathbb{N}^m$)

$$\mathcal{A}_{n,k} = \{a \in \mathcal{A} : \ |a| = n, \chi(a) = k\}, \qquad a_{n,k} = |\mathcal{A}_{n,k}|$$

Uniform distribution:

$$P[X_n = k] = \frac{a_{n,k}}{a_n} \text{ where } a_n = \sum_k a_{n,k}$$

Then $(\mathcal{A}, \chi)$ corresponds to

$$A(z, u) = \sum_{n,k} a_{n,k} z^n u^k.$$

What is the limiting distribution of X_n?
E.g.: X_n asymptotically Gaussian if $\exists \mu_n, \sigma_n$, $\sigma_n \to \infty$, s.t.

$$\sum_{k \le \mu_n + x\sigma_n} a_{n,k} = a_n \Phi(x) + o(a_n)$$

as $n \to \infty$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

Question: Is X_n asymptotically Gaussian?
Distributional quantities encoded in $A(z,u)$:
$p_n(u) = [z^n]A(z,u)/[z^n]A(z,1)$ Thus

$$\mathbf{E} X_n = \frac{[z^n]A_u(z,1)}{[z^n]A(z,1)}, \quad \mathbf{Var} X_n = \frac{[z^n][A_{uu}(z,1) + A_u(z,1)]}{[z^n]A(z,1)} - (\mathbf{E}\, X_n)^2$$

$$m_n(t) = \frac{[z^n]A(z,e^t)}{[z^n]A(z,1)}, \qquad \phi_n(t) = \frac{[z^n]A(z,e^{it})}{[z^n]A(z,1)}$$

Example (Cycles in permutations)

$$\mathcal{F} = \mathtt{Set}(\mu\mathtt{Cyc}(\mathcal{Z})) \implies f(z,u) = \exp\left(u \log \frac{1}{1-z)}\right) = (1-z)^{-u}$$

Thus
$$p_n(u) = \binom{n+u-1}{n} = \frac{\Gamma(n+u)}{\Gamma(u)n!}$$

If $|u-1| < \varepsilon$ then
$$p_n(u) = \frac{n^{u-1}}{\Gamma(u)}\left(1 + O\left(\frac{1}{n}\right)\right) \sim (e^{u-1})^{\log n}$$

Example (Cycles in permutations – cont'd)

For the expectation we get

$$\mathbb{E}X_n = \frac{[z^n]\frac{1}{1-z}\log\frac{1}{1-z}}{[z^n]\frac{1}{1-z}} \sim \log n$$

and similarly $\mathbf{Var}X_n \sim \log n$. Let $\phi_n^*(t) = \mathbb{E}\exp\left(it\frac{X_n-\mathbb{E}X_n}{\sqrt{\mathbf{Var}X_n}}\right)$, then

$$\phi_n^*(t) \sim \frac{e^{-it\sqrt{\log n}}}{\Gamma(e^{it/\sqrt{\log n}})}\exp\left((e^{it/\sqrt{\log n}}-1)\log n\right) \sim e^{-\frac{t^2}{2}+O\left(\frac{1}{\sqrt{\log n}}\right)}.$$

Theorem (Goncharov 1944)

The number of cycles in random permutions of size n is asymptotically normal with mean and variance proportional to $\log n$

Important inequalities for random variables:

Theorem (Markov's inequality)

If $X > 0$ *and* $\mu = \mathbf{E}X$ *then* $\mathbf{P}\{X \geq t\mu\} \leq 1/t$.

Theorem (Chebyshev's inequality)

If X *is real-valued with* $\mu = \mathbf{E}X$ *and* $\sigma^2 = \mathbf{Var}X$, *then*

$$\mathbf{P}\{|X - \mu| \geq t\sigma\} \leq 1/t^2.$$

Definition

A sequence $(X_n)_{n\geq 0}$ of random variables with $\mathbf{E}X_n = \mu_n$ is called asymptotically concentrated, if

$$\forall \varepsilon > 0 \ : \ \lim_{n\to\infty} \mathbf{P}\left\{1 - \varepsilon < \frac{X_n}{\mu_n} < 1 + \varepsilon\right\} = 1.$$

Theorem

Let $(X_n)_{n\geq 0}$ be a sequence of random variables, $\mu_n = \mathbf{E}X_n$, $\sigma_n^2 = \mathbf{Var}X_n$, satisfying $\lim_{n\to\infty} \sigma_n/\mu_n = 0$. Then X_n is asymptotically concentrated.

Proof: Set $\tilde{X}_n = X_n/\mu_n$. Then $\mathbf{E}\tilde{X}_n = 1$ and $\tilde{\sigma}_n = \sigma_n/\mu_n = o(1)$. Therefore $\mathbf{P}\left\{|\tilde{X}_n - 1| \geq \varepsilon\right\} \leq \tilde{\sigma}_n^2/\varepsilon^2 = o(1)$. □

Example (Number of leaves in plane trees)

$$\mathcal{F} = \mu\{\circ\} \cup \{\circ\} \times \mathrm{Seq}_{\geq 1}(\mathcal{F})$$

implies

$$F(z,u) = zu + \frac{zF(z,u)}{1 - F(z,u)}$$

Example (Number of leaves in plane trees – cont'd)

and thus

$$F(z,u) = \frac{1}{2}\left(1 + (u-1)z - \sqrt{1 - 2(u+1)z - (u-1)^2 z^2}\right)$$
$$F(z,1) = \frac{1}{2}\left(1 - \sqrt{1-4z}\right)$$
$$F_u(z,1) = \frac{1}{2}\left(z + \frac{z}{\sqrt{1-4z}}\right)$$

$$[z^n]F(z) = \frac{1}{n}\binom{2n-2}{n-1} \sim \frac{4^{n-1}}{\sqrt{\pi n^3}}$$
$$[z^n]F_u(z,1) = [z^{n-1}]\frac{1}{2\sqrt{1-4z}} \sim \frac{1}{2}\frac{4^{n-1}}{\sqrt{\pi n}}$$

Thus $\mu \sim n/2$. Similarly $\sigma_n = O\left(\sqrt{n}\right) \implies$ asymptocally concentrated.