

Analytic Combinatorics: Complex-analytic Methods and Applications

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Multivariate Hayman admissibility

Notations

$\mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_d^{n_d}$

For a function $y(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}^d$:

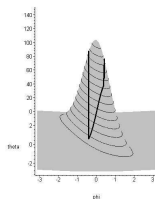
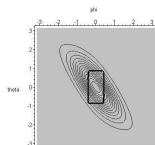
$\mathbf{a}(\mathbf{x}) = (a_j(\mathbf{x}))_{j=1, \dots, d}$ vector of the logarithmic (partial) derivatives, i.e.,

$$a_j(\mathbf{x}) = \frac{x_j y_{x_j}(\mathbf{x})}{y(\mathbf{x})},$$

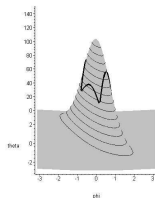
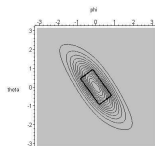
$B(\mathbf{x}) = (B_{jk}(\mathbf{x}))_{j,k=1, \dots, d}$ matrix of the second logarithmic (partial) derivatives of $y(\mathbf{x})$, i.e.,

$$B_{jk}(\mathbf{x}) = \frac{x_j x_k y_{x_j x_k}(\mathbf{x}) + \delta_{jk} x_j y_{x_j}(\mathbf{x})}{y(\mathbf{x})} - \frac{x_j x_k y_{x_j}(\mathbf{x}) y_{x_k}(\mathbf{x})}{y(\mathbf{x})^2},$$

Multivariate Hayman admissibility



The typical shape of $|y(re^{i\varphi}, se^{i\theta})|$ (for fixed r and s)



$B(\mathbf{r})$ positive definite

$\mathbf{v}_1(\mathbf{r}), \dots, \mathbf{v}_d(\mathbf{r})$ orthonormal basis of eigenvectors

Definition (Multivariate Hayman admissible function)

A function

$$y(\mathbf{x}) = \sum_{n_1, \dots, n_d \geq 0} y_{n_1 \dots n_d} x_1^{n_1} \cdots x_d^{n_d},$$

with real coefficients $y_{n_1 \dots n_d}$ is called **H-admissible** in $\mathcal{R} \subset \mathbb{R}^d$ if it is entire and positive in $\mathcal{R}$ (for some fixed $R_0 > 0$) and satisfies

- (I) $B(\mathbf{r})$ is positive definite and there exists a function $\delta : \mathbb{R}^d \rightarrow [-\pi, \pi]^d$ such that

$$y(\mathbf{r}e^{i\theta}) \sim y(\mathbf{r}) \exp \left(i\theta \mathbf{a}(\mathbf{r})^t - \frac{\theta B(\mathbf{r}) \theta^t}{2} \right), \text{ as } \mathbf{r} \rightarrow \infty \text{ in } \mathcal{R},$$

uniformly for $\mathbf{r} \in \Delta(\mathbf{r})$

$$\theta \in \Delta(\mathbf{r}) = \left\{ \sum_{j=1}^d \mu_j \mathbf{v}_j(\mathbf{r}) \text{ with } |\mu_j| \leq \delta_j(\mathbf{r}), \text{ for } j = 1, \dots, d \right\}.$$

Definition (Mv Hayman admissible function – cont'd)

(II) The asymptotic relation

$$y(\mathbf{r}e^{i\theta}) = o\left(\frac{y(\mathbf{r})}{\sqrt{\det B(\mathbf{r})}}\right), \text{ as } \mathbf{r} \rightarrow \infty \text{ in } \mathcal{R},$$

holds uniformly for $\theta \notin \Delta(\mathbf{r})$.

(III) The eigenvalues $\lambda_1(\mathbf{r}), \dots, \lambda_d(\mathbf{r})$ of $B(\mathbf{r})$ satisfy

$$\lambda_i(\mathbf{r}) \rightarrow \infty, \text{ as } \mathbf{r} \rightarrow \infty \text{ in } \mathcal{R}, \text{ for all } i = 1, \dots, d.$$

(IV) We have $B_{ii}(\mathbf{r}) = o(a_i(\mathbf{r})^2)$, as $\mathbf{r} \rightarrow \infty$ in $\mathcal{R}$.

