

Analytic Combinatorics: Complex-analytic Methods and Applications

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Singularity analysis for implicit functions

Theorem (Implicit function theorem)

If $F(z, y)$ is analytic, $F(z_0, y_0) = 0$, $F_y(z_0, y_0) \neq 0$, then there is a unique analytic function $y(z)$, defined in a neighbourhood of z_0 such that $F(z, y(z)) \equiv 0$.

Thus, the solution $y(z)$ of $F(z, y(z)) \equiv 0$ can only have a singularity at z_0 if $F(z_0, y_0) = 0$, but $F_y(z_0, y_0) = 0$. Then

$$\begin{aligned}
 F(z, y) = & F(z_0, y_0) + F_z(z_0, y_0)(z - z_0) + F_y(z_0, y_0)(y - y_0) \\
 & + \frac{F_{yy}(z_0, y_0)(y - y_0)^2}{2} + F_{yz}(z_0, y_0)(y - y_0)(z - z_0) \\
 & + \frac{F_{zz}(z_0, y_0)(z - z_0)^2}{2} + O(|z - z_0|^3 + |y - y_0|^3)
 \end{aligned}$$

Theorem

Assume that $F(z, y)$ is analytic and that $y(z)$ is a solution of the functional equation $F(z, y) = 0$. Moreover, $F(z_0, y_0) = 0$, $F_y(z_0, y_0) = 0$, $F_z(z_0, y_0) \neq 0$, and $F_{yy}(z_0, y_0) \neq 0$. Then, locally around z_0 , the function $y(z)$ admits the expansion

$$y(z) = y_0 \pm \sqrt{\frac{2z_0 F_z(z_0, y_0)}{F_{yy}(z_0, y_0)}} \sqrt{1 - \frac{z}{z_0}} + O\left(\left|1 - \frac{z}{z_0}\right|\right).$$

as $z \rightarrow z_0$.

Proof (sketch): Considerations of the previous slide \rightsquigarrow main term

bootstrapping \rightsquigarrow error term



Applications

The practical procedure is as follows:

- 1 Preparation:
 - a) Locate the dominant singularities.
 - b) Check for analytic continuation.
- 2 Determine the singular expansions.
- 3 Apply transfers.
- 4 In case of more than one dominant singularity: Collect contributions and sum up

Example (Motzkin trees)

$$\mathcal{M} = \mathcal{Z} + \mathcal{Z} \times \mathcal{M} + \mathcal{Z} \times \mathcal{M} \times \mathcal{M},$$

thus $M(z) = z(1 + M(z) + M(z)^2)$ and

$$M(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z}.$$

Singularities are at $z = -1$ and $z = 1/3$.

Analytic continuation possible. For $z \rightarrow 1/3$ we have

$$M(z) = 1 - \sqrt{3}\sqrt{1-3z} + O((1-3z)^{3/2})$$

Apply transfer:

$$m_n = \frac{-\sqrt{3} \cdot 3^n}{n^{3/2} \Gamma(-\frac{1}{2})} + O(3^n n^{-5/2}) = \frac{3^n \sqrt{3}}{2\sqrt{\pi} n^3} + O(3^n n^{-5/2})$$

Example (2-regular graphs)

Recall

$$\mathcal{G} = \text{Set}(\text{Cyc}_{\geq 3}^{(u)}(\mathcal{Z}))$$

and

$$g(z) = \exp\left(\frac{1}{2}\left(\log\frac{1}{1-z} - z - \frac{z^2}{2}\right)\right) = \frac{e^{-\frac{z}{2} - \frac{z^2}{4}}}{\sqrt{1-z}}.$$

as well as

$$e^{-\frac{z}{2} - \frac{z^2}{4}} = e^{-3/4} \left(1 - (z-1) + \frac{1}{4}(z-1)^2 + O((z-1)^3)\right).$$

Thus

$$\frac{e^{-\frac{z}{2} - \frac{z^2}{4}}}{\sqrt{1-z}} = \frac{e^{-3/4}}{\sqrt{1-z}} + O(\sqrt{1-z})$$

Example (2-regular graphs – cont'd)

$$\frac{e^{-\frac{z}{2} - \frac{z^2}{4}}}{\sqrt{1-z}} = \frac{e^{-3/4}}{\sqrt{1-z}} + O(\sqrt{1-z})$$

