Analytic Combinatorics: Complex-analytic Methods and Applications

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Singularity Analysis

Preliminaries

Assumptions:

- $f : \Omega \to \mathbb{C}$ is analytic at $z = 0$, $0 \in \Omega$.
- The dominant singularity z_0 is not a pole.
- Near $z = z_0$ we have $f(z) = g((z - z_0)^\alpha)$ where $\alpha \notin \mathbb{Z}$ and g is analytic at 0.
- z_0 is the unique singularity in $|z| \leq |z_0| + \eta$.
- W.l.o.g. let $z_0 = 1$, otherwise consider $f(zz_0)$.

Lemma

Given sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ with $a_n = O\left(n^{-\gamma}\right)$ $(\gamma > 0)$ and $b_n = O\left(\theta^n\right)$ $(0 < \theta < 1)$ then

$$\sum_{k=0}^{n} a_k b_{n-k} = O\left(n^{-\gamma}\right)$$

Proof:

$$\sum_{k=0}^{n/2} a_k b_{n-k} \leq \max_{0\leq k\leq n/2} |a_k| \cdot \sum_{k=0}^{n/2} C \cdot \theta^{n-k} \leq C_1 C \theta^{n/2} \leq C_2 \tilde{\theta}^n = O\left(n^{-\gamma}\right)$$

$$\sum_{k=n/2}^{n} a_k b_{n-k} \leq \max_{n/2\leq k\leq n} |a_k| \sum_{k=n/2}^{n} C \cdot \theta^{n-k} \leq C_3 \cdot n^{-\gamma} \cdot 1.$$

Lemma

Let $\beta \notin \mathbb{N}$. Then

$$[z^n](1-z)^\beta \sim \frac{n^{-\beta-1}}{\Gamma(-\beta)}.$$

Proof: Later.

Lemma

Let $v(z)$ be analytic in $|z| < 1 + \eta$ and $u(z) = (1-z)^\gamma v(z)$. Then

$$[z^n]u(z) = O\left(n^{-\gamma-1}\right).$$

Proof: $a_n := [z^n](1-z)^\gamma = O\left(n^{-\gamma-1}\right)$, $b_n := [z^n]v(z) = O\left(\theta^n\right)$ for some $0 < \theta < 1$. Now apply first lemma.

Theorem (Darboux)

Let $v(z)$ be analytic in $|z| < 1 + \eta$ and locally around $z = 1$ let the asymptotic expansion $v(z) = \sum_{j\geq 0} v_j(1-z)^j$ hold. Let $\beta \notin \mathbb{N}$. Then we have

$$[z^n](1-z)^\beta v(z) = [z^n]\sum_{j=0}^{m} v_j(1-z)^{\beta+j} + O\left(n^{-m-\beta-2}\right)$$

$$= \sum_{j=0}^{m} v_j \binom{n-\beta-j-1}{n} + O\left(n^{-m-\beta-2}\right).$$

Proof:

$$(1-z)^\beta v(z) = \sum_{j=0}^{m} v_j(1-z)^{\beta+j} + \underbrace{\sum_{j>m} v_j(1-z)^{\beta+j}}_{(1-z)^{\beta+m+1}\tilde{v}(z)}$$

Analyticity domains of $v(z)$ and $\tilde{v}(z)$ identical, so last lemma completes the proof. $\square$

Theorem (Darboux's theorem)

If f is analytic in the disk $|z| < 1$ and k times continuously differentiable on its boundary, then $[z^n]f(z) = o(n^k)$.

Proof: $k = 0$:

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z^{n+1}} \,\mathrm{d}z = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} \,\mathrm{d}t \to 0$$

by the Riemann-Lebesgue lemma.
$k > 0$: For $k = 1$ integration by parts gives

$$\int_0^{2\pi} f(e^{it}) e^{-int} \,\mathrm{d}t = 0 + \frac{1}{in} \int_0^{2\pi} f'(e^{it}) e^{-int} \,\mathrm{d}t = o\left(\frac{1}{n}\right),$$

for larger k iterate. $\square$

Example (2-regular graphs)

