Analytic Combinatorics: Complex-analytic Methods and Applications

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Singularity analysis for implicit functions

Theorem (Implicit function theorem)

If $F(z,y)$ is analytic, $F(z_0,y_0)=0$, $F_y(z_0,y_0)\neq 0$, then there is a unique analytic function $y(z)$, defined in a neighbourhood of z_0 such that $F(z,y(z))\equiv 0$.

Thus, the solution of a function can only have a singularity at z_0 if $F_y(z_0,y_0)=0$. Then

$$
\begin{aligned}
F(z,y) = {} & F(z_0,y_0) + F_z(z_0,y_0)(z-z_0) + F_y(z_0,y_0)(y-y_0) \\
& + \frac{F_{yy}(z_0,y_0)(y-y_0)^2}{2} + F_{yz}(z_0,y_0)(y-y_0)(z-z_0) \\
& + \frac{F_{zz}(z_0,y_0)(z-z_0)^2}{2} + O\left(|z-z_0|^3 + |y-y_0|^3\right)
\end{aligned}
$$

Theorem

Assume that $F(z,y)$ is analytic and that $y(z)$ is a solution of the functional equation $F(z,y)=0$. Moreover, $F(z_0,y_0)=0$, $F_y(z_0,y_0)=0$, $F_z(z_0,y_0)\neq 0$, and $F_{yy}(z_0,y_0)\neq 0$. Then, locally around z_0, the function $y(z)$ admits the expansion

$$y(z)=y_0\pm\sqrt{\frac{2z_0F_z(z_0,y_0)}{F_{yy}(z_0,y_0)}}\sqrt{1-\frac{z}{z_0}}+O\left(\left|1-\frac{z}{z_0}\right|\right).$$

as $z\to z_0$.

Proof (sketch): Considerations of the previous slide $\leadsto$ main term
bootstrapping $\leadsto$ error term □

Applications

The practical procedure is as follows:

1. Preparation:
   a) Locate the dominant singularities.
   b) Check for analytic continuation.
2. Determine the singular expansions.
3. Apply transfers.
4. In case of more than one dominant singularity: Collect contributions and sum up

Example (Motzkin trees)

$$\mathcal{M} = \mathcal{Z} + \mathcal{Z} \times \mathcal{M} + \mathcal{Z} \times \mathcal{M} \times \mathcal{M},$$

thus $M(z) = z(1 + M(z) + M(z)^2$ and

$$M(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z}.$$

Singularities are at $z = -1$ and $z = 1/3$.
Analytic continuation possible. For $z \to 1/3$ we have

$$M(z) = 1 - \sqrt{3}\sqrt{1-3z} + O((1-3z)^{3/2})$$

Apply transfer:

$$m_n = \frac{-\sqrt{3} \cdot 3^n}{n^{3/2}\Gamma\left(-\frac{1}{2}\right)} + O(3^n n^{-5/2}) = \frac{3^n\sqrt{3}}{2\sqrt{\pi n^3}} + +O(3^n n^{-5/2})$$

Example (2-regular graphs)

Recall

$$\mathcal{G} = \mathtt{Set}(\mathtt{Cyc}^{(u)}_{\geq 3}(\mathcal{Z}))$$

and

$$g(z) = \exp\left(\frac{1}{2}\left(\log\frac{1}{1-z} - z - \frac{z^2}{2}\right)\right) = \frac{e^{-\frac{z}{2}-\frac{z^2}{4}}}{\sqrt{1-z}}.$$

as well as

$$e^{-\frac{z}{2}-\frac{z^2}{4}} = e^{-3/4}\left(1-(z-1)+\frac{1}{4}(z-1)^2 + O\left((z-1)^3\right)\right).$$

Thus

$$\frac{e^{-\frac{z}{2}-\frac{z^2}{4}}}{\sqrt{1-z}} = \frac{e^{-3/4}}{\sqrt{1-z}} + O(\sqrt{1-z})$$

Example (2-regular graphs – cont'd)

$$\frac{e^{-\frac{z}{2}-\frac{z^2}{4}}}{\sqrt{1-z}} = \frac{e^{-3/4}}{\sqrt{1-z}} + O(\sqrt{1-z})$$

Applying transfers gives

$$[z^n]g(z) = \frac{g_n}{n!} = \frac{e^{-3/4}n^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} + O(n^{-3/2})$$

and

$$g_n = \frac{n!e^{-3/4}}{\sqrt{\pi n}}\left(1 + O\left(\frac{1}{n}\right)\right).$$

More precisely:

$$g_n = \frac{n!e^{-3/4}}{\sqrt{\pi n}}\left(1 - \frac{5}{8n} + \frac{1}{128n^2} + O\left(n^{-3}\right)\right)$$

