

Analytic Combinatorics: Complex-analytic Methods and Applications

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└ The saddle point method

The saddle point method

Typical application in combinatorics:

$$f_n = [z^n]f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z^{n+1}} dz$$

Lemma

If $f(z)$ is a generating function which is not a polynomial, its radius of convergence R is larger than zero, and $\lim_{z \rightarrow R-} f(z) = \infty$, then

$$\lim_{z \rightarrow 0+} \frac{f(z)}{z^{n+1}} = \lim_{z \rightarrow R-} \frac{f(z)}{z^{n+1}} = \infty \implies \exists! \zeta \in (0, R) : g'(\zeta) = 0$$

where $g(z) = f(z)z^{-n-1}$.

Proof: $(f(z)z^{-n-1})'' > 0$ for $0 < z < R$, hence it is convex. □

Corollary (Saddle point bound)

f analytic at $z = 0$, $f_n \geq 0$, radius of convergence R . Assume $\lim_{z \rightarrow R-} f(z) = \infty$ and that $f(z)$ is not a polynomial. Let ζ be the unique positive solution of $\frac{zf'(z)}{f(z)} = n + 1$. Then

$$f_n = [z^n]f(z) \leq \frac{f(\zeta)}{\zeta^n}$$

Proof: ζ is saddle point if $\left(\frac{f(z)}{z^{n+1}}\right)' = 0$:

$$\left(\frac{f(z)}{z^{n+1}}\right)' = \frac{f'(z)}{z^{n+1}} - (n+1)\frac{f(z)}{z^{n+2}} = 0 \iff \frac{zf'(z)}{f(z)} = n+1 \iff z = \zeta.$$

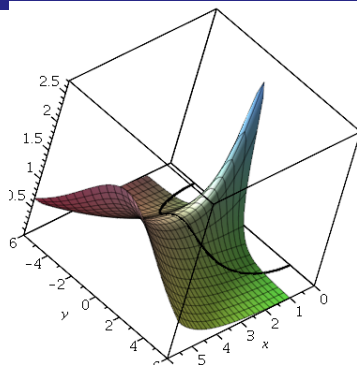
Choose γ to be the circle $|z| = \zeta$ and use the estimate

$$\left|\frac{f(z)}{z^{n+1}}\right| \leq \frac{f(\zeta)}{\zeta^{n+1}} \text{ for the integration.}$$



The saddle point method

Proof idea:



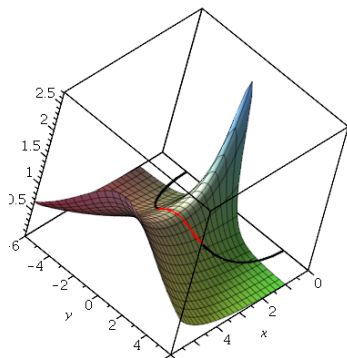
Example

$$f(z) = e^z \implies zf'(z)/f(z) = z = n+1$$

$$\frac{1}{2\pi i} \oint_{|z|=n+1} \frac{e^z}{z^{n+1}} dz = \frac{e^{n+1}}{(n+1)^n} \asymp \frac{\sqrt{n}}{n!}.$$

The saddle point method

Idea of the saddle point method to compute $\int_A^B e^{f(z)} dz$:
 (Note: saddle points of $e^{f(z)}$ and of $f(z)$ coincide!)



- Choose integration contour through a saddle point.
- Cut off the tails.
- Approximate the integrand:

$$f(z) = f(\zeta) + \frac{1}{2}f''(\zeta)(z-\zeta)^2 + O(\eta_n)$$

$\eta_n \rightarrow 0$ uniformly.

- Add new tails, using the simplified integrand:

$$\int_{\gamma} \asymp \int_{-\infty}^{\infty} e^{-|f''(\zeta)|z^2/2} dz$$

Example (Stirling's formula revisited)

$$n! \sim \frac{n^n}{e^n} \sqrt{2\pi n}$$

$$\frac{1}{n!} = [x^n] e^x = \frac{1}{2\pi i} \oint \frac{e^x}{x^{n+1}} dx = \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} e^{re^{i\theta}} e^{-in\theta} d\theta$$

$$re^{i\theta} = r + i\theta r - \frac{\theta^2}{2}r + O(\theta^3 r)$$

$$e^{re^{i\theta}} \sim e^r \exp\left(i\theta r - \frac{\theta^2}{2}r\right), \text{ as } r \rightarrow \infty, \text{ uniformly for } \theta = o(r^{-1/3})$$

