Analytic Combinatorics: Complex-analytic Methods and Applications

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Complex Analysis

Generating functions as functions

So far: $f(z) = \sum_n f_n z^n$ formal power series
Examples:

$$f(z) = \frac{1 - \sqrt{1 - 4z}}{2}, \qquad g(z) = \frac{e^{-z}}{1 - z}.$$

Constructed from the known series

$$(1-y)^{-1} = \sum_{n \geq 0} y^n, \quad (1-y)^{1/2} = \sum_{n \geq 0} \binom{1/2}{n} y^n, \quad e^y = \sum_{n \geq 0} \frac{1}{n!} y^n.$$

explicit coefficients: $f_n = \frac{1}{2n}\binom{2n-2}{n-1} \sim \frac{4^{n-1}}{\sqrt{\pi n^3}}$,
similarly: $g_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \sim \frac{1}{e} \approx 0.36787$.

In general

$$f(z) = \sum_n f_n z^n$$

is also a function defined on a disk in $\mathbb{C}$:

Theorem (Hadamard's theorem)

Given the series $f(z) = \sum_n f_n z^n$, let

$$R = \frac{1}{\limsup_{n\to\infty} |f_n|^{1/n}}.$$

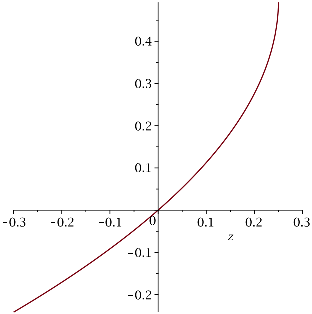
Then the series converges for all $z \in \mathbb{C}$ with $|z| < R$ and diverges for $|z| > R$.

Remark: R is called the radius of convergence of the power series.

The function

$$f(z) = \frac{1 - \sqrt{1 - 4z}}{2}$$

on the reals.



singularity at $z = 1/4$, because $f'(1/4) = \infty$.

Recall:

$$f_n = [z^n]f(z) \sim \frac{4^n}{\sqrt{\pi n^3}}, \qquad g_n = [z^n]g(z) \sim \frac{1}{e}.$$

These follow the asymptotic scheme $a_n \sim A^n \theta(n)$:

$A^n \ldots$ exponential growth (we write $a_n \bowtie A^n$)

$\theta(n) \ldots$ subexponential (polynomial) modulation

Two principles

Principle 1: The location of the singularity determines the exponential growth

Principle 2: The nature of the singularity determines the subexponential modulation

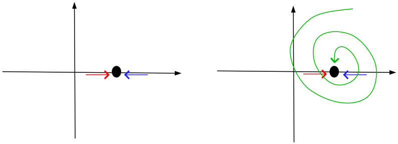
Analytic and meromorphic functions

Definition

A region is a connected open subset of $\mathbb{C}$.

Definition

$f : \Omega \to \mathbb{C}$ is differentiable at z_0 if $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.



Theorem

$f : \Omega \to \mathbb{C}$ is differentiable $\Longleftrightarrow$ f is infinitely often differentiable.

Definition

A function $f : \Omega \to \mathbb{C}$ is analytic at z_0 if there is a disk $D(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$ in which f is expressible as

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$

f is analytic in Ω if f is analytic at z_0 for all $z_0 \in \Omega$.

Theorem

$f : \Omega \to \mathbb{C}$ is differentiable $\Longleftrightarrow$ f is analytic.

Theorem (Closure properties)

If f and g are analytic, then so are

$$f \pm g, \quad f \cdot g, \quad f', \quad \frac{f}{g} ⚠, \quad f \circ g ⚠$$

Theorem (Identity theorem)

If f and g are analytic in Ω and coincide on a set with an accumulation point, then $f(z) = g(z)$ for all $z \in \Omega$.

Theorem

If $f : \Omega \to \mathbb{C}$ is analytic at z_0, then the power series $\sum_{n \geq 0} a_n (z - z_0)^n$ expressing f converges on every disk $D \subseteq \Omega$ centered in z_0.

