

Analytic Combinatorics: Complex-analytic Methods and Applications

Bernhard Gittenberger

TU Wien

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Tauberian theorems

Tauber's theorem

Recall:

Theorem (Abel's limit theorem)

If

$$\sum_{n \geq 0} a_n = s$$

*then $f(z) = \sum_{n \geq 0} a_n z^n$ defines a continuous function f on $[0, 1]$
and $f(1) = s$.*

What about the converse?

Given the function, statement about the series.

Theorem (Tauber's theorem (1897))

Given $f(z) = \sum_{n \geq 0} a_n z^n$ such that $\lim_{z \rightarrow 1-} f(z) = s$ and

$$a_n = o\left(\frac{1}{n}\right) \quad \text{Tauberian condition}$$

Then $\sum_{n \geq 0} a_n = s$

Example (Example without side condition)

$$f(z) = \frac{1}{1+z} = \sum_{n \geq 0} (-1)^n z^n.$$

Clearly, $\lim_{z \rightarrow 1-} f(z) = \frac{1}{2},$

but $\sum_{n \geq 0} (-1)^n$ is divergent!

Theorem (Hardy & Littlewood (1914), Karamata (1930))

Let $f(z) = \sum_{n \geq 0} f_n z^n$ be a power series with radius of convergence equal to 1 and, as $z \rightarrow 1-$, (*no Δ -analyticity!*)

$$f(z) \sim \frac{1}{(1-z)^\alpha} L\left(\frac{1}{1-z}\right).$$

If $f_n \geq 0$ (Tauberian condition), then

$$\sum_{k=0}^n f_k \sim \frac{n^\alpha}{\Gamma(\alpha+1)} L(n).$$

If moreover f_n is monotone, then

$$f_n \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} L(n).$$

Proof: Set $U(t) = \sum_{k=0}^{\lfloor t \rfloor} f_k$, which is the improper distribution function of a measure U on $[0, \infty)$.

Let

$$\omega(s) = \mathbf{E} e^{-sU} = \int_0^\infty e^{-sx} U(\mathrm{d}x) = \sum_{k \geq 0} e^{-sk} f_k = f(e^{-s})$$

be its Laplace transform.

Let $\tau > 0$ and $t = 1/\tau$. Then

$$\omega(\tau s) = \int_0^\infty e^{-\tau s x} U(\mathrm{d}x) = \int_0^\infty e^{-sx/t} U(\mathrm{d}x) = \int_0^\infty e^{-sy} U(t \mathrm{d}y)$$

is the Laplace transform of the measure with distribution function $U(tx)$.

Let V be the measure with distribution function $U(tx)/\omega(\tau)$.

For $\tau \rightarrow 0$ we have

$$\omega(\tau) = f(e^{-\tau}) \sim f(1 - \tau) \sim \tau^{-\alpha} L\left(\frac{1}{\tau}\right) = t^{\alpha} L(t) \text{ and}$$

$$\mathbf{E}e^{-sV} = \frac{\omega(\tau s)}{\omega(\tau)} \sim \frac{f(1 - \tau s)}{f(1 - \tau)} \sim \frac{1}{s^{\alpha}} \frac{L\left(\frac{t}{s}\right)}{L(t)} \longrightarrow \frac{1}{s^{\alpha}} = \mathcal{L}\left(\frac{x^{\alpha-1}}{\Gamma(\alpha)}\right)$$

Now let W denote the measure with **density** $x^{\alpha-1}/\Gamma(\alpha)$. Then, by the continuity theorem for Laplace transforms,

$$\frac{U(tx)}{\omega(\tau)} \longrightarrow W([0, x]) = \int_0^x \frac{x^{\alpha-1}}{\Gamma(\alpha)} dx = \frac{x^{\alpha}}{\Gamma(\alpha+1)}.$$

But

$$\frac{U(tx)}{\omega(\tau)} = \frac{1}{\omega(\tau)} \sum_{k \leq \lfloor tx \rfloor} f_k \sim \frac{1}{t^{\alpha}} \sum_{k \leq \lfloor tx \rfloor} f_k,$$

so set $x = 1$ and $t = n$.