Example (Stirling's formula revisited – cont'd)

$$\left| e^{re^{i\theta}} \right| = e^{\Re(re^{i\theta})} = e^{r \cos \theta}$$

$$\cos \theta \sim 1 - \frac{\theta^2}{2}, \quad \theta \text{ small}$$

$$\implies \left| e^{re^{i\theta}} \right| \leq e^r \exp \left(-\frac{\theta_0^2}{2} r \right), \text{ as } r \rightarrow \infty,$$

uniformly for $\theta_0 \leq |\theta| \leq \pi$

└ The saddle point method

Example (Stirling's formula revisited – cont'd)

If $x = re^{i\theta}$, then with $\delta(r) = r^{-2/5}$

$$e^x \sim e^r e^{i\theta r - \theta^2 r/2}, \quad \text{as } r \rightarrow \infty, \text{ uniformly for } |\theta| \leq \delta(r)$$

and

$$|e^x| = O\left(e^r \exp\left(-\frac{r^{1/5}}{2}\right)\right) = o\left(\frac{e^r}{\sqrt{r}}\right), \quad r \rightarrow \infty, |\theta| \geq \delta(r)$$

Hence

$$\int_{-\pi}^{\pi} e^{re^{i\theta}} e^{-in\theta} d\theta = \underbrace{\int_{-\delta}^{\delta} e^{re^{i\theta}} e^{-in\theta} d\theta}_{(e^r \cdot \sqrt{2\pi/r}) e^{-(r-n)^2/2r} + O(e^{-Cr^{1/5}})} + \underbrace{\int_{|\theta| \geq \delta} e^{re^{i\theta}} e^{-in\theta} d\theta}_{o(e^r/\sqrt{r})}$$

Example (Stirling's formula revisited – cont'd)

We get

$$r^n [x^n] e^x \sim \frac{e^r}{\sqrt{2\pi r}} \exp\left(-\frac{(r-n)^2}{2r}\right)$$

for $r \rightarrow \infty$ and uniformly for all integers n .

Inserting $r = n \implies$ Stirling's formula.

Large powers

Large powers appear frequently in combinatorial enumeration:

- $\mathcal{A} = \text{Seq}_k(\mathcal{B})$, then

$$[z^n]A(z) = [z^n]B(z)^k.$$

- If $f(z) = z\phi(f(z))$ then by Lagrange inversion we get

$$[z^n]f(z) = \frac{1}{n}[z^{n-1}]\phi(z)^n.$$

Theorem

Let $f(z) = \sum_{n \geq 0} f_n z^n$, $f_n \geq 0$, suppose there are $n_1 < n_2 < n_3$ such that $f_{n_1} f_{n_2} f_{n_3} > 0$ and $\gcd(n_2 - n_1, n_3 - n_1) = 1$.

Then, for all $\varepsilon > 0$,

$$f_{n,k} = [z^n] f(z)^k = \frac{f(\rho)}{\rho^n \sqrt{2\pi\sigma^2 k}} \left(\exp\left(-\frac{r^2}{2\sigma^2 k}\right) + O\left(k^{-\frac{1}{2}+\varepsilon}\right) \right),$$

as $k \rightarrow \infty$ and uniformly for μ and r such that $n = \mu k + r$ and $\mu \in [a, b]$, where $a > 0$ and $b > 0$. ρ and σ^2 are given by

$$\frac{\rho f'(\rho)}{f(\rho)} = \mu, \quad \sigma^2 = \mu - \mu^2 + \frac{\rho^2 f''(\rho)}{f(\rho)}.$$

Proof: Use Cauchy's integral formula with $|z| = \rho$, ρ being the saddle point of $\frac{f(z)^k}{z^{\mu k + 1}}$.

Large powers

Then, as $\mu k + 1 \sim n$, we get

$$f_{n,k} \sim \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\rho e^{it})^k \rho^{-n} e^{-itn} dt$$

The maximum of $|f(\rho e^{it})|$ is at $t = 0$, thus for $\delta > 0$ there is $0 < \alpha < 1$ such that

$$|f(\rho e^{it})| \leq \alpha f(\rho)$$

when $\rho \in [\rho_a, \rho_b]$ and $\delta \leq |t| \leq \pi$.

We choose ρ_a, ρ_b so that $\rho \in [\rho_a, \rho_b]$ is equivalent to $\mu \in [a, b]$.