Labelled undirected 2-regular simple graphs consist of cycles only, each of length at least 3:

$$\mathcal{G} = \mathtt{Set}(\mathtt{Cyc}^{(u)}_{\geq 3}(\mathcal{Z}))$$

where $\mathcal{Z}$ is the class of a labelled atom and $\mathtt{Cyc}^{(u)}_{\geq 3}$ means the undirected cycle construction.
Thus the EGF satisfies

$$g(z) = \exp\left(\frac{1}{2}\left(\log\frac{1}{1-z} - z - \frac{z^2}{2}\right)\right) = \frac{e^{-\frac{z}{2}-\frac{z^2}{4}}}{\sqrt{1-z}}.$$

$\Longrightarrow$ singularity at $z = 1$, $e^{-\frac{z}{2}-\frac{z^2}{4}}$ is analytic at $z = 1$

Example (2-regular graphs – cont'd)

We have

$$e^{-\frac{z}{2}-\frac{z^2}{4}} = e^{-3/4} + e^{-3/4}(1-z) + \frac{e^{-3/4}}{4}(1-z)^2 + O\left((1-z)^3\right)$$

So,
$g(z) = \frac{e^{-3/4}}{\sqrt{1-z}} + e^{-3/4}\sqrt{1-z} + \frac{e^{-3/4}}{4}(1-z)^{3/2} + O\left((1-z)^{5/2}\right)$

Thus by Darboux's theorem we obtain

$$\frac{g_n}{n!} = e^{-3/4}\left(\binom{n-\frac{1}{2}}{n} + \binom{n-\frac{3}{2}}{n} + \frac{1}{4}\binom{n-\frac{5}{2}}{n} + o\left(n^{-2}\right)\right).$$

$$\implies g_n = \frac{n!e^{-3/4}}{\sqrt{n\pi}}\left(1 - \frac{5}{8n} + \frac{1}{128n^2} + o\left(n^{-3/2}\right)\right)$$

Several singularities:

Theorem (Szegő's theorem)

Let $h(z)$ analytic in $|z| < 1$ and $e^{i\phi_1}, \ldots, e^{i\phi_r}$ be its singularities on $|z| = 1$. Assume that locally around $e^{i\phi_k}$ the expansion

$$h(z) = \sum_{\ell \geq 0} c_\ell^{(k)} \left(1 - ze^{-i\phi_k}\right)^{\alpha_k + \ell\beta_k}, \qquad \text{for some } \beta_k > 0,$$

holds.
Then, as $n \to \infty$,

$$[z^n]h(z) \sim \sum_{\ell \geq 0} \sum_{k=1}^{r} c_\ell^{(k)} \binom{\alpha_k + \ell\beta_k}{n} \left(-e^{-i\phi_k}\right)^n$$

Definition

The Gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, \mathrm{d}t, \qquad \Re z > 0.$$

It can be analytically continued to $\mathbb{C} \setminus \{0, -1, -2, \dots\}$.

Lemma (Stirling's formula)

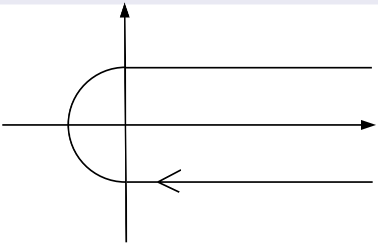
For $\Re z > 0$ we have

$$\Gamma(z+1) \sim \left(\frac{z}{e}\right)^z \sqrt{2\pi z}, \quad \textit{as } z \to \infty.$$

Lemma (Hankel's representation of $\Gamma(z)$)

Let $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^- \cup \mathcal{H}^\circ$ *where* $\mathcal{H}^+ = \{t + i \mid t \geq 0\}$, $\mathcal{H}^- = \{t - i \mid t \geq 0\}$, $\mathcal{H}^\circ = \left\{e^{i\varphi} \mid \frac{\pi}{2} \leq \varphi \leq \frac{3\pi}{2} i\right\}$ *and the curve encircles the origin clockwise. Then*

$$\frac{1}{\Gamma(\alpha)} = \int_{\mathcal{H}} (-t)^{-\alpha} e^{-t} \, \mathrm{d}t.$$



The method of singularity analysis

The task: Given OGF/EGF, determine its coefficients

The goal:

- Find function class for comparison with given function.
- coefficient extraction for that class is easy.
- Transfers applicable.
  - $f(z) = O(g(z)) \implies f_n = O(g_n)$
  - $f(z) = o(g(z)) \implies f_n = o(g_n)$
  - $f(z) \sim g(z) \implies f_n \sim g_n$