(V) For $\mathbf{r}$ sufficiently large and $\theta \in [-\pi, \pi]^d \setminus \{\mathbf{0}\}$:

$$|y(\mathbf{r}e^{i\theta})| < y(\mathbf{r}).$$

Remark: (I)–(III) imply $\|B(\mathbf{r})\| = o(\|\mathbf{a}(\mathbf{r})\|^2)$, as $\mathbf{r} \rightarrow \infty$ in $\mathcal{R}$.

Theorem

Let $y(\mathbf{x})$ be H -admissible. Then as $\mathbf{r} \rightarrow \infty$ we have

$$y_{\mathbf{n}} \sim \frac{y(\mathbf{r})}{\mathbf{r}^{\mathbf{n}} (2\pi)^{d/2} \sqrt{\det B(\mathbf{r})}} \exp \left(-\frac{1}{2} (\mathbf{a}(\mathbf{r}) - \mathbf{n}) B(\mathbf{r})^{-1} (\mathbf{a}(\mathbf{r}) - \mathbf{n})^t \right),$$

uniformly for all $\mathbf{n} \in \mathbb{Z}^d$.

Proof (sketch): Let $\mathcal{E} = \left\{ \sum_j \mu_j \mathbf{v}_j \mid |\mu_j| \leq \delta_j \right\}$.

Then we have $y_{\mathbf{n}} \mathbf{r}^{\mathbf{n}} = I_1 + I_2$ with

$$I_1 = \frac{1}{(2\pi)^d} \int \cdots \int_{\mathcal{E}} \frac{y(\mathbf{r} e^{i\theta})}{e^{i\mathbf{n}\theta^t}} d\theta_1 \cdots d\theta_d$$

and

$$I_2 = \frac{1}{(2\pi)^d} \int \cdots \int_{[-\pi, \pi]^d \setminus \mathcal{E}} \frac{y(\mathbf{r} e^{i\theta})}{e^{i\mathbf{n}\theta^t}} d\theta_1 \cdots d\theta_d = o \left(\frac{y(\mathbf{r})}{\sqrt{\det B(\mathbf{r})}} \right)$$

Corollary

Let $y(x)$ be an H -admissible function. If $n_1, \dots, n_d \rightarrow \infty$ in such a way that all coordinates of the solution $\rho_{\mathbf{n}}$ of $\mathbf{a}(\rho_{\mathbf{n}}) = \mathbf{n}$ tend to infinity as well, then we have

$$y_{\mathbf{n}} \sim \frac{y(\rho_{\mathbf{n}})}{\rho_{\mathbf{n}}^{\mathbf{n}} \sqrt{(2\pi)^d \det B(\rho_{\mathbf{n}})}},$$

where $\rho_{\mathbf{n}}$ is uniquely defined for sufficiently large $\mathbf{n}$, i.e., $\min_j n_j > N_0$ for some $N_0 > 0$.

A Class of H-admissible Functions

Theorem

Let $P(\mathbf{z}) = \sum_{\mathbf{m} \in M} b_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$ be a polynomial in $\mathbf{z}$ with real coefficients and

$$y(\mathbf{z}) = \sum_{\mathbf{n} \in \mathbf{n}^d} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} = e^{P(\mathbf{z})}$$

Then the following conditions are equivalent

1 $\forall \vartheta \in [-\pi, \pi]^d \setminus \mathbf{0}$ we have:

$$\left| y(\mathbf{r} e^{i\vartheta}) \right| < y(\mathbf{r}), \text{ for sufficiently large } \mathbf{r} \text{ (in } \mathcal{R})$$

2 $y(\mathbf{z})$ is H-admissible in $\mathcal{R}$.

Theorem

Let $P(z_1, \dots, z_d) = \sum_{j=1}^L a_j z_1^{k_{1j}} \cdots z_d^{k_{dj}}$ be a polynomial in d variables and with positive coefficients $a_j > 0$

$\mathbf{K}_j := (k_{1j}, \dots, k_{dj})$: exponent vector of the j th monomial

Then $e^{P(z_1, \dots, z_d)}$ is admissible if and only if the span of the

$\mathbf{K}_j := (k_{1j}, \dots, k_{dj})$ over $\mathbb{Z}$ equals $\mathbb{Z}^d$.