Theorem (analytic continuation)

If $f : \Omega \to \mathbb{C}$ is analytic at some $z \in \partial\Omega$, then f can be uniquely analytically continued to $\Omega \cup D(z, r)$ for some disk $D(z, r)$ centered in z.
If $\Omega' \supseteq \Omega$ and $f : \Omega \to \mathbb{C}$ and $g : \Omega' \to \mathbb{C}$ are analytic and coincide on Ω, then g is uniquely determined by f.

Definition

f has a singularity at z_0 if f is not analytic at z_0.

Definition

A singularity z_0 is called isolated singularity if f is analytic in $D \setminus \{z_0\}$, where D is an open set. It is called

- removable if $\lim_{z \to z_0} f(z)$ exists,
- a pole if $(z - z_0)^k f(z)$ is analytic at z_0 for some positive integer k,
- an essential singularity otherwise.

Examples

$$\frac{e^z - 1}{z} \quad (z_0 = 0), \qquad \frac{1}{1 - 2z - 3z^2} \quad (z_0 = -1), \qquad e^{\frac{1}{z}} \quad (z_0 = 0).$$

Theorem

If z_0 is an isolated singularity of f, then f can be expressed as a Laurent series

$$\sum_{n \geq -K} a_n (z - z_0)^n.$$

Remark: If K is integer, then we have a pole of order K, otherwise $K = \infty$ and the singularity is essential.

Examples

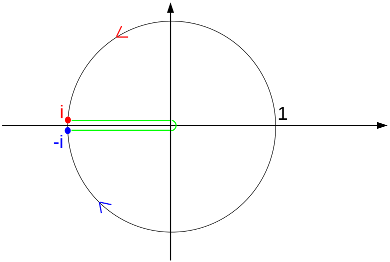
$$\frac{e^z - 1}{z} = \sum_{n \geq 0} \frac{z^n}{(n+1)!}, \qquad \frac{1}{1 - 2z - 3z^2} = \sum_{n \geq -1} \frac{3^{n+1}}{4^{n+2}} (z+1)^n,$$

$$e^{\frac{1}{z}} = \sum_{n=-\infty}^{0} \frac{z^n}{n!}$$

Definition

A singularity z_0 is called a branch point if it going around a circle encircling z_0 yields a new element of $f(z)$, i.e., f is multi-valued.

Example: $f(z) = \sqrt{z}$ and $z_0 = 0$.



Note: $\sqrt{e^{i\varphi}} = e^{i\varphi/2}$.

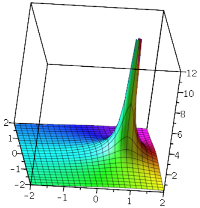
Remark: One distinguishes

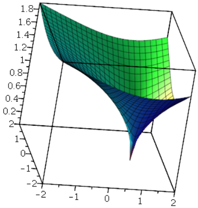
- algebraic branch points (Puiseux expansion with finitely many terms with negative powers),
- transcendental branch points (Puiseux expansion with infinitely many terms with negative powers),
- and logarithmic branch points (Riemann surface with infinitely many sheets).

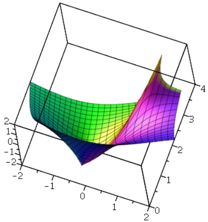
Examples

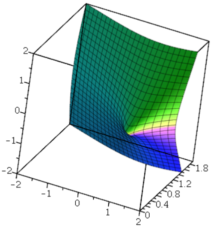
e^z	no singularities, entire
$1/(1-z)$	simple pole
$\sqrt{1-z}$	algebraic branch point
$\log(1-z)$	logarithmic branch point
$\exp(1/\sqrt{z})$	transcendental branch point

$$\frac{1}{1-z} \qquad \sqrt{1-z} \qquad \log(1-z)$$









Definition

A function $f : \Omega \to \mathbb{C}$ is called meromorphic if f is analytic in Ω with the exception of poles.

Theorem

If $f(z)$ is analytic at $z_0 = 0$ and its power series has a finite radius of convergence, say R, then there is at least one singularity on the circle $|z| = R$.

Theorem (Pringsheim's theorem)

If $f(z)$ is analytic at $z = 0$ and has a power series expansion

$$f(z) = \sum_{n \geq 0} f_n z^n \text{ with } f_n \geq 0 \text{ for all } n,$$

then $z = R$ is a singularity of $f(z)$.

Theorem

If $f(z)$ is analytic at $z_0 = 0$ and the radius of convergence of its power series is R, then $f_n = [z^n]f(z) \bowtie \frac{1}{R^n}$

Proof: By definition, $\limsup_{n\to\infty} |f_n|^{1/n} = \frac{1}{R}$.
Thus, for all $\varepsilon > 0$ we must have

$$|f_n| \geq \frac{1}{(R+\varepsilon)^n} \quad \text{infinitely often.}$$

On the other hand:

$$|f_n| = o\left(\frac{1}{(R-\varepsilon)^n}\right)$$

since the series converges for $|z| < R$. □

Theorem

If $f(z)$ is analytic at $z_0 = 0$ and the radius of convergence of its power series is R, then $f_n = [z^n]f(z) \bowtie \frac{1}{R^n}$

Simplest singular case: $f(z)$ rational $\implies$

$$f(z) = \sum_{\rho} \sum_{j} \frac{a_{\rho,j}}{(z-\rho)^j}$$

Then

$$[z^n]\frac{1}{(z-\rho)^j} = [z^n]\frac{(-1)^j}{\rho^j\left(1-\frac{z}{\rho}\right)^j} = (-1)^j\binom{n+j-1}{j-1}\rho^{-n-j}$$

Theorem

If $f(z)$ is analytic at $z_0 = 0$ and the radius of convergence of its power series is R, then $f_n = [z^n]f(z) \bowtie \frac{1}{R^n}$

Definition

Assume $f(z)$ is analytic at $z = 0$. A singularity is called a dominant singularity if its modulus is minimal among all singularities.

Two strategies for meromorphic functions:

1. Subtraction of singularities
2. contour integration

Example

Consider

$$
\begin{aligned}
f(z) &= \frac{e^{-z}}{1-z} \\
&= \frac{e^{-1}}{1-z} + \underbrace{\frac{e^{-z}-e^{-1}}{1-z}}_{\text{analytic for } |z|<1+\varepsilon} \\
&= \frac{e^{-1}}{1-z} + \underbrace{\frac{e^{-1}\sum_{n\geq 1}\frac{(1-z)^n}{n!}}{1-z}}_{=:h(z)}
\end{aligned}
$$

$$
\Longrightarrow \quad f_n = \frac{1}{e}[z^n]\frac{1}{1-z} + [z^n]h(z) = \frac{1}{e} + O\left((1-\varepsilon)^n\right) \sim \frac{1}{e}
$$

In general: For a dominant polar singularity z_0, expand

$$f(z) = \sum_{j=-r}^{\infty} a_j (z - z_0)^j = \sum_{j=1}^{r} \frac{a_{-j}}{(z - z_0)^j} + h(z)$$

and subtract the principal part.
Dealing with several singularities:

Theorem

Assume that f has the singularities $z_0, z_1, \ldots, z_s$ with $|z_i| = R$, $i = 0, \ldots, s$ and let $g_i(z)$ be the principal part of the Laurent series of f at z_i, $i = 0, \ldots, s$.
Then

$$h(z) = f(z) - g_0(z) - g_1(z) - \cdots - g_s(z)$$

is analytic in the domain $|z| \leq R + \varepsilon$.

Theorem

If $f(z)$ is meromorphic, but analytic in $|z| < R$ and has a unique dominant polar singularity z_0 on $|z| = R$, then

$$[z^n]f(z) = [z^n]\sum_{j=1}^{r} \frac{a_{-j}}{(z-z_0)^j} + O\left(\left(\frac{1}{R}+\varepsilon\right)^n\right)$$

$$= \mathrm{Poly}_{\text{degree } r-1}(n) z_0^{-n} + O\left(\left(\frac{1}{R+\varepsilon}\right)^n\right).$$

Several singularities $z_0, z_1, \ldots, z_s$:

$$[z^n]f(z) = \sum_{j=0}^{s} P_{j,\, r_j-1}(n) z_j^{-n} + O\left(\left(\frac{1}{R'+\varepsilon}\right)^n\right)$$

where $R' = \min\limits_{z \text{ pole of } f,\ z \notin \{z_0, \ldots, z_s\}} |z|$.