Applying transfers gives

$$[z^n]g(z) = \frac{g_n}{n!} = \frac{e^{-3/4}n^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} + O(n^{-3/2})$$

and

$$g_n = \frac{n!e^{-3/4}}{\sqrt{\pi n}} \left(1 + O\left(\frac{1}{n}\right)\right).$$

More precisely:

$$g_n = \frac{n!e^{-3/4}}{\sqrt{\pi n}} \left(1 - \frac{5}{8n} + \frac{1}{128n^2} + O\left(n^{-3}\right)\right)$$

Example (Children's rounds)

$$\mathcal{R} = \text{Set}(\mathcal{Z} * \text{Cyc}(\mathcal{Z}))$$

Thus

$$R(z) = \exp\left(z \log \frac{1}{1-z}\right) = (1-z)^{-z}.$$

Simplifying gives

$$\begin{aligned} R(z) &= \frac{1}{1-z} \exp\left(- (1-z) \log \frac{1}{1-z}\right) \\ &= \frac{1}{1-z} - \log \frac{1}{1-z} + O\left((1-z) \log^2 \frac{1}{1-z}\right) \\ &= \frac{1}{1-z} - \log \frac{1}{1-z} + O\left(\sqrt{1-z}\right) \end{aligned}$$

$R(z)$ has singularity at $z = 1$ and is analytic in $\mathbb{C} \setminus [1, \infty)$.

Example (Children's rounds – cont'd)

$$\frac{r_n}{n!} = 1 - \frac{1}{n} + O\left(n^{-3/2}\right)$$

Remark: Better estimate is

$$\frac{r_n}{n!} = 1 - \frac{1}{n} + O\left(\frac{\log n}{n^2}\right)$$

In general: $m, k \in \mathbb{N}$ then

$$[z^n](1-z)^m \left(\log \frac{1}{1-z}\right)^k \sim \frac{(\log n)^{k-1}}{n^{m+1}}$$

Example (Permutations with all cycles of odd length)

$$\mathcal{F} = \text{Set}(\text{Cyc}_{\text{odd}}(\mathcal{Z}))$$

Thus

$$\begin{aligned} f(z) &= \exp\left(\frac{\log \frac{1}{1-z} - \log \frac{1}{1+z}}{2}\right) = \exp\left(\frac{1}{2} \log \frac{1+z}{1-z}\right) \\ &= \sqrt{\frac{1+z}{1-z}} \end{aligned}$$

Singularities are $z = \pm 1$,

$f(z)$ is analytic on $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$.

Example (Permutations with all cycles of odd length – cont'd)

The singular expansions are

$$f(z) = \frac{\sqrt{2}}{\sqrt{1-z}} - \frac{1}{2\sqrt{2}}\sqrt{1-z} + O(|1-z|^{3/2}), \text{ as } z \rightarrow 1,$$

$$f(z) = \frac{1}{\sqrt{2}}\sqrt{1+z} + O(|1+z|^{3/2}), \text{ as } z \rightarrow -1.$$

and hence we get

$$\frac{f_n}{n!} = \frac{\sqrt{2}}{\sqrt{\pi n}} - \frac{(-1)^n + \frac{1}{2}}{2\sqrt{2\pi n^3}} + O(n^{-5/2})$$

Example (Number of cycles in a random mapping – cont'd)

When marking cycles, then

$$\mathcal{B} = \text{Set}(\mu_{\text{Cyc}}(\mathcal{T})), \quad \text{where } \mathcal{T} = \mathcal{Z} * \text{Set}(\mathcal{T}).$$

Thus, with $b_{n,k} = \#$ functional digraphs with n vertices and k nodes, the bivariate GF $B(z, u) = \sum_{n \geq 0} \sum_{k \geq 0} b_{n,k} u^k \frac{z^n}{n!}$ satisfies

$$B(z, u) = \exp\left(u \log \frac{1}{1 - T(z)}\right), \quad \text{where } T(z) = ze^{T(z)}.$$

The average number of cycles in \mathcal{B}_n is then

$$\frac{1}{n^n} \sum_{k \geq 0} kb_{n,k} = \frac{n!}{n^n} [z^n] \frac{d}{du} B(z, u) \Big|_{u=1}.$$

Example (Number of cycles in a random mapping – cont'd)

Since

$$\frac{d}{du} B(z, u) \Big|_{u=1} = \frac{1}{1 - T(z)} \log \frac{1}{1 - T(z)}$$

we are searching for

$$\frac{n!}{n^n} [z^n] \frac{1}{1 - T(z)} \log \frac{1}{1 - T(z)}.$$

To find the singularities, we must solve

$$F(z, T) = T - ze^T = 0, \quad F_T(z, T) = 1 - ze^T = 0,$$

which has only one solution: $z_0 = 1/e$, $T_0 = T(z_0) = 1$.