Equivalently, this means that

$$\mathbf{K}_j \theta^T \equiv 0 \pmod{2\pi}, \quad j = 1 \dots, L,$$

has only the trivial solution $\theta \equiv \mathbf{0} \pmod{2\pi}$.

Closure Properties

- If $y(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$, then $e^{y(\mathbf{x})} \in \mathcal{H}_{\mathcal{R}}$.
- If $y_1(\mathbf{x}), y_2(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$ and $\det(B_1 + B_2) \leq C \min(\det B_1, \det B_2)$ and B_1 and B_2 have same eigenvectors, then $y_1(\mathbf{x})y_2(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$.
- If $y(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$, $p(\mathbf{x})$ polynomial with positive coefficients, then $y(\mathbf{x})p(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$.
- $y(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$, $f(\mathbf{x})$ analytic, real if $\mathbf{x} \in \mathbb{R}^d$ and

$$\max_{x_i=r_i, i=1, \dots, d} |f(\mathbf{x})| = O\left(y(\mathbf{r})^{1-\delta}\right), \text{ as } \mathbf{r} \rightarrow \infty.$$

Then $y(\mathbf{x}) + f(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$.

- If $y(x) \in \mathcal{H}$, then $e^{uy(x)} \in \mathcal{H}_{\mathcal{R}}$ with suitable $\mathcal{R}$.
- If $y(\mathbf{x}) \in \mathcal{H}_{\mathcal{R}}$, then $\exp(y(\mathbf{x}_1)y(\mathbf{x}_2)) \in \mathcal{H}_{\tilde{\mathcal{R}}}$ with suitable $\tilde{\mathcal{R}}$.

Example (Stirling numbers of the second kind)

gen. function $y(z, u) = e^{u(e^z - 1)}$

Example (Permutations with bounded cycle length)

cycle length $\leq \ell \implies$ gen. function

$$y(z, u) = \exp \left(u \sum_{i=1}^{\ell} \frac{z^i}{i} \right)$$

exponent is polynomial $\rightarrow$ check conditions of theorem

Example (Partitions of a set of partitions)

partitions of the set of subsets of a given partition

$$y(z, u) = \exp \left(u \left(e^{e^z - 1} - 1 \right) \right)$$

Example (Set partitions with bounded block size)

$$y(z, u) = \exp \left(u \sum_{i=1}^{\ell} \frac{z^i}{i!} \right)$$

Example (Coverings of complete bipartite graphs with complete bipartite graphs)

$$y(z, u) = \exp ((e^z - 1)(e^u - 1))$$

Example (Partitions of sets with coloured elements (out of d colours))

$S \subseteq \mathbb{Z}^d$, S finite, in each block the vector of the number of elements per colour must be an element of S .

$$y(\mathbf{z}) = \exp \left(\sum_{\mathbf{n} \in S} \frac{\mathbf{z}^{\mathbf{n}}}{n_1! \cdots n_d!} \right)$$

Rule of thumb

- Small singularities:
 - $f(z)$ grows subexponentially when z approaches the singularity
 - Use singularity analysis
- Large singularities:
 - $f(z)$ grows at least exponentially when z approaches the singularity
 - Use saddle point method

Multivariate asymptotics and limiting distributions

Introductory remarks

X ... random variable on probability space Ω .

Assumption: $X(\Omega) \subseteq \mathbb{N}$; $p_k := \mathbf{P}\{X = k\}$

distribution function: $F(x) = \mathbf{P}\{X \leq x\} = \sum_{k \leq x} p_k$.

probability generating function: $p(u) = \sum_{k \geq 0} p_k u^k$;

analytic for $|u| < 1$ since $p(1) = 1$.

$$p_k = \frac{1}{k!} p^{(k)}(0).$$

Moments: $\mathbf{E}X = \sum_{k \geq 0} k p_k$, generally: $\mathbf{E}h(X) = \sum_{k \geq 0} h(k) p_k$;

for $h(x) = e^{itx}$: characteristic function $\phi_X(t) = \mathbf{E}e^{itX}$.