Complex contour integrals

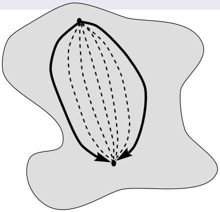
Theorem

Let $f : \Omega \to \mathbb{C}$ be analytic and γ and γ' be simple paths from a to b ($a, b \in \Omega$) such that

- *γ and γ' are homotopic;*
- *the set encircled by the closed curve $\gamma \cup \gamma'$ is a subset of Ω.*

Then

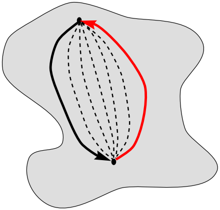
$$\int_{\gamma} f(z)\,\mathrm{d}z = \int_{\gamma'} f(z)\,\mathrm{d}z.$$



Corollary (Cauchy's integral theorem)

If $f : \Omega \to \mathbb{C}$ analytic and γ is a simple closed curve encircling a subset of Ω, then

$$\oint_{\gamma} f(z)\,\mathrm{d}z = 0.$$



Definition

If $f : \Omega \to \mathbb{C}$ is meromorphic at z_0 and $f(z) = \sum_{n \geq -M} a_n (z - z_0)^n$. Then the residue of f at z_0 is defined by $\mathrm{Res}(f; z_0) = a_{-1}$

Theorem (Residue theorem)

If $f : \Omega \to \mathbb{C}$ is meromorphic and γ is a simple closed curve encircling a region $\mathcal{D} \subseteq \Omega$ exactly once such that

- *there is no singularity on γ;*
- *$\mathcal{D}$ contains the poles $z_1, z_2, \ldots, z_k$.*

Then

$$\oint_\gamma f(z)\, \mathrm{d}z = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(f; z_j).$$

Theorem (Residue theorem)

If $f : \Omega \to \mathbb{C}$ is meromorphic and γ is a simple closed curve encircling a region $\mathcal{D} \subseteq \Omega$ clockwise exactly once such that

- *there is no singularity on γ;*
- *$\mathcal{D}$ contains the poles $z_1, z_2, \ldots, z_k$.*

Then

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(f; z_j).$$

Proof:
Assume $k = 1$ and $z_1 = 0$.
Choose $\gamma = \{re^{i\varphi} \mid 0 \le \varphi < 2\pi\}$
Integrate $f(z) = \sum_{n \ge -M} f_n z^n$ term by term

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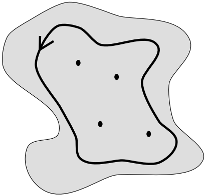

$$\oint_\gamma f(z)\,\mathrm{d}z$$

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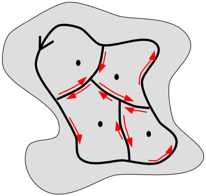

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_j \oint_{\gamma_j} f(z)\,\mathrm{d}z$$
$$= \sum_j \oint_{\gamma_j} \sum_{n \geq -M_j} a_{n,j}(z - z_j)^n\,\mathrm{d}z$$

Theorem (Cauchy's integral formula)

If $f : \Omega \to \mathbb{C}$ is analytic at $z = 0$ ($0 \in \Omega$) and γ a simple closed curve encircling the origin clockwise exactly once, then

$$f_n = [z^n]f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z^{n+1}} \,\mathrm{d}z$$

Proof:

$$f(z) = \sum_{\ell \geq 0} f_\ell z^\ell$$

$$\Longrightarrow \qquad \frac{f(z)}{z^{n+1}} = \sum_{\ell \geq 0} f_\ell z^{\ell-n-1}$$

$$\Longrightarrow \qquad \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}; 0\right) = f_n \quad \square$$

Theorem

If $f : \Omega \to \mathbb{C}$ is analytic at $z = 0$ ($0 \in \Omega$) and meromorphic in Ω, then

$$f_n = - \sum_{\rho \text{ singularity of } f \text{ with } |\rho|<r} \operatorname{Res}\left(\frac{f(z)}{z^{n+1}}; \rho\right) + O\left(r^{-n}\right)$$

Proof:

$$\frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{n+1}} \,\mathrm{d}z = f_n + \sum_{\rho \text{ singularity of } f \text{ with } |\rho|<r} \operatorname{Res}\left(\frac{f(z)}{z^{n+1}}; \rho\right)$$

and

$$\frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{n+1}} \,\mathrm{d}z = O\left(r^{-n}\right). \quad \square$$

Example

The EGF of the class of ordered set partitions (surjections) is

$$f(z) = \sum_{n\geq 0} f_n \frac{z^n}{n!} = \sum_{n\geq 0} \sum_{k=0}^{n} k! S(n,k) \frac{z^n}{n!}$$

$$\begin{aligned}
\sum_{k\geq 0} S(n,k) y^k &= n![z^n] e^{y(e^z-1)} = \sum_{k\geq 0} \frac{y^k}{k!} n! [z^n] (e^z-1)^k \\
&= \sum_{k\geq 0} \frac{y^k}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n = \sum_{j\geq 0} \sum_{k\geq j} y^k \frac{1}{(k-j)!j!} j^n (-1)^{k-j} \\
&= \sum_{j\geq 0} \frac{j^n}{j!} y^j \sum_{k\geq j} \frac{y^{k-j}}{(k-j)!} (-1)^{k-j} = e^{-y} \sum_{j\geq 0} \frac{j^n}{j!} y^j \\
\implies \qquad f_n &= \sum_{k\geq 0} S(n,k) k! = \sum_{k\geq 0} S(n,k) \int_0^\infty y^k e^{-y} \,\mathrm{d}y \\
&= \sum_{j\geq 0} \frac{j^n}{j!} \int_0^\infty y^j e^{-2y} \,\mathrm{d}y = \sum_{j\geq 0} \frac{j^n}{2^{j+1}}
\end{aligned}$$

Example

The EGF of the class of ordered set partitions (surjections) is

$$f(z) = \sum_{n\geq 0} f_n \frac{z^n}{n!} = \sum_{n\geq 0} \sum_{k=0}^{n} k! S(n,k) \frac{z^n}{n!}$$

$$f(z) = \sum_{n\geq 0} \sum_{j\geq 0} \frac{j^n}{2^{j+1}} \frac{z^n}{n!} = \sum_{j\geq 0} \frac{1}{2^{j+1}} \sum_{n\geq 0} \frac{(jz)^n}{n!}$$

$$= \frac{1}{2} \sum_{j\geq 0} \left(\frac{e^z}{2}\right)^r = \frac{1}{2 - e^z} \qquad \text{(pedestrian way)}$$

Symbolic method: $\mathtt{Seq}(\mathtt{Set}_{\geq 1}(\mathcal{Z})) \implies f(z) = \frac{1}{1-(e^z-1)}$

Example

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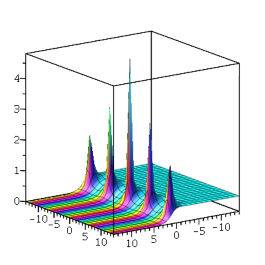
$$f(z) = \sum_{n\geq 0}\sum_{k=0}^{n} k!S(n,k)\frac{z^n}{n!} = \frac{1}{2-e^z}$$



singularities at $\log 2 \pm 2k\pi i$, $k \in \mathbb{N}$.
dominant singularity: $z = \log 2$

$$f(z) = \frac{-1/2}{z-\log 2} + \frac{1}{4} - \frac{1}{24}(z-\log 2) + \ldots.$$

$$f_n = \frac{1}{2(\log 2)^{n+1}} + O((\rho^{-n})$$

$$\#surjections = \frac{n!}{2(\log 2)^{n+1}} + O\left(n!\rho^{-n}\right)$$

with $\rho < \sqrt{\log^2 2 + 4\pi^2}$