Example (Children's rounds)

$$\mathcal{R} = \texttt{Set}(\mathcal{Z} * \texttt{Cyc}(\mathcal{Z}))$$

Thus
$$R(z) = \exp\left(z \log \frac{1}{1-z}\right) = (1-z)^{-z}.$$

Simplifying gives

$$\begin{aligned} R(z) &= \frac{1}{1-z} \exp\left(-(1-z)\log\frac{1}{1-z}\right) \\ &= \frac{1}{1-z} - \log\frac{1}{1-z} + O\left((1-z)\log^2\frac{1}{1-z}\right) \\ &= \frac{1}{1-z} - \log\frac{1}{1-z} + O\left(\sqrt{1-z}\right) \end{aligned}$$

$R(z)$ has singularity at $z = 1$ and is analytic in $\mathbb{C} \setminus [1, \infty)$.

Example (Children's rounds – cont'd)

$$\frac{r_n}{n!} = 1 - \frac{1}{n} + O\left(n^{-3/2}\right)$$

Remark: Better estimate is

$$\frac{r_n}{n!} = 1 - \frac{1}{n} + O\left(\frac{\log n}{n^2}\right)$$

In general: $m, k \in \mathbb{N}$ then

$$[z^n](1-z)^m \log^k \frac{1}{1-z} \sim \frac{(\log n)^{k-1}}{n^{m+1}}$$

Example (Permutations with all cycles of odd length)

$$\mathcal{F} = \mathtt{Set}(\mathtt{Cyc}_{\mathrm{odd}}(\mathcal{Z}))$$

Thus

$$f(z) = \exp\left(\frac{\log\frac{1}{1-z} - \log\frac{1}{1+z}}{2}\right) = \exp\left(\frac{1}{2}\log\frac{1+z}{1-z}\right)$$
$$= \sqrt{\frac{1+z}{1-z}}$$

Singularities are $z = \pm 1$,
$f(z)$ is analytic on $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$.

Example (Permutations with all cycles of odd length – cont'd)

The singular expansions are

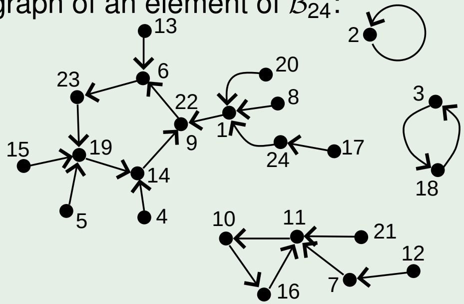
$$f(z) = \frac{\sqrt{2}}{\sqrt{1-z}} - \frac{1}{2\sqrt{2}}\sqrt{1-z} + O\left(|1-z|^{3/2}\right), \text{ as } z \to 1,$$
$$f(z) = \frac{1}{\sqrt{2}}\sqrt{1+z} + O\left(|1+z|^{3/2}\right), \text{ as } z \to -1.$$

and hence we get

$$\frac{f_n}{n!} = \frac{\sqrt{2}}{\sqrt{\pi n}} - \frac{(-1)^n + \frac{1}{2}}{2\sqrt{2\pi n^3}} + O\left(n^{-5/2}\right)$$

Example (Number of cycles in a random mapping)

Let $\mathcal{B} = \bigcup_{n\geq 0} \mathcal{B}_n$ where $\mathcal{B}_n = \{f : \{1,\ldots,n\} \to \{1,\ldots,n\}\}$, equipped with the uniform distribution.
Clearly: $|\mathcal{B}_n| = n^n$.
Functional digraph of an element of $\mathcal{B}_{24}$:



We see that

$$\mathcal{B} = \mathrm{Set}(\mathrm{Cyc}(\mathcal{T})), \qquad \text{where } \mathcal{T} = \mathcal{Z} * \mathrm{Set}(\mathcal{T}).$$

Example (Number of cycles in a random mapping – cont'd)

When marking cycles, then

$$\mathcal{B} = \mathtt{Set}(\mu\mathtt{Cyc}(\mathcal{T})), \quad \text{where } \mathcal{T} = \mathcal{Z} * \mathtt{Set}(\mathcal{T}).$$

Thus, with $b_{n,k} = \#$ functional digraphs with n vertices and k nodes, the bivariate GF $B(z,u) = \sum_{n\geq 0}\sum_{k\geq 0} b_{n,k} u^k \frac{z^n}{n!}$ satisfies

$$B(z,u) = \exp\left(u \log \frac{1}{1-T(z)}\right), \quad \text{where } T(z) = ze^{T(z)}.$$