In particular:

$$\mu_r = \mathbf{E}X^r = \left(u \frac{\partial}{\partial u}\right)^r p(u) \big|_{u=1};$$

$$\mu = \mu_1 = p'(1), \quad \sigma^2 = \mu_2 - \mu^2 = p''(1) + p'(1) - p'(1)^2.$$

In combinatorics: $\mathcal{A}$ combinatorial structure;
probability spaces $(\mathcal{A}_n, P_n)$, where
 P_n is a probability distribution on $\mathcal{A}_n$,

$X_n(a) = \chi(a)$ for some random $a \in \mathcal{A}_n$ (recall: $\chi : \mathcal{A} \rightarrow \mathbb{N}$, or $\mathbb{N}^m$)

$$\mathcal{A}_{n,k} = \{a \in \mathcal{A} : |a| = n, \chi(a) = k\}, \quad a_{n,k} = |\mathcal{A}_{n,k}|$$

Uniform distribution:

$$P[X_n = k] = \frac{a_{n,k}}{a_n} \text{ where } a_n = \sum_k a_{n,k}$$

Then $(\mathcal{A}, \chi)$ corresponds to

$$A(z, u) = \sum_{n,k} a_{n,k} z^n u^k.$$

What is the limiting distribution of X_n ?

E.g.: X_n asymptotically Gaussian if $\exists \mu_n, \sigma_n, \sigma_n \rightarrow \infty$, s.t.

$$\sum_{k \leq \mu_n + x\sigma_n} a_{n,k} = a_n \Phi(x) + o(a_n)$$

as $n \rightarrow \infty$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt.$$

Question: Is X_n asymptotically Gaussian?

Distributional quantities encoded in $A(z, u)$:

$p_n(u) = [z^n]A(z, u)/[z^n]A(z, 1)$ Thus

$$\mathbf{E}X_n = \frac{[z^n]A_u(z, 1)}{[z^n]A(z, 1)}, \quad \mathbf{Var}X_n = \frac{[z^n][A_{uu}(z, 1) + A_u(z, 1)]}{[z^n]A(z, 1)} - (\mathbf{E}X_n)^2$$

$$m_n(t) = \frac{[z^n]A(z, e^t)}{[z^n]A(z, 1)}, \quad \phi_n(t) = \frac{[z^n]A(z, e^{it})}{[z^n]A(z, 1)}$$

Example (Cycles in permutations)

$$\mathcal{F} = \text{Set}(\mu_{\text{Cyc}}(\mathcal{Z})) \implies f(z, u) = \exp\left(u \log \frac{1}{1-z}\right) = (1-z)^{-u}$$

Thus

$$p_n(u) = \binom{n+u-1}{n} = \frac{\Gamma(n+u)}{\Gamma(u)n!}$$

If $|u-1| < \varepsilon$ then

$$p_n(u) = \frac{n^{u-1}}{\Gamma(u)} \left(1 + O\left(\frac{1}{n}\right)\right) \sim (e^{u-1})^{\log n}$$

Example (Cycles in permutations – cont'd)

For the expectation we get

$$\mathbb{E}X_n = \frac{[z^n] \frac{1}{1-z} \log \frac{1}{1-z}}{[z^n] \frac{1}{1-z}} \sim \log n$$

and similarly $\text{Var}X_n \sim \log n$. Let $\phi_n^*(t) = \mathbb{E} \exp \left(it \frac{X_n - \mathbb{E}X_n}{\sqrt{\text{Var}X_n}} \right)$, then

$$\phi_n^*(t) \sim \frac{e^{-it\sqrt{\log n}}}{\Gamma(e^{it/\sqrt{\log n}})} \exp \left((e^{it/\sqrt{\log n}} - 1) \log n \right) \sim e^{-\frac{t^2}{2} + O\left(\frac{1}{\sqrt{\log n}}\right)}.$$

Theorem (Goncharov 1944)

The number of cycles in random permutations of size n is asymptotically normal with mean and variance proportional to $\log n$

Important inequalities for random variables:

Theorem (Markov's inequality)

If $X \geq 0$ and $\mu = \mathbf{E}X$ then $\mathbf{P}\{X \geq t\mu\} \leq 1/t$.