We find $F_z(1/e, 1) = -e$, $F_{TT}(1/e, 1) = -1$ and thus

$$T(z) = 1 - \sqrt{2}\sqrt{1 - ez} + O(1 - ez), \quad \text{as } z \rightarrow \frac{1}{e}.$$

Example (Number of cycles in a random mapping – cont'd)

Hence

$$\begin{aligned}
 & \frac{n!}{n^n} [z^n] \frac{1}{1 - T(z)} \log \frac{1}{1 - T(z)} \\
 & \sim \frac{n!}{n^n} [z^n] \frac{1}{\sqrt{2}\sqrt{1 - ez}} \log \frac{1}{\sqrt{2}\sqrt{1 - ez}} \\
 & \sim \sqrt{2\pi n} e^{-n} \cdot \frac{e^n}{\sqrt{2}} [z^n] \frac{1}{\sqrt{1 - z}} \cdot \frac{1}{2} \log \frac{1}{1 - z} \\
 & \sim \sqrt{\pi n} \cdot \frac{1}{2} \cdot \frac{n^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2})} \log n \\
 & = \frac{1}{2} \log n
 \end{aligned}$$

Example (Pólya trees)

$$\mathcal{P} = \{o\} \times \text{MSet}(\mathcal{P})$$

Thus

$$P(z) = z \exp \left(\sum_{\ell \geq 1} \frac{P(z^\ell)}{\ell} \right) = ze^{P(z)} Q(z)$$

Searching for dominant singularities: The above equation implies $\rho \leq \frac{1}{e} < 1$, thus $Q(z)$ is analytic for $|z| = \rho$.

The system $P = ze^P Q$, $1 = ze^P Q$ has one solution $(\rho, 1)$.

$P(\rho) = 1$ implies $\rho e = 1/Q(\rho) < 1$ and thus $\rho < 1/e$.

Comparing with the GF for the Catalan numbers:

$P(1/4) < C(1/4) = 1/2 < 1$ implies $\rho > 1/4$.

Example (Pólya trees – cont'd)

We have $F(z, P(z)) = 0$ with $F(z, P) = ze^P Q - P$.

$$P(z) \sim 1 - \sqrt{\frac{2\rho F_z(\rho, 1)}{F_{PP}(\rho, 1)}} \sqrt{1 - \frac{z}{\rho}}.$$

We compute $F_{PP}(\rho, 1) = (ze^P Q)(\rho, 1) = 1$ and

$$\begin{aligned} \rho F_z(\rho, 1) &= \rho \frac{\partial}{\partial z} (ze^P Q - P) = \rho(\rho e^P Q' + e^P Q) \\ &= 1 + \sum_{i \geq 2} P'(\rho^i) \rho^i. \end{aligned}$$

Thus

$$[z^n]P(z) = \frac{-\sqrt{2} \cdot \sqrt{1 + \sum_{i \geq 2} P'(\rho^i) \rho^i}}{\Gamma(-1/2) \sqrt{n^3}} \rho^{-n} = \frac{\sqrt{1 + \sum_{i \geq 2} P'(\rho^i) \rho^i}}{\sqrt{2\pi n^3} \rho^n}$$

Example (Extremal parameters: Average height of binary trees (sketch))

$$\mathcal{B}_0 = \emptyset, \quad \mathcal{B}_1 = \{\square\}, \quad \mathcal{B}_{h+1} = \{\square\} \cup \{\circ\} \times \mathcal{B}_h \times \mathcal{B}_h$$

and thus $B_1(z) = 1$, $B_{k+1}(z) = 1 + zB_k(z)^2$

$$\begin{aligned} \sum_{h \geq 0} h(\# \text{ trees w. height } h) &= \sum_{h \geq 0} (\# \text{ trees w. height } \geq h) \\ &= [z^n] \underbrace{\sum_{h \geq 0} (B_\infty(z) - B_h(z))}_{f(z)} \end{aligned}$$

$$\implies \text{average height} = \frac{f_n}{\frac{1}{n+1} \binom{2n}{n}} \sim \frac{f_n}{4^n} \sqrt{\pi n^3}$$