The average number of cycles in $\mathcal{B}_n$ is then

$$\frac{1}{n^n}\sum_{k\geq 0} k b_{n,k} = \frac{n!}{n^n}[z^n]\frac{\mathrm{d}}{\mathrm{d}u}B(z,u)\Big|_{u=1}.$$

Example (Number of cycles in a random mapping – cont'd)

Since

$$\frac{\mathrm{d}}{\mathrm{d}u}B(z,u)\Big|_{u=1} = \frac{1}{1-T(z)}\log\frac{1}{1-T(z)}$$

we are searching for

$$\frac{n!}{n^n}[z^n]\frac{1}{1-T(z)}\log\frac{1}{1-T(z)}.$$

To find the singularities, we must solve

$$F(z,T) = T - ze^T = 0, \qquad F_T(z,T) = 1 - ze^T = 0,$$

which has only one solution: $z_0 = 1/e$, $T_0 = T(z_0) = 1$.

We find $F_z(1/e,1) = -e$, $F_{TT}(1/e,1) = -1$ and thus

$$T(z) = 1 - \sqrt{2}\sqrt{1-ez} + O(1-ez), \quad \text{as } z \to \frac{1}{e}.$$

Example (Number of cycles in a random mapping – cont'd)

Hence

$$\begin{aligned}
&\frac{n!}{n^n}[z^n]\frac{1}{1-T(z)}\log\frac{1}{1-T(z)} \\
&\quad\sim \frac{n!}{n^n}[z^n]\frac{1}{\sqrt{2}\sqrt{1-ez}}\log\frac{1}{\sqrt{2}\sqrt{1-ez}} \\
&\quad\sim \sqrt{2\pi n}e^n\cdot\frac{e^n}{\sqrt{2}}[z^n]\frac{1}{\sqrt{1-z}}\cdot\frac{1}{2}\log\frac{1}{1-z} \\
&\quad\sim \sqrt{\pi n}\cdot\frac{1}{2}\cdot\frac{n^{\frac{1}{2}-1}}{\Gamma\left(\frac{1}{2}\right)}\log n \\
&\quad= \frac{1}{2}\log n
\end{aligned}$$

Example (Pólya trees)

$$\mathcal{P} = \{\circ\} \times \mathtt{MSet}(\mathcal{P})$$

Thus

$$P(z) = z \exp\left(\sum_{\ell \geq 1} \frac{P(z^i)}{i}\right) = ze^{P(z)}Q(z)$$

Searching for dominant singularities: The above equation implies $\rho \leq \frac{1}{e} < 1$, thus $Q(z)$ is analytic for $|z| = \rho$.

The system $P = ze^P Q$, $1 = ze^P Q$ has one solution $(\rho, 1)$.

$P(\rho) = 1$ implies $\rho e = 1/Q(\rho) < 1$ and thus $\rho < 1/e$.

Comparing with the GF for the Catalan numbers:
$P(1/4) < C(1/4) = 1/2 < 1$ implies $\rho > 1/4$.

Example (Pólya trees – cont'd)

We have $F(z, P(z)) = 0$ with $F(z, P) = ze^P Q - P$.

$$P(z) \sim 1 - \sqrt{\frac{2\rho F_z(\rho, 1)}{F_{PP}(\rho, 1)}} \sqrt{1 - \frac{z}{\rho}}.$$

We compute $F_{PP}(\rho, 1) = (ze^P Q)(\rho, 1) = 1$ and

$$\begin{aligned} \rho F_z(\rho, 1) &= \rho \frac{\partial}{\partial z}(ze^P Q - P) = \rho(\rho e Q' + eQ) \\ &= 1 + \sum_{i \geq 2} P'(\rho^i)\rho^i. \end{aligned}$$

Thus

$$[z^n]P(z) = \frac{-\sqrt{2} \cdot \sqrt{1 + \sum_{i \geq 2} P'(\rho^i)\rho^i}}{\Gamma(-1/2)\sqrt{n^3}} \rho^{-n} = \frac{\sqrt{1 + \sum_{i \geq 2} P'(\rho^i)\rho^i}}{\sqrt{2\pi n^3}\rho^n}$$

Example (Extremal parameters: Average height of binary trees (sketch))

$$\mathcal{B}_0 = \emptyset, \qquad \mathcal{B}_1 = \{\Box\}, \qquad \mathcal{B}_{h+1} = \{\Box\} \cup \{\circ\} \times \mathcal{B}_h \times \mathcal{B}_h$$

and thus $B_1(z) = 1$, $B_{k+1}(z) = 1 + zB_k(z)^2$

$$\sum_{h\geq 0} h(\text{\# trees w. height } h) = \sum_{h\geq 0}(\text{\# trees w. height } \geq h)$$

$$= [z^n] \underbrace{\sum_{h\geq 0}(B_\infty(z) - B_h(z))}_{f(z)}$$

$$\Longrightarrow \text{ average height} = \frac{f_n}{\frac{1}{n}\binom{2n-2}{n-1}} \sim \frac{f_n}{4^n}\sqrt{\pi n^3}$$