Example (Permutations with distinct cycle lengths)

The EGF of $\text{Cyc}_k(\mathcal{Z})$ is z^k/k . Thus the EGF of the class of permutations with distinct cycle lengths is

$$f(z) = \prod_{k \geq 1} \left(1 + \frac{z^k}{k}\right).$$

Therefore

$$\begin{aligned} \log f(z) &= \log \prod_{k \geq 1} \left(1 + \frac{z^k}{k}\right) = \sum_{k \geq 1} \log \left(1 + \frac{z^k}{k}\right) = \sum_{k \geq 1} \sum_{n \geq 1} \frac{(-1)^{n-1} z^{kn}}{k^n n} \\ &= \log \frac{1}{1-z} + \sum_{k \geq 1} \left(\log \left(1 + \frac{z^k}{k}\right) - \frac{z^k}{k} \right) \\ &= \log \frac{1}{1-z} + \underbrace{\log(1+z) - z + \sum_{k \geq 2} \sum_{n \geq 2} \frac{(-1)^{n-1} z^{kn}}{k^n n}}_{=: g(z)} \end{aligned}$$

Example (Permutations with distinct cycle lengths – cont'd)

$$g(z) = \log(1+z) - z + \sum_{k \geq 2} \sum_{n \geq 2} \frac{(-1)^{n-1} z^{kn}}{k^n n} = \sum_{k \geq 1} \left(\log \left(1 + \frac{z^k}{k} \right) - \frac{z^k}{k} \right)$$

$g(1)$ convergent?

$$\begin{aligned} g(1) &= \lim_{z \rightarrow 1^-} g(z) = \sum_{k \geq 1} \left(\log \left(1 + \frac{1}{k} \right) - \frac{1}{k} \right) \\ &= -\log 1 + \lim_{n \rightarrow \infty} (\log(n+1) - H_n) = -\gamma \end{aligned}$$

Thus

$$f(z) \sim \frac{e^{-\gamma}}{1-z} \text{ for } z \rightarrow 1.$$

This implies $\sum_{k=0}^n f_k \sim ne^{-\gamma}$.

Example (Permutations with distinct cycle lengths – cont'd)

What about f_n ?

Write $f(z) = \frac{1+z}{1-z} e^{\tilde{g}(z)} = \frac{1+z}{1-z} h(z)$ with

$$\tilde{g}(z) = g(z) - \log(1+z) = -z + \sum_{m \geq 2} \frac{(-1)^{m-1}}{m} \sum_{k \geq 2} \frac{z^{km}}{k^m}$$

So,

$$f(z) = \frac{1}{1-z} \sum_{n \geq 0} h_n z^n + \frac{z}{1-z} \sum_{n \geq 0} h_n z^n$$

and thus $f_n = 2h_0 + \cdots + 2h_{n-1} + h_n$.

Our goal: Show that f_n **converges**, then $\sum_{k=0}^n f_k \sim ne^{-\gamma}$ implies $f_n \sim e^{-\gamma}$.

Example (Permutations with distinct cycle lengths – cont'd)

If $|[z^n]\tilde{g}(z)| \leq [z^n]v(z)$ then $|h_n| \leq [z^n]e^{v(z)}$.

For $n \geq 2$ we have

$$[z^n]\tilde{g}(z) = [z^n] \sum_{m \geq 2} \frac{(-1)^{m-1}}{m} \sum_{k \geq 2} \frac{z^{km}}{k^m} = \sum_{m \geq 2, m|n, m < n} \frac{(-1)^{m-1}}{m} \left(\frac{m}{n}\right)^m.$$

Moreover,

$$\frac{d}{dx} \left(\frac{x}{n}\right)^x = \left(\frac{x}{n}\right)^x (1 + \log x - \log n) < 0$$

if $1 \leq x \leq n/e$. Thus $\left(\frac{m}{n}\right)^m$ is decreasing for $1 \leq m \leq n/e$.