Moreover,

$$\log f(\rho e^{it}) = \log f(\rho) + i\mu t - \frac{\sigma^2 t^2}{2} + g(t^3)$$

and choose δ so small that $|g(t^3)| < \frac{\sigma^2 t^2}{3}$, uniformly for $\rho \in [\rho_a, \rho_b]$, if $|t| < \delta$.

$$\begin{aligned}
f_{n,k} &= \frac{1}{2\pi} \int_{-\delta}^{\delta} f\left(\rho e^{it}\right)^k \rho^{-n} e^{-int} dt + O\left(f(\rho)^k \rho^{-n} \alpha^k\right) \\
&= \frac{f(\rho)^k}{2\pi \rho^n} \left(\int_{-\delta}^{\delta} \exp\left(-irt - \frac{\sigma^2 k}{2} t^2 + kg(t^3)\right) dt + O\left(\alpha^k\right) \right) \\
&= \frac{f(\rho)^k}{2\pi \rho^n} \left(\int_{-k^{-\frac{1}{2}+\varepsilon}/\sigma}^{k^{-\frac{1}{2}+\varepsilon}/\sigma} \exp\left(-irt - \frac{\sigma^2 k}{2} t^2 + O\left(k^{-\frac{1}{2}+3\varepsilon}\right)\right) dt \right. \\
&\quad \left. + O\left(\alpha^k\right) + O\left(\delta \exp\left(-\frac{k^{2\varepsilon}}{6}\right)\right) \right)
\end{aligned}$$

Now substitute $u = \sqrt{\sigma^2 k} \cdot t$ and obtain

$$f_{n,k} = \frac{f(\rho)^k}{2\pi \rho^n \sqrt{\sigma^2 k}} \int_{-k^\varepsilon}^{k^\varepsilon} \exp\left(-i \frac{r}{\sqrt{\sigma^2 k}} u - \frac{u^2}{2}\right) \left(1 + O\left(k^{-\frac{1}{2}+3\varepsilon}\right)\right) du.$$

└ Large powers

$$f_{n,k} = \frac{f(\rho)^k}{2\pi\rho^n\sqrt{\sigma^2k}} \int_{-k^\varepsilon}^{k^\varepsilon} \exp\left(-i\frac{r}{\sqrt{\sigma^2k}}u - \frac{u^2}{2}\right) \left(1 + O\left(k^{-\frac{1}{2}+3\varepsilon}\right)\right) du.$$

Finally, use

$$\int_{-\infty}^{\infty} e^{-Cx^2-Dx} dx = \frac{\exp\left(\frac{D^2}{4C}\right)}{\sqrt{C}} \underbrace{\int_{-\infty}^{\infty} e^{-v^2} dv}_{=\sqrt{\pi}}$$

with

$$C = \frac{1}{2} \text{ and } D = \frac{ir}{\sqrt{\sigma^2k}}.$$



Hayman admissible functions

$$y_n = [x^n]y(x) = \frac{1}{2\pi i} \int_{|x|=r} \frac{y(x)}{x^{n+1}} dx$$

If $y(x)x^{-n}$ is concentrated at a saddle point ζ_n :

$$y_n \sim \frac{y(\zeta_n)\zeta_n^{-n}}{\sqrt{2\pi\sigma^2(\zeta_n)}}, \quad (n \rightarrow \infty)$$

where

$$\sigma^2(\zeta) = \left[\frac{\partial^2}{\partial v^2} \log y(\zeta e^v) \right]_{v=0} = \frac{\zeta^2 y''(\zeta)}{y(\zeta)} - \left(\frac{\zeta y'(\zeta)}{y(\zeta)} \right)^2 + \frac{\zeta y'(\zeta)}{y(\zeta)}.$$

Definition (Hayman admissible function)

- 1 $f(x)$ analytic in $|x| < R$, positive in (R_0, R)
- 2 $\exists \delta : (R_0, R) \rightarrow (0, \pi)$ such that in (R_0, R) :

$$f(re^{i\theta}) \sim f(r) \exp\left(i\theta a(r) - \frac{\theta^2}{2} b(r)\right), r \rightarrow R,$$

uniformly for $|\theta| \leq \delta(r)$, where
 $a(r) = rf'(r)/f(r)$, $b(r) = ra'(r)$

- 3 In (R_0, R)

$$f(re^{i\theta}) = o\left(\frac{f(r)}{\sqrt{b(r)}}\right), r \rightarrow R,$$

uniformly for $\delta(r) \leq |\theta| \leq \pi$

- 4 $b(r) \rightarrow \infty$ as $r \rightarrow R$.

Theorem (Hayman 1956)

Let $f(x)$ be H -admissible ($\in \mathcal{H}_R$). Then, as $r \rightarrow R$, we have

$$f_n = \frac{f(r)}{r^n \sqrt{2\pi b(r)}} \left(\exp \left(-\frac{(a(r) - n)^2}{2b(r)} \right) + o(1) \right),$$

uniformly for all integers n .

Examples: e^x , e^{e^x} , $\exp \left(\frac{1}{1-x} \right)$ are admissible.

Corollary

Let $f(z) \in \mathcal{H}_R$, then

$$f_n \sim \frac{f(\zeta_n)}{\sqrt{2\pi b(\zeta_n)} \zeta_n^n}$$

where ζ_n is defined by $a(\zeta_n) = n$.

Remark: For sufficiently large n , ζ_n is unique.

Proof: Let $\delta = \delta(r)$

$$\begin{aligned} f_n r^n &= \frac{1}{2\pi} \int_{-\delta}^{2\pi-\delta} f(re^{i\theta}) e^{-in\theta} d\theta \\ &= \underbrace{\frac{1}{2\pi} \int_{-\delta}^{\delta} f(re^{i\theta}) e^{-in\theta} d\theta}_{I_1} + \underbrace{\frac{1}{2\pi} \int_{\delta}^{2\pi-\delta} f(re^{i\theta}) e^{-in\theta} d\theta}_{I_2} \end{aligned}$$

Clearly, $I_2 = o\left(\frac{f(r)}{\sqrt{b(r)}}\right)$.

$$\begin{aligned} I_1 &= \frac{f(r)}{2\pi} \int_{-\delta}^{\delta} \exp\left(i(a(r) - n)\theta - \frac{b(r)}{2}\theta^2\right) (1 + o(1)) d\theta \\ &= \frac{f(r)}{2\pi} \left(\int_{-\delta}^{\delta} \exp\left(i(a(r) - n)\theta - \frac{b(r)}{2}\theta^2\right) d\theta + o\left(\frac{1}{\sqrt{b(r)}}\right) \right) \end{aligned}$$

└ Hayman admissible functions

Remaining part:

$$\frac{f(r)}{2\pi} \int_{-\delta}^{\delta} \exp \left(i(a(r) - n)\theta - \frac{b(r)}{2}\theta^2 \right) d\theta$$

$$\frac{|f(re^{i\delta})|}{f(r)} \sim e^{-b(r)\delta^2/2} = o\left(\frac{1}{\sqrt{b(r)}}\right) \implies b(r)\delta^2 \rightarrow \infty$$

Substitute $y = \theta\sqrt{b(r)/2}$:

$$\begin{aligned} I_1 &\sim \frac{f(r)}{\pi\sqrt{2b(r)}} \int_{-\delta\sqrt{b(r)/2}}^{\delta\sqrt{b(r)/2}} \exp \left(-y^2 + iy \frac{a(r) - n}{\sqrt{b(r)}} \sqrt{2} \right) dy \\ &\sim \frac{f(r)}{\pi\sqrt{2b(r)}} \underbrace{\int_{-\infty}^{\infty} \exp \left(-y^2 + iy \frac{a(r) - n}{\sqrt{b(r)}} \sqrt{2} \right) dy}_{\sqrt{\pi} \exp(-(a(r)-n)^2/(2b(r)))} \quad \square \end{aligned}$$

Consequences:

- Admissible functions have at least exponential growth, their maximal modulus is on the positive real line.
- "Distribution" of partial sums asymptotically Gaussian, i.e.,

$$\sum_{n \leq a(r) + \omega \sqrt{2b(r)}} a_n r^n \sim \frac{f(r)}{\sqrt{\pi}} \int_{-\infty}^{\omega} e^{-t^2} dt.$$

- Growth of derivatives: $f^{(k)}(r) \sim f(r)(a(r)/r)^k$, as $r \rightarrow R$.

Theorem (Closure Properties, Hayman 1956)

- 1 $p(x)$ real polynomial, $e^{p(x)} = \sum a_n x^n$, almost all a_n positive, then $e^{p(x)} \in \mathcal{H}$
- 2 $f_1(x), f_2(x) \in \mathcal{H} \implies e^{f_1(x)} \in \mathcal{H}$ and $f_1(x)f_2(x) \in \mathcal{H}$
- 3 $f(x) \in \mathcal{H}_R$, $p(x)$ polynomial with positive leading coefficient and $p(R) > 0 \implies f(x)p(x) \in \mathcal{H}_R$
- 4 $f(x) \in \mathcal{H}$ and $g(x) = O(f^{1-\delta}) \implies f(x) + g(x) \in \mathcal{H}$

Remark: Maple packages by Salvy et al. (Algolib)

Example (Stirling's formula)

Find asymptotic of $n!$:

$$\frac{1}{n!} = [z^n]f(z) = [z^n]e^z$$

$$e^z \in \mathcal{H},$$

$$a(r) = \frac{rf'(r)}{f(r)} = r, \quad b(r) = ra'(r) = r.$$

Saddle point equation: $a(\zeta_n) = n$.

This implies $\zeta_n = n$ and thus

$$f_n = \frac{1}{n!} \sim \frac{1}{\sqrt{2\pi b(\zeta_n)}} \frac{f(\zeta_n)}{\zeta_n^n} = \frac{1}{\sqrt{2\pi n}} \frac{e^n}{n^n}.$$

Example (Bell numbers)

$e^z \in \mathcal{H}$, likewise $e^z - 1 \in \mathcal{H}$ and thus $f(z) = e^{e^z - 1} \in \mathcal{H}$.
This function corresponds to $\text{Set}(\text{Set}(\mathcal{Z}) \setminus \{\varepsilon\})$.

We get

$$a(r) = \frac{rf'(r)}{f(r)} = \frac{re^r f(r)}{f(r)} = re^r, \quad b(r) = ra'(r) = r(1+r)e^r,$$

Saddle point equation: $\zeta_n e^{\zeta_n} = n$.

Taking logarithms and then using $\zeta_n = \log n - \log \zeta_n$ gives

$$\zeta_n \sim \log n - \log \log n + o(1).$$

Thus

$$B_n = n![z^n]f(z) \sim \frac{n!}{\sqrt{2\pi\zeta_n^2 e^{\zeta_n}}} \cdot \frac{e^{e^{\zeta_n} - 1}}{\zeta_n^n} \sim \frac{n!}{e\sqrt{2\pi n \log n}} \cdot \frac{e^{n/\log n}}{\zeta_n^n}$$

Example (Involutions)

An involution is a permutation with no cycles of length 3 or more. Thus

$$f(z) = e^{z + \frac{z^2}{2}} = \sum_{n \geq 0} f_n \frac{z^n}{n!} \in \mathcal{H}$$

We compute

$$a(r) = r + r^2, \quad b(r) = r + 2r^2, \quad \zeta_n^2 + \zeta_n = n$$

and so $\zeta_n = \sqrt{n} - \frac{1}{2} + \frac{1}{8\sqrt{n}} + O(n^{-3/2})$.

This implies

$$f(\zeta_n) = e^{\zeta_n + \frac{1}{2}\zeta_n^2} = e^{\frac{1}{2}(\zeta_n + n)} = \exp\left(\frac{\sqrt{n}}{2} - \frac{1}{4} + O(n^{-1/2})\right)$$

and $b(\zeta_n) = \zeta_n + 2\zeta_n^2 \sim 2\zeta_n^2 \sim 2n$.

Example (Involutions – cont'd)

$$\begin{aligned}
\zeta_n^n &= \left(\sqrt{n} - \frac{1}{2} + \frac{1}{8\sqrt{n}} + O(n^{-3/2}) \right)^n \\
&= n^{n/2} \left(1 - \frac{1}{2\sqrt{n}} + \frac{1}{8n} + O(n^{-2}) \right)^n \\
&= n^{n/2} \exp \left(n \left[\left(-\frac{1}{2\sqrt{n}} + \frac{1}{8n} \right) - \frac{1}{2} \left(-\frac{1}{2\sqrt{n}} + \frac{1}{8n} \right)^2 + O\left(\frac{1}{n^{3/2}}\right) \right] \right) \\
&= n^{n/2} \exp \left(-\frac{\sqrt{n}}{2} + O(n^{-1/2}) \right) \sim n^{n/2} e^{-\sqrt{n}/2}
\end{aligned}$$

Thus

$$\frac{f_n}{n!} \sim \frac{f(\zeta_n)}{\sqrt{2\pi b(\zeta_n)} \zeta_n^n} \sim \frac{\exp\left(\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right)}{2n^{n/2} \sqrt{\pi n}}$$