Example (Extremal parameters: Average height of binary trees (sketch) – cont'd)

Flajolet & Odlyzko 1982 showed:

$$f(z) = 2\log\frac{1}{1-4z} + O\left((1-4z)^{1/4+\varepsilon}\right)$$

$$\implies 4^{-n} f_n = \frac{2}{n} + O\left(n^{-5/4+\varepsilon}\right)$$

$$\implies \quad \text{average height} = \sqrt{\pi n} + O\left(n^{1/4+\varepsilon}\right)$$

Example (Extremal parameters: Longest cycle in a permutation (sketch))

$\mathcal{A}\ldots$ cyclic permutations, GF $A(z) = \log \frac{1}{1-z} = \sum_{\ell \geq 1} \frac{z^\ell}{\ell}$

$\mathcal{B}\ldots$ all permutations, GF $B(z) = e^{A(z)}$, because $\mathcal{B} = \texttt{Set}(\mathcal{A})$

longest cycle length less than k: GF $B_k(z) = \exp\left(\sum_{j=1}^{k-1} \frac{z^j}{j}\right)$.

Thus consider

$$f(z) = \frac{1}{1-z} \sum_{k \geq 0} \left(1 - \exp\left(-\sum_{j \geq k} \frac{z^j}{j}\right)\right)$$

Rem.: $f_n = \mathbb{E}(\text{length of longest cycle in } \mathcal{B}_n)$

Example (Extremal parameters: Longest cycle in a permutation (sketch) – cont'd)

Substitute $z = e^{-t}$, then $t \to 0 \implies z \to 1$, $z \sim 1 - t$:

$$\sum_{j \geq k} \frac{z^j}{j} = t \sum_{j \geq k} \frac{e^{-tj}}{tj} \sim t \int_k^\infty \frac{e^{-tu}}{tu} \, \mathrm{d}u = \int_{kt}^\infty \frac{e^{-u}}{u} \, \mathrm{d}u$$

Example (Extremal parameters: Longest cycle in a permutation (sketch) – cont'd)

Substitute $z = e^{-t}$, then $t \to 0 \implies z \to 1$, $z \sim 1 - t$:

$$\sum_{j \geq k} \frac{z^j}{j} = t \sum_{j \geq k} \frac{e^{-tj}}{tj} \sim t \int_k^{\infty} \frac{e^{-tu}}{tu} \, \mathrm{d}u = \int_{kt}^{\infty} \frac{e^{-u}}{u} \, \mathrm{d}u$$

$$(1 - z) f(z) = \sum_{k \geq 0} \left(1 - \exp \left(- \sum_{j \geq k} \frac{z^j}{j} \right) \right)$$

Example (Extremal parameters: Longest cycle in a permutation (sketch) – cont'd)

Substitute $z = e^{-t}$, then $t \to 0 \implies z \to 1$, $z \sim 1 - t$:

$$\sum_{j \geq k} \frac{z^j}{j} = t \sum_{j \geq k} \frac{e^{-tj}}{tj} \sim t \int_k^\infty \frac{e^{-tu}}{tu} \, \mathrm{d}u = \int_{kt}^\infty \frac{e^{-u}}{u} \, \mathrm{d}u$$

$$(1 - z) f(z) \sim \int_0^\infty \left(1 - \exp\left(\int_{xt}^\infty \frac{e^{-u}}{u} \, \mathrm{d}u \right) \right) \mathrm{d}x$$

Example (Extremal parameters: Longest cycle in a permutation (sketch) – cont'd)

Substitute $z = e^{-t}$, then $t \to 0 \implies z \to 1$, $z \sim 1 - t$:

$$\sum_{j \geq k} \frac{z^j}{j} = t \sum_{j \geq k} \frac{e^{-tj}}{tj} \sim t \int_k^\infty \frac{e^{-tu}}{tu} \,\mathrm{d}u = \int_{kt}^\infty \frac{e^{-u}}{u} \,\mathrm{d}u$$

$$(1 - z) f(z) = \frac{1}{t} \underbrace{\int_0^\infty \left(1 - \exp\left(-\int_x^\infty \frac{e^{-u}}{u} \,\mathrm{d}u\right)\right) \mathrm{d}x}_{\text{constant } C}$$

$$\implies f(z) \sim \frac{C}{(1 - z)^2} \qquad \implies f_n \sim Cn.$$