Extension:

$$f(z) = h_0(z) + h_1(z) + \cdots + h_k(z) + O(h_{k+1}(z)),$$

where $h_i(z) \gg h_{i+1}(z)$

$$\implies f_n = h_{0,n} + h_{1,n} + \cdots + h_{k,n} + O(h_{k+1,n}) \quad \text{and } h_{i,n} \gg h_{i+1,n}$$

Example (2-regular graphs – revisited)

We got

$$f(z) = \frac{e^{-\frac{z}{2}-\frac{z^2}{4}}}{\sqrt{1-z}} = \frac{e^{-3/4}}{\sqrt{1-z}} + e^{-3/4}\sqrt{1-z} + O\left((1-z)^{3/2}\right),$$

$$\frac{f_n}{n!} = \frac{e^{-3/4}}{\sqrt{\pi n}} - \frac{5e^{-3/4}}{8\sqrt{\pi n^3}} + O\left(n^{-2}\right).$$

Lemma

Let $\alpha \in \mathbb{C} \setminus \mathbb{N}$*. Then*

$$[z^n](1-z)^\alpha \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}.$$

Proof:

$$[z^n](1-z)^\alpha = (-1)^n \binom{\alpha}{n} = \binom{n-\alpha-1}{n} = \frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)n!}$$

then apply Stirling's formula

$$\Gamma(z+1) \sim \left(\frac{z}{e}\right)^z \sqrt{2\pi z}$$

to $\Gamma(n-\alpha)$. □

Lemma (Flajolet, Odlyzko 1990)

Let $\alpha \in \mathbb{C} \setminus \mathbb{N}$. *Then*

$$[z^n](1-z)^\alpha \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \left(1 + \sum_{k \geq 1} \frac{e_k(\alpha)}{n^k}\right).$$

where

$$e_k(\alpha) = \sum_{\ell=k}^{2k} (\alpha+1)(\alpha+2)\cdots(\alpha+\ell) \underbrace{[w^k x^\ell] e^x (1+xw)^{-1-\frac{1}{w}}}_{c_{k,\ell}}$$

Proof: Start with

$$[z^n](1-z)^\alpha = \frac{1}{2\pi i}\oint_C \frac{(1-z)^\alpha}{z^{n+1}}\,\mathrm{d}z$$

with

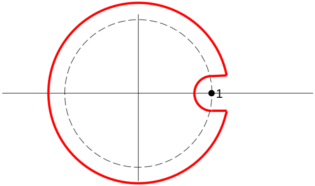
$$C = \left\{ z \,:\, |z| = \frac{1}{2} \right\}$$

Behaviour near the singularity determines coefficient asymptotics.
$\longrightarrow$ So, deform C appropriately.

Let $\mathcal{H}_n = \mathcal{H}_n^- \cup \mathcal{H}_n^+ \cup \mathcal{H}_n^\circ$

$$\mathcal{H}_n^- = \left\{ t - \frac{i}{n} \mid 0 \le t \le c_R \right\}, \quad \mathcal{H}_n^\circ = \left\{ 1 - \frac{e^{i\varphi}}{n} \mid |\varphi| \le \frac{\pi}{2} \right\},$$

$$\mathcal{H}_n^+ = \left\{ t + \frac{i}{n} \mid 0 \le t \le c_R \right\}, \quad \mathcal{H}_n^c = \left\{ z \mid |z| = R, |\Im z| \ge \frac{i}{n} \text{ if } \Re z > 0 \right\}.$$



$$\int_{\mathcal{H}_n^c} \frac{|1-z|^\alpha}{|z|^{n+1}} \, |\mathrm{d}z| = O(R^{-n})$$

Thus

$$[z^n](1-z)^\alpha \sim \frac{1}{2\pi i} \int_{\mathcal{H}_n} (1-z)^\alpha \, \mathrm{d}z \sim \overbrace{\frac{n^{-\alpha-1}}{2\pi i} \int_{\mathcal{H}} (-t)^\alpha \left(1 + \frac{t}{n}\right)^{-n-1} \mathrm{d}t}^{z=1+\frac{t}{n}}.$$

We have

$$[z^n](1-z)^\alpha \sim \frac{n^{-\alpha-1}}{2\pi i}\int_{\mathcal{H}}(-t)^\alpha\left(1+\frac{t}{n}\right)^{-n-1}\mathrm{d}t.$$

Now use

$$\left(1+\tfrac{t}{n}\right)^{-n-1}=e^{-t}e^t\left(1+\tfrac{t}{n}\right)^{-1-n}=e^{-t}\sum_{k,\ell\geq 0}c_{k,\ell}t^\ell n^{-k} \qquad \left(x=t, w=\tfrac{1}{n}\right)$$

Thus

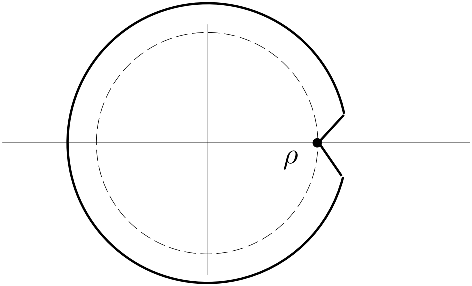
$$[z^n](1-z)^\alpha \sim \frac{1}{n^{\alpha+1}}\sum_{k,\ell\geq 0}(-1)^\ell c_{k,\ell}n^{-k}\underbrace{\frac{1}{2\pi i}\int_{\mathcal{H}}(-t)^{\alpha+\ell}e^{-t}\,\mathrm{d}t}_{1/\Gamma(-\alpha-\ell)}$$

$$=\frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\sum_{k,\ell\geq 0}c_{k,\ell}\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+\ell)}{n^k} \qquad \square$$

Definition

A function is f called Δ-analytic if there are $\eta > 0$ and $0 < \phi < \frac{\pi}{2}$ such that f is analytic in $\Delta \setminus \{1\}$ where

$$\Delta = \Delta(\eta, \phi) = \{z \mid |z| \leq 1 + \eta, |\arg(z - 1)| \geq \phi\}$$



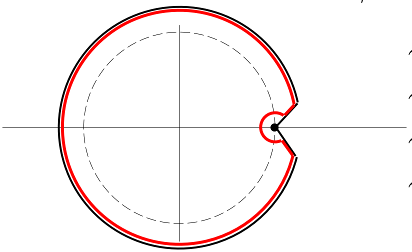
Remark: Δ is often called Camembert.

Theorem (O-transfer; Flajolet, Odlyzko 1990)

If $f(z)$ is Δ-analytic and $f(z) = O\left(|1-z|^{\alpha}\right)$ in Δ, then

$$f_n = [z^n]f(z) = O\left(n^{-\alpha-1}\right)$$

Proof:



$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ where

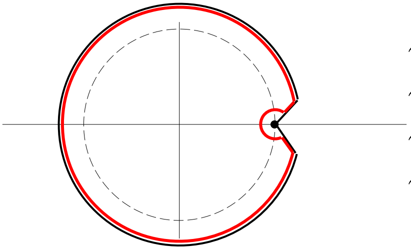
$$\gamma_1 = \left\{z \mid |z-1| = \tfrac{1}{n},\ |\arg(z-1)| \geq \phi\right\}$$

$$\gamma_2 = \left\{z \mid \tfrac{1}{n} \leq |z-1|,\ |z| \leq 1+\eta,\ \arg(z-1) = \phi\right\}$$

$$\gamma_3 = \{z \mid |z| = 1+\eta,\ |\arg(z-1)| \geq \phi\}$$

$$\gamma_4 = \overline{\gamma_2}$$

Proof:



$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ where

$$\gamma_1 = \left\{ z \mid |z-1| = \tfrac{1}{n}, \ |\arg(z-1)| \geq \phi \right\}$$
$$\gamma_2 = \left\{ z \mid \tfrac{1}{n} \leq |z-1|, \ |z| \leq 1+\eta, \ \arg(z-1) = \phi \right\}$$
$$\gamma_3 = \{ z \mid |z| = 1+\eta, \ |\arg(z-1)| \geq \phi \}$$
$$\gamma_4 = \overline{\gamma_2}$$

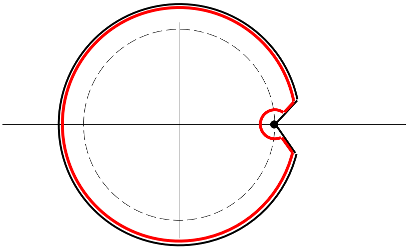
By assumption: $|f(z)| \leq K|1-z|^{\alpha}$

Set $f_n^{(j)} := \int_{\gamma_j} \frac{|f(z)|}{|z|^{n+1}} |\mathrm{d}z| \implies f_n \leq f_n^{(1)} + f_n^{(2)} + f_n^{(3)} + f_n^{(4)}$

Small circle γ_1: $|z|^{n+1} \geq \left(1 - \frac{1}{n}\right)^{n+1}$, $|f(z)| \leq Kn^{-\alpha}$

$$\implies f_n^{(1)} \leq Kn^{-\alpha} \left(1 - \frac{1}{n}\right)^{-n-1} \frac{2\pi}{n} = O\left(n^{-\alpha-1}\right)$$

Proof:



$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ where

$$\gamma_1 = \left\{ z \mid |z-1| = \tfrac{1}{n},\ |\arg(z-1)| \geq \phi \right\}$$

$$\gamma_2 = \left\{ z \mid \tfrac{1}{n} \leq |z-1|,\ |z| \leq 1+\eta,\ \arg(z-1) = \phi \right\}$$

$$\gamma_3 = \{ z \mid |z| = 1+\eta,\ |\arg(z-1)| \geq \phi \}$$

$$\gamma_4 = \overline{\gamma_2}$$

$$|f(z)| \leq K|1-z|^\alpha$$

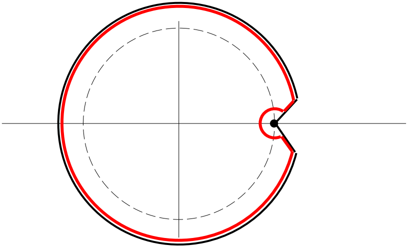
Rectilinear parts γ_2 and γ_4: Let E be such that $|1 + e^{i\phi}E| = 1 + \eta$. Then $f_n^{(2)}$ is bounded by

$$\int_1^{En} K \left(\frac{t}{n}\right)^\alpha \left|1 + \frac{e^{i\phi}t}{n}\right|^{-n-1} \frac{\mathrm{d}t}{n} \leq \frac{K}{n^{\alpha+1}} \int_1^\infty t^\alpha \left|1 + \frac{e^{i\phi}t}{n}\right|^{-n-1} \mathrm{d}t$$

Rem.: $\left(1 + \frac{e^{i\phi}t}{n}\right)^{-n-1} \sim \exp\left(-e^{i\phi}t\right)$.

Proof:

$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ where



$$\gamma_1 = \left\{ z \mid |z-1| = \tfrac{1}{n},\ |\arg(z-1)| \ge \phi \right\}$$

$$\gamma_2 = \left\{ z \mid \tfrac{1}{n} \le |z-1|,\ |z| \le 1+\eta,\ \arg(z-1) = \phi \right\}$$

$$\gamma_3 = \{ z \mid |z| = 1+\eta,\ |\arg(z-1)| \ge \phi \}$$

$$\gamma_4 = \overline{\gamma_2}$$

$$|f(z)| \le K|1-z|^\alpha$$

Large circle γ_3: $|z| = 1+\eta$, thus $|1-z| \le 3$.

$$f_n^{(3)} \le K3^\alpha (1+\eta)^{-n-1} 2\pi (1+\eta) = O\left(n^{-\alpha-1}\right) \qquad \square$$

Extensions:

Theorem

If $f(z)$ is Δ-analytic and $f(z) = O\left(|1-z|^{\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}\right)$ in Δ, then

$$f_n = [z^n]f(z) = O\left(n^{-\alpha-1}(\log n)^{\beta}\right)$$

Small circle: $\left|\log\frac{1}{1-z}\right| \leq \sup_{\theta}|\log(ne^{-i\theta})| = O(\log n)$

Rectilinear parts: $z = 1 + e^{i\phi}\frac{t}{n}$, $t \in [1, En]$

Split into $[1, \log^2 n] \cup [\log^2 n, En]$

$\log\frac{1}{1-z} = \log\left(\frac{ne^{-i\phi}}{-t}\right) \sim \log n$ for $t = O\left(\log^2 n\right)$.

Larger t: $\log\frac{1}{1-z} = O(\log n)$;

$\int |z|^{-n-1}\,\mathrm{d}z = O\left(ne^{-c\log^2 n}\right) = o(1/n) = O\left((\log n)^{\beta}\right)$

Large circle: $\log\frac{1}{1-z}$ bounded; $\int |z|^{-n-1}\,\mathrm{d}z = O\left((1+\eta)^{-n}\right)$

Theorem

If $f(z)$ is Δ-analytic and

$$f(z) = O\left(|1-z|^{\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}\left(\log\log\frac{1}{1-z}\right)^{\gamma}\right)$$

in Δ, then

$$f_n = [z^n]f(z) = O\left(n^{-\alpha-1}(\log n)^{\beta}(\log\log n)^{\gamma}\right)$$

Corollary (o-transfer)

If $f(z)$ is Δ-analytic and $f(z) = o\left(|1-z|^{\alpha}\right)$ in Δ, then

$$f_n = [z^n]f(z) = o\left(n^{-\alpha-1}\right)$$

Proof (sketch): Same idea, but instead of $|f(z)| \leq K|1-z|^{\alpha}$ use $|f(z)| \leq \varepsilon|1-z|^{\alpha}$ for z sufficiently close to 1.

Show $|f_n| < \varepsilon n^{-\alpha-1}$ for n sufficiently large. Fiddling with ε and δ yields the proof. □

$\sim$-transfer: $f(z) \sim (1-z)^{\alpha} \iff f(z) = (1-z)^{\alpha} + o\left((1-z)^{\alpha}\right)$

Theorem

Let $\alpha \notin \mathbb{N}$, $\beta \in \mathbb{C}$. *Then*

$$[z^n](1-z)^\alpha \left(\frac{1}{z}\log\frac{1}{1-z}\right)^\beta \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}(\log n)^\beta \left(1 + \sum_{k\geq 1} \frac{C_k(\alpha,\beta)}{(\log n)^k}\right)$$

where $C_k(\alpha,\beta) = \binom{\beta}{k}\Gamma(-\alpha)\frac{\mathrm{d}^k}{\mathrm{d}s^k}\frac{1}{\Gamma(s)}\Big|_{s=-\alpha}$.

Sketch of the proof: Use same contour as for $(1-z)^{-\alpha}$ $(\mathcal{H}_n \cup \mathcal{H}_n^c)$ and substitute $z = 1 + \frac{t}{n}$ on $\mathcal{H}_n$. This gives

$$z^{-n-1} f(z) \sim e^{-t}(-t)^\alpha n^{-\alpha} \log^\beta n \qquad \underbrace{(1 - \log(-t)/\log n)^\beta}_{\text{expand and use derivatives of Hankel's formula:}}$$

$$\frac{1}{2\pi i}\int_{\mathcal{H}} (-t)^\alpha e^{-t} \log^k(-t)\,\mathrm{d}t = \frac{\mathrm{d}^k}{\mathrm{d}\alpha^k}\frac{1}{\Gamma(-\alpha)} \qquad \square$$

The extra factor $1/z$ does not disturb, since

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{k\geq 0}(1-z)^k = 1 + O(1-z)\,.$$

$$\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{\beta} = \left(\log\frac{1}{1-z}\right)^{\beta} + O\left((1-z)\left(\log\frac{1}{1-z}\right)^{\beta}\right)$$

Thus transfers also apply to functions of the form

$$(1-z)^{\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}$$

Theorem

If $f(z)$ is Δ-analytic and

$$f(z) \sim (1-z)^{\alpha} L\left(\frac{1}{1-z}\right)$$

where $L(u) = (\log u)^{\beta} (\log\log u)^{\gamma}$, then

$$f_n \sim n^{-\alpha-1} L(n).$$

Likewise, the statement holds for o- and Σ-transfers.

Definition

A function $L : \mathbb{C} \to \mathbb{C}$ is called slowly varying if

1. There is $x_0 > 0$ with $0 \leq \phi \leq \frac{\pi}{2}$ such that $L(x) \neq 0$ for $-(\pi - \phi) \leq \arg(x - x_0) \leq \pi - \phi$;
2. There is a function $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{x \to \infty} \varepsilon(x) = 0$ such that for all $\theta \in [-(\pi - \phi), \pi - \phi]$ and for all $x \geq x_0$ we have

$$\left| \frac{L\left(xe^{i\theta}\right)}{L(x)} - 1 \right| < \varepsilon(x) \text{ and } \left| \frac{L\left(x \log^2 x\right)}{L(x)} - 1 \right| < \varepsilon(x).$$

Theorem

If $f(z)$ is Δ-analytic and $f(z) = O\left((1-z)^\alpha L\left(\frac{1}{1-z}\right)\right)$ where $L(u)$ is slowly varying, then $f_n = O(n^{-\alpha-1} L(n))$.
This holds for o- and $\sim$-transfers as well.