Theorem (Chebyshev's inequality)

If X is real-valued with $\mu = \mathbf{E}X$ and $\sigma^2 = \mathbf{Var}X$, then

$$\mathbf{P}\{|X - \mu| \geq t\sigma\} \leq 1/t^2.$$

Definition

A sequence $(X_n)_{n \geq 0}$ of random variables with $\mathbf{E}X_n = \mu_n$ is called **asymptotically concentrated**, if

$$\forall \varepsilon > 0 : \lim_{n \rightarrow \infty} \mathbf{P}\left\{1 - \varepsilon < \frac{X_n}{\mu_n} < 1 + \varepsilon\right\} = 1.$$

Theorem

Let $(X_n)_{n \geq 0}$ be a sequence of random variables, $\mu_n = \mathbf{E}X_n$, $\sigma_n^2 = \mathbf{Var}X_n$, satisfying $\lim_{n \rightarrow \infty} \sigma_n / \mu_n = 0$. Then X_n is asymptotically concentrated.

Proof: Set $\tilde{X}_n = X_n / \mu_n$. Then $\mathbf{E}\tilde{X}_n = 1$ and $\tilde{\sigma}_n = \sigma_n / \mu_n = o(1)$. Therefore $\mathbf{P}\left\{|\tilde{X}_n - 1| \geq \varepsilon\right\} \leq \tilde{\sigma}_n^2 / \varepsilon^2 = o(1)$. □

Example (Number of leaves in plane trees)

$$\mathcal{F} = \mu\{\circ\} \cup \{\circ\} \times \text{Seq}_{\geq 1}(\mathcal{F})$$

implies

$$F(z, u) = zu + \frac{zF(z, u)}{1 - F(z, u)}$$

Example (Number of leaves in plane trees – cont'd)

and thus

$$F(z, u) = \frac{1}{2} \left(1 + (u-1)z - \sqrt{1 - 2(u+1)z - (u-1)^2 z^2} \right)$$

$$F(z, 1) = \frac{1}{2} \left(1 - \sqrt{1 - 4z} \right)$$

$$F_u(z, 1) = \frac{1}{2} \left(z + \frac{z}{\sqrt{1 - 4z}} \right)$$

$$[z^n]F(z) = \frac{1}{n} \binom{2n-2}{n-1} \sim \frac{4^{n-1}}{\sqrt{\pi n^3}}$$

$$[z^n]F_u(z, 1) = [z^{n-1}] \frac{1}{2\sqrt{1-4z}} \sim \frac{1}{2} \frac{4^{n-1}}{\sqrt{\pi n}}$$

Thus $\mu \sim n/2$. Similarly $\sigma_n = O(\sqrt{n}) \implies$ asymptotically concentrated.

Theorem (Number of components in sequence constructions)

Let $\mathcal{F} = \text{Seq}(\mu\mathcal{A})$ and assume that there is $\rho > 0$ such that $A(z)$ is analytic in $|z| \leq \rho$, $A(\rho) = 1$, $A(0) = 0$, and $A(z)$ is aperiodic. Then the number of components in a random sequence of objects from $\mathcal{A}$ which is of size n satisfies

$$\mu_n = \frac{n}{\rho A'(\rho)} + O(1), \quad \sigma_n^2 = n \cdot \frac{\rho A''(\rho) + A'(\rho) - \rho A'(\rho)^2}{\rho^2 A'(\rho)^3} + O(1).$$

Proof: We have

$$F(z, u) = \frac{1}{1 - uA(z)}$$

and hence

$$F_u(z, 1) = \frac{A(z)}{(1 - A(z))^2} \text{ and } F_{uu}(z, 1) = \frac{2A(z)^2}{(1 - A(z))^3}$$

Moreover,

$$F(z, 1) = \frac{1}{A'(\rho)(\rho - z)} + O(1) \quad \left[\text{in fact } \frac{1}{A'(\rho)(\rho - z)} \left(1 + \frac{A''(\rho)}{2A'(\rho)}(\rho - z) + O((\rho - z)^2) \right) \right]$$

Therefore

$$F_u(z, 1) = \frac{1}{A'(\rho)^2(\rho - z)^2} + \frac{A''(\rho) - A'(\rho)^2}{A'(\rho)^3(\rho - z)} + O(1)$$

$$F_{uu}(z, 1) = \frac{2}{A'(\rho)^3(\rho - z)^3} + \frac{3A''(\rho) - 2A'(\rho)^2}{A'(\rho)^4(\rho - z)^2} + O\left(\frac{1}{\rho - z}\right)$$

By aperiodicity $F(z, 1)$ has only one dominant singularity. □