Example (Permutations with distinct cycle lengths – cont'd)

Consequently,

$$\begin{aligned} |[z^n]\tilde{g}(z)| &\leq \frac{1}{2} \left(\frac{2}{n}\right)^2 + \sum_{3 \leq m \leq n/3, m|n} \frac{1}{m} \left(\frac{3}{n}\right)^3 + \frac{2}{n} 2^{-n/2} \\ &\leq \frac{2}{n^2} + \frac{3}{n^2} + \frac{2}{n^2} < \frac{10}{n^2}. \end{aligned}$$

Thus we can take $w(z) = e^{v(z)} := \exp\left(10 \sum_{\ell \geq 1} \frac{z^\ell}{\ell^2}\right)$.

As $w' = v'w$, we have $nw_n = 10 \sum_{k=0}^{n-1} \frac{w_k}{n-k}$.

Note: $v(1) < \infty$, $w_n \geq 0$, $w_n \leq w(1)$.

Choose $C > 0$, then for $1 \leq n \leq C$ and sufficiently large B we have $w_n \leq B/n^2$.

Example (Permutations with distinct cycle lengths – cont'd)

Assume $w_m \leq B/m^2$ for $1 \leq m \leq n$ and some $n \geq C$ and proceed by induction on n .

Choose $C_0 < \frac{C}{2}$: Then

$$\frac{w_k}{n-k} \leq w(1) \cdot \frac{2}{n} \text{ for } 0 \leq k \leq C_0,$$

$$\frac{w_k}{n-k} \leq \frac{B}{k^2} \cdot \frac{1}{n-k} \text{ for } C_0 \leq k \leq n/2.$$

With this we get

$$\sum_{C_0 \leq k \leq n/2} \frac{w_k}{n-k} \leq \frac{2B}{n} \sum_{C_0 \leq k \leq n/2} \frac{1}{k^2} = \frac{2BS_n}{n}$$

$$\sum_{n/2 < k \leq n-1} \frac{w_k}{n-k} \leq \frac{4B}{n^2} \sum_{n/2 < k \leq n-1} \frac{1}{n-k} \leq \frac{4BH_n}{n^2} \leq \frac{4(1 + \log n)B}{n^2}$$

Example (Permutations with distinct cycle lengths – cont'd)

Altogether this gives

$$nw_n \leq \frac{20C_0 w(1)}{n} + \left(\frac{20S_n}{n} + \frac{40(1 + \log n)}{n^2} \right) B \stackrel{!}{\leq} \frac{B}{n}$$

if

$$B \geq \frac{20C_0 w(1)}{1 - 20S_n - \frac{40(1 + \log n)}{n}}$$

This can be achieved as $n \geq C$; so choose C sufficiently large to guarantee that S_n can be made small (by choosing C_0 large enough) and the denominator is positive.

Then choose B sufficiently large.

Finally, we obtain $h_n \leq w_n = O(1/n^2)$ and so f_n converges. □

Saddle point techniques

Laplace's method

Goal: Evaluation of the parameter integral

$$I(\lambda) = \int_a^b f(z) e^{-\lambda h(z)} dz, \quad \text{as } \lambda \rightarrow \infty,$$

where

- f and h are analytic and real-valued for real z .
- $\lambda > 0$, $a, b \in \mathbb{R}$.
- There is a $a < t_0 < b$ such that $h'(t_0) = 0$ and $h''(t_0) > 0$.