Example (Extremal parameters: Average height of binary trees (sketch) – cont'd)

Flajolet & Odlyzko 1982 showed:

$$f(z) = 2 \log \frac{1}{1-4z} + O\left((1-4z)^{1/4+\varepsilon}\right)$$

$$\implies 4^{-n}f_n = \frac{2}{n} + O\left(n^{-5/4+\varepsilon}\right)$$

$$\implies \text{average height} = 2\sqrt{\pi n} + O\left(n^{1/4+\varepsilon}\right)$$

Example (Extremal parameters: Longest cycle in a permutation (sketch))

\mathcal{A} . . . cyclic permutations, GF $A(z) = \log \frac{1}{1-z} = \sum_{\ell \geq 1} \frac{z^\ell}{\ell}$

\mathcal{B} . . . all permutations, GF $B(z) = e^{A(z)}$, because $\mathcal{B} = \text{Set}(\mathcal{A})$

longest cycle length less than k : GF $B_k(z) = \exp \left(\sum_{j=1}^{k-1} \frac{z^j}{j} \right)$.

Thus consider

$$f(z) = \frac{1}{1-z} \sum_{k \geq 0} \left(1 - \exp \left(- \sum_{j \geq k} \frac{z^j}{j} \right) \right)$$

Rem.: $f_n = \mathbb{E}(\text{length of longest cycle in } \mathcal{B}_n)$

Example (Extremal parameters: Longest cycle in a permutation (sketch) – cont'd)

Substitute $z = e^{-t}$, then $t \rightarrow 0 \implies z \rightarrow 1$, $z \sim 1 - t$:

$$\sum_{j \geq k} \frac{z^j}{j} = t \sum_{j \geq k} \frac{e^{-tj}}{tj} \sim t \int_k^\infty \frac{e^{-tu}}{tu} du = \int_{kt}^\infty \frac{e^{-u}}{u} du$$

Example (Extremal parameters: Longest cycle in a permutation (sketch) – cont'd)

Substitute $z = e^{-t}$, then $t \rightarrow 0 \implies z \rightarrow 1$, $z \sim 1 - t$:

$$\sum_{j \geq k} \frac{z^j}{j} = t \sum_{j \geq k} \frac{e^{-tj}}{tj} \sim t \int_k^\infty \frac{e^{-tu}}{tu} du = \int_{kt}^\infty \frac{e^{-u}}{u} du$$

$$(1 - z)f(z) = \sum_{k \geq 0} \left(1 - \exp \left(- \sum_{j \geq k} \frac{z^j}{j} \right) \right)$$

Example (Extremal parameters: Longest cycle in a permutation (sketch) – cont'd)

Substitute $z = e^{-t}$, then $t \rightarrow 0 \implies z \rightarrow 1$, $z \sim 1 - t$:

$$\sum_{j \geq k} \frac{z^j}{j} = t \sum_{j \geq k} \frac{e^{-tj}}{tj} \sim t \int_k^\infty \frac{e^{-tu}}{tu} du = \int_{kt}^\infty \frac{e^{-u}}{u} du$$

$$(1 - z)f(z) \sim \int_0^\infty \left(1 - \exp \left(\int_{xt}^\infty \frac{e^{-u}}{u} du \right) \right) dx$$

Example (Extremal parameters: Longest cycle in a permutation (sketch) – cont'd)

Substitute $z = e^{-t}$, then $t \rightarrow 0 \implies z \rightarrow 1$, $z \sim 1 - t$:

$$\sum_{j \geq k} \frac{z^j}{j} = t \sum_{j \geq k} \frac{e^{-tj}}{tj} \sim t \int_k^\infty \frac{e^{-tu}}{tu} du = \int_{kt}^\infty \frac{e^{-u}}{u} du$$

$$(1 - z)f(z) = \frac{1}{t} \underbrace{\int_0^\infty \left(1 - \exp \left(- \int_x^\infty \frac{e^{-u}}{u} du \right) \right) dx}_{\text{constant } C}$$

$$\implies f(z) \sim \frac{C}{(1 - z)^2} \quad \implies f_n \sim Cn.$$