Idea: Let $f(t_0) \neq 0$ and estimate

$$\begin{aligned} I(\lambda) &\sim \int_a^b f(t_0) \exp \left(-\lambda \left(h(t_0) + h''(t_0) \frac{(t-t_0)^2}{2} \right) \right) dt \\ &\sim f(t_0) e^{-\lambda h(t_0)} \int_{-\infty}^{\infty} \exp \left(-\frac{(t-t_0)^2}{2} \lambda h''(t_0) \right) dt \\ &\sim f(t_0) e^{-\lambda h(t_0)} \sqrt{\frac{2\pi}{\lambda h''(t_0)}} \end{aligned}$$

where we used

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k e^{-tx^2} dx = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \frac{t^{-(k+1)/2} k!}{(k/2)! \cdot 2^{k+1/2}} & \text{if } k \text{ is even.} \end{cases}$$

Note: k fixed, $\alpha > 0$, then $\int_{\theta}^{\infty} x^k e^{-\alpha x^2} dx = O\left(e^{-\alpha \theta^2}\right)$.

⚠ Simultaneous limits: $\lambda \rightarrow \infty$, $t \rightarrow t_0$

Theorem (Laplace's method)

Let $-\infty \leq a < b \leq \infty$ and

$$I(x) = \int_a^b f(t) e^{-xh(t)} dt.$$

Assume

- (1) f is once and h twice continuously differentiable;
- (2) t_0 unique minimum of h , with $h''(t_0) > 0$, $f(t_0) \neq 0$;
- (3) $h(t) = h(t_0) + \frac{h''(t_0)}{2}(t - t_0)^2 + O(|t - t_0|^3)$;
- (4) $I(x)$ convergent for $x \geq x_0$.

Then

$$I(x) = f(t_0) \sqrt{\frac{2\pi}{xh''(t_0)}} e^{-xh(t_0)} \left(1 + O\left(\frac{1}{\sqrt{x}}\right) \right), \text{ as } x \rightarrow \infty.$$

Proof: W.l.o.g. $t_0 = 0$ ($a < 0$, $b > 0$), $h(t_0) = 0$ and $f(t) \equiv 1$ (otherwise $f(t) = f(t_0) + O(|t - t_0|)$)

By (2), we have $|t| < \delta \implies \left| h(t) - h''(0)\frac{t^2}{2} \right| \leq c|t|^3$. Write

$$I(x) = \underbrace{\int_{-\delta}^{\delta} e^{-xh(t)} dt}_{I_1(x)} + \underbrace{\int_{|t|>\delta} e^{-xh(t)} dt}_{I_2(x)}.$$

With $\mu := \inf_{|t|>\delta} h(t) > 0$ we get

$$|I_2(x)| \leq \int_{|t|>\delta} e^{-x_0 h(t) - (x-x_0)h(t)} dt \leq e^{-(x-x_0)\mu} \int_{|t|>\delta} e^{-x_0 h(t)} dt = O(e^{-\beta x})$$

for some $\beta > 0$.

Turn to $I_1(x)$:

$$I_1(x)$$

$$= \int_{-\delta}^{\delta} e^{-xh''(0)\frac{t^2}{2}} dt + \int_{-\delta}^{\delta} e^{-xh''(0)\frac{t^2}{2}} \left(e^{-x\left(h(t)-h''(0)\frac{t^2}{2}\right)} - 1 \right) dt$$

$$= \sqrt{\frac{2\pi}{h''(0)x}} + O\left(e^{-x\delta^2}\right) + J$$

As

$$|e^t - 1| = \left| \sum_{n \geq 1} \frac{t^n}{n!} \right| < \sum_{n \geq 1} \frac{|t|^n}{(n-1)!} = |t|e^{|t|}$$

and $|h(t) - h''(0)\frac{t^2}{2}| \leq c|t^3|$, we have

$$|J| \leq 2cx \int_0^{\delta} t^3 \exp\left(-xh''(0)\frac{t^2}{2} + cxt^3\right) dt$$

We had

$$I_1(x) = \sqrt{\frac{2\pi}{h''(t_0)x}} + O\left(e^{-x\delta^2}\right) + J$$

with

$$|J| \leq 2cx \int_0^\delta t^3 \exp\left(-xh''(t_0)\frac{t^2}{2} + cxt^3\right) dt$$

and thus we get

$$|J| \leq 2cx \int_0^\delta t^3 \exp\left(-xt^2\left(\frac{h''(t_0)}{2} - \delta c\right)\right) dt \leq 2cx \int_0^\delta t^3 \exp(-xt^2b) dt$$

for $b = \frac{h''(t_0)}{2} - \delta c$. Substitution $u = \sqrt{xb}t$ gives

$$|J| \leq \frac{2c}{b^2x} \int_0^\infty u^3 e^{-u^2} du = O\left(\frac{1}{x}\right)$$

and therefore $I_1(x) = \sqrt{\frac{2\pi}{h''(t_0)x}} \left(1 + O\left(\frac{1}{\sqrt{x}}\right)\right)$.



Saddle points

Let f be analytic in some region Ω , $z_0 \in \Omega$

z_0 is an **ordinary point**: $f(z_0) \neq 0$, $f'(z_0) \neq 0$

For $z = z_0 + re^{i\theta}$ and $\frac{f'(z_0)}{f(z_0)} = \lambda e^{i\phi}$ we have

$$|f(z)| = |f(z_0) + re^{i\theta} f'(z_0) + O(r^2)| \sim |f(z_0)| \cdot |1 + r\lambda \cos(\theta + \phi)|$$

Let r be sufficiently small, while θ varies. Then

$$|f(z)| \sim |f(z_0)|(1 \pm \lambda r) \quad \text{for } \theta = -\phi, \theta = -\phi + \pi, \text{ resp.}$$

(line of steepest ascent resp. descent)

$$|f(z)| \sim |f(z_0)| + o(r) \quad \text{for } \theta = -\phi \pm \frac{\pi}{2}$$

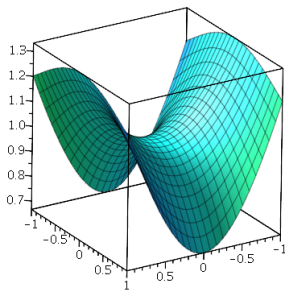
(level set, perpendicular to gradient)

z_0 is a **zero**: $f(z_0) = 0, f'(z_0) \neq 0$ if first order.

Then $|f(z)| \sim r|f'(z_0)| = O(r)$. For m th order: $|f(z)| = O(r^m)$.

z_0 is a **simple saddle point**: $f(z_0) \neq 0, f'(z_0) = 0, f''(z_0) \neq 0$

$$\begin{aligned} \text{Set } \lambda e^{i\phi} &= \frac{1}{2} \frac{f''(z_0)}{f(z_0)}. & |f(z)| &= \left| f(z_0) + \frac{r^2}{2} e^{2i\theta} f''(z_0) + O(r^3) \right| \\ & & &= |f(z_0)| \left| 1 + \lambda r^2 e^{i(2\theta + \phi)} + O(r^3) \right| \\ & & &\sim |f(z_0)| (1 + \lambda r^2 \cos(2\theta + \phi)) \end{aligned}$$



$|f(z)|$ maximal for $\theta = -\phi/2$

$|f(z)|$ minimal for $\theta = -\phi/2 + \pi/2$

$|f(z)| \sim |f(z_0)|$ for $\theta = -\phi/2 + \pi/4$

and for $\theta = -\phi/2 + 3\pi/4$

Pattern repeats once. ($0 \leq \theta \leq 2\pi$)

p -fold saddles: $f'(z_0) = f''(z_0) = \dots = f^{(p)}(z_0) = 0$,
 $f^{(p+1)}(z_0) \neq 0$

Example: $p = 2$ and $p = 4$:

