Analytic Combinatorics: Complex-analytic Methods and Applications

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Tauberian theorems

Tauber's theorem

Recall:

Theorem (Abel's limit theorem)

If

$$\sum_{n\geq 0} a_n = s$$

then $\sum_{n\geq 0} a_n z^n$ *defines a continuous function* f *on* $[0,1]$ *and* $f(1) = s$.

What about the converse?
Given the function, statement about the series.

Theorem (Tauber's theorem (1897))

Given $f(z) = \sum_{n\geq 0} a_n z^n$ *such that* $\lim_{z\to 1-} f(z) = s$ *and*

$$a_n = o\left(\frac{1}{n}\right) \qquad \textit{Tauberian condition}$$

Then $\sum_{n\geq 0} a_n = s$

Example (Example without side condition)

$$f(z) = \frac{1}{1+z} = \sum_{n\geq 0} (-1)^n z^n.$$

Clearly, $\lim_{z\to 1-} f(z) = \frac{1}{2}$,
but $\sum_{n\geq 0}(-1)^n$ is divergent!

Theorem (Hardy & Littlewood (1914), Karamata (1930))

Let $f(z) = \sum_{n\geq 0} f_n z^n$ *be a power series with radius of convergence equal to 1 and, as* $z \to 1-$, *(no* Δ*-analyticity!)*

$$f(z) \sim \frac{1}{(1-z)^\alpha} L\left(\frac{1}{1-z}\right).$$

If $f_n \geq 0$ *(Tauberian condition), then*

$$\sum_{k=0}^{n} f_k \sim \frac{n^\alpha}{\Gamma(\alpha+1)} L(n).$$

If moreover f_n *is monotone, then*

$$f_n \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} L(n).$$

Proof: Set $U(t) = \sum_{k=0}^{\lfloor t \rfloor} f_k$, which is the improper distribution function of a measure U on $[0, \infty)$.
Let

$$\omega(s) = \mathbf{E} e^{-sU} = \int_0^\infty e^{-sx} U(\mathrm{d}x) = \sum_{k \geq 0} e^{-sk} f_k = f(e^{-s})$$

be its Laplace transform.
Let $\tau > 0$ and $t = 1/\tau$. Then

$$\omega(\tau s) = \int_0^\infty e^{-\tau s x} U(\mathrm{d}x) = \int_0^\infty e^{-sx/t} U(\mathrm{d}x) = \int_0^\infty e^{-sy} U(t\,\mathrm{d}y)$$

is the Laplace transform of the measure with distribution function $U(tx)$.

Let V be the measure with distribution function $U(tx)/\omega(\tau)$.

For $\tau \to 0$ we have
$\omega(\tau) = f(e^{-\tau}) \sim f(1-\tau) \sim \tau^{-\alpha} L\left(\frac{1}{\tau}\right) = t^{\alpha} L(t)$ and

$$\mathbf{E} e^{-sV} = \frac{\omega(\tau s)}{\omega(\tau)} \sim \frac{f(1-\tau s)}{f(1-\tau)} \sim \frac{1}{s^{\alpha}} \frac{L\left(\frac{t}{s}\right)}{L(t)} \longrightarrow \frac{1}{s^{\alpha}} = \mathcal{L}\left(\frac{x^{\alpha-1}}{\Gamma(\alpha)}\right)$$

Now let W denote the measure with density $x^{\alpha-1}/\Gamma(\alpha)$. Then, by the continuity theorem for Laplace transforms,

$$\frac{U(tx)}{\omega(\tau)} \longrightarrow \int_0^x \frac{x^{\alpha-1}}{\Gamma(\alpha)} \,\mathrm{d}x = \frac{x^{\alpha}}{\Gamma(\alpha+1)}.$$

But

$$\frac{U(tx)}{\omega(\tau)} = \frac{1}{\omega(\tau)} \sum_{k \le \lfloor tx \rfloor} f_k \sim \frac{1}{t^{\alpha}} \sum_{k \le \lfloor tx \rfloor} f_k,$$

so set $x = 1$ and $t = n$. $\square$

Example (Permutations with distinct cycle lengths)

The EGF of $\mathrm{Cyc}_k(\mathcal{Z})$ is z^k/k. Thus the EGF of the class of permutations with distinct cycle lengths is

$$f(z) = \prod_{k\geq 1}\left(1+\frac{z^k}{k}\right).$$

Therefore

$$\begin{aligned}\log f(z) &= \log \prod_{k\geq 1}\left(1+\frac{z^k}{k}\right) = \sum_{k\geq 1}\log\left(1+\frac{z^k}{k}\right) = \sum_{k\geq 1}\sum_{n\geq 1}\frac{(-1)^{n-1}z^{kn}}{k^n n} \\ &= \log\frac{1}{1-z} + \sum_{k\geq 1}\left(\log\left(1+\frac{z^k}{k}\right)-\frac{z^k}{k}\right) \\ &= \log\frac{1}{1-z} + \underbrace{\log(1+z) - z + \sum_{k\geq 2}\sum_{n\geq 2}\frac{(-1)^{n-1}z^{kn}}{k^n n}}_{=:g(z)}\end{aligned}$$

Example (Permutations with distinct cycle lengths – cont'd)

$$g(z) = \log(1+z) - z + \sum_{k\geq 2}\sum_{n\geq 2}\frac{(-1)^{n-1}z^{kn}}{k^n n} = \sum_{k\geq 1}\left(\log\left(1+\frac{z^k}{k}\right) - \frac{z^k}{k}\right)$$

$g(1)$ convergent?

$$\begin{aligned} g(1) &= \lim_{z\to 1-} g(z) = \sum_{k\geq 1}\left(\log\left(1+\frac{1}{k}\right) - \frac{1}{k}\right) \\ &= -\log 1 + \lim_{n\to\infty}(\log(n+1) - H_n) = -\gamma \end{aligned}$$

Thus

$$f(z) \sim \frac{e^{-\gamma}}{1-z} \text{ for } z \to 1.$$

This implies $\sum_{k=0}^{n} f_k \sim ne^{-\gamma}$.

Example (Permutations with distinct cycle lengths – cont'd)

What about f_n?
Write $f(z) = \frac{1+z}{1-z} e^{\tilde{g}(z)} = \frac{1+z}{1-z} h(z)$ with

$$\tilde{g}(z) = g(z) - \log z = -z + \sum_{m \geq 2} \frac{(-1)^{m-1}}{m} \sum_{k \geq 2} \frac{z^{km}}{k^m}$$

So,

$$f(z) = \frac{1}{1-z} \sum_{n \geq 0} h_n z^n + \frac{z}{1-z} \sum_{n \geq 0} h_n z^n$$

and thus $f_n = 2h_0 + \cdots + 2h_{n-1} + h_n$.
Our goal: Show that f_n converges, then $\sum_{k=0}^{n} f_k \sim n e^{-\gamma}$ implies $f_n \sim e^{-\gamma}$.

Example (Permutations with distinct cycle lengths – cont'd)

If $|[z^n]\tilde{g}(z)| \leq [z^n]v(z)$ then $|h_n| \leq [z^n]e^{v(z)}$.
For $n \geq 2$ we have

$$[z^n]\tilde{g}(z) = [z^n] \sum_{m\geq 2} \frac{(-1)^{m-1}}{m} \sum_{k\geq 2} \frac{z^{km}}{k^m} = \sum_{m\geq 2,\ m|n,\ m<n} \frac{(-1)^{m-1}}{m} \left(\frac{m}{n}\right)^m .$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{x}{n}\right)^x = \left(\frac{x}{n}\right)^x (1 + \log x - \log n) < 0$$

if $1 \leq x \leq n/e$. Thus $\left(\frac{m}{n}\right)^m$ is decreasing for $1 \leq m \leq n/e$.

Example (Permutations with distinct cycle lengths – cont'd)

Consequently,

$$\begin{aligned}|[z^n]\tilde{g}(z)| &\le \frac{1}{2}\left(\frac{2}{n}\right)^2 + \sum_{3\le m\le n/3,\ m|n} \frac{1}{m}\left(\frac{3}{n}\right)^3 + \frac{2}{n}2^{-n/2} \\ &\le \frac{2}{n^2} + \frac{3}{n^2} + \frac{2}{n^2} < \frac{10}{n^2}.\end{aligned}$$

Thus we can take $w(z) = e^{v(z)} := \exp\left(10 \sum_{\ell\ge 1} \frac{z^\ell}{\ell^2}\right)$.

As $w' = v'w$, we have $\boxed{nw_n = 10\sum_{k=0}^{n-1} \frac{w_k}{n-k}}$.

Note: $v(1) < \infty$, $w_n \ge 0$, $w_n \le w(1)$.
Choose $C > 0$, then for $1 \le n \le C$ and sufficiently large B we have $w_n \le B/n^2$.

Example (Permutations with distinct cycle lengths – cont'd)

Assume $w_m \le B/m^2$ for $1 \le m \le n$ and some $n \ge C$ and proceed by induction on n.
Choose $C_0 < \frac{C}{2}$: Then

$$\frac{w_k}{n-k} \le w(1) \cdot \frac{2}{n} \text{ for } 0 \le k \le C_0,$$
$$\frac{w_k}{n-k} \le \frac{B}{k^2} \cdot \frac{1}{n-k} \text{ for } C_0 \le k \le n/2.$$

With this we get

$$\sum_{C_0 \le k \le n/2} \frac{w_k}{n-k} \le \frac{2B}{n} \sum_{C_0 \le k \le n/2} \frac{1}{k^2} = \frac{2BS_n}{n}$$
$$\sum_{n/2 < k \le n-1} \frac{w_k}{n-k} \le \frac{4B}{n^2} \sum_{n/2 < k \le n-1} \frac{1}{n-k} = \frac{4BH_n}{n^2} \le \frac{4(1+\log n)B}{n^2}$$

Example (Permutations with distinct cycle lengths – cont'd)

Altogether this gives

$$nw_n \le \frac{20C_0 w(1)}{n} + \left(\frac{20S_n}{n} + \frac{40(1+\log n)}{n^2}\right) B \overset{!}{\le} \frac{B}{n}$$

if

$$B \ge \frac{20C_0 w(1)}{1 - 20S_n - \frac{40(1+\log n)}{n}}$$

This can be achieved as $n \ge C$; so choose C sufficiently large to guarantee that S_n can be made small (by choosing C_0 large enough) and the denominator is positive.
Then choose B sufficiently large.

Finally, we obtain $h_n \le w_n = O\left(1/n^2\right)$ and so f_n converges. □

Saddle point techniques

Laplace's method

Goal: Evaluation of the parameter integral

$$I(\lambda) = \int_a^b f(z) e^{-\lambda h(z)} \,\mathrm{d}z, \qquad \text{as } \lambda \to \infty,$$

where

- f and h are analytic and real-valued for real z.
- $\lambda > 0$, $a, b \in \mathbb{R}$.
- There is a $a < t_0 < b$ such that $h'(t_0) = 0$ and $h''(t_0) > 0$.

Idea: Let $f(t_0) \neq 0$ and esimate

$$
\begin{aligned}
I(\lambda) &\sim \int_a^b f(t_0) \exp\left(-\lambda\left(h(t_0) + h''(t_0)\frac{(t-t_0)^2}{2}\right)\right) \mathrm{d}t \\
&\sim f(t_0) e^{-\lambda h(t_0)} \int_{-\infty}^{\infty} \exp\left(-\frac{(t-t_0)^2}{2}\lambda h''(t_0)\right) \mathrm{d}t \\
&\sim f(t_0) e^{-\lambda h(t_0)} \sqrt{\frac{2\pi}{\lambda h''(t_0)}}
\end{aligned}
$$

where we used

$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k e^{-tx^2} \,\mathrm{d}x = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \frac{t^{-(k+1)/2} k!}{(k/2)! \cdot 2^{k+1/2}} & \text{if } k \text{ is even.} \end{cases}
$$

Note: k fixed, $\alpha > 0$, then $\int_\theta^\infty x^k e^{-\alpha x^2} \,\mathrm{d}x = O\left(e^{-\alpha\theta^2}\right)$.

⚠ Simultaneous limits: $\lambda \to \infty$, $t \to t_0$

Theorem (Laplace's method)

Let $-\infty \leq a < b \leq \infty$ *and*

$$I(x) = \int_a^b f(t) e^{-xh(t)} \, \mathrm{d}t.$$

Assume

(1) f *is once and* h *twice continuously differentiable;*

(2) t_0 *unique minimum of* h*, with* $h''(t_0) > 0$, $f(t_0) \neq 0$*;*

(3) $h(t) = h(t_0) + \frac{h''(t_0)}{2}(t - t_0)^2 + O\left(|t - t_0|^3\right)$*;*

(4) $I(x)$ *convergent for* $x \geq x_0$.

Then

$$I(x) = f(t_0) \sqrt{\frac{2\pi}{xh''(t_0)}} e^{-xh(t_0)} \left(1 + O\left(\frac{1}{\sqrt{x}}\right)\right), \text{ as } x \to \infty.$$

Proof: W.l.o.g. $t_0 = 0$ ($a < 0$, $b > 0$), $h(t_0) = 0$ and $f(t) \equiv 1$ (otherwise $f(t) = f(t_0) + O(|t - t_0|)$)
By (2), we have $|t| < \delta \implies \left|h(t) - h''(0)\frac{t^2}{2}\right| \leq c|t|^3$. Write

$$I(x) = \underbrace{\int_{-\delta}^{\delta} e^{-xh(t)}\,\mathrm{d}t}_{I_1(x)} + \underbrace{\int_{|t|>\delta} e^{-xh(t)}\,\mathrm{d}t}_{I_2(x)}.$$

With $\mu := \inf_{|t|>\delta} h(t) > 0$ we get

$$|I_2(x)| \leq \int_{|t|>\delta} e^{-x_0h(t)-(x-x_0)h(t)}\,\mathrm{d}t \leq e^{-(x-x_0)\mu}\int_{|t|>\delta} e^{-x_0h(t)}\,\mathrm{d}t = O\left(e^{-\beta x}\right)$$

for some $\beta > 0$.

Turn to $I_1(x)$:

$$
\begin{aligned}
I_1(x) \\
&= \int_{-\delta}^{\delta} e^{-xh''(0)\frac{t^2}{2}}\,\mathrm{d}t + \int_{-\delta}^{\delta} e^{-xh''(0)\frac{t^2}{2}} \left(e^{-x\left(h(t)-h''(0)\frac{t^2}{2}\right)} - 1 \right) \mathrm{d}t \\
&= \sqrt{\frac{2\pi}{h''(0)x}} + O\left(e^{-x\delta^2}\right) + J
\end{aligned}
$$

As

$$
|e^t - 1| = \left| \sum_{n\geq 1} \frac{t^n}{n!} \right| < \sum_{n\geq 1} \frac{|t|^n}{(n-1)!} = |t| e^{|t|}
$$

and $|h(t) - h''(0)\frac{t^2}{2}| \leq c|t^3|$, we have

$$
|J| \leq 2cx \int_0^{\delta} t^3 \exp\left(-xh''(0)\frac{t^2}{2} + cxt^3 \right) \mathrm{d}t
$$

We had

$$I_1(x) = \sqrt{\frac{2\pi}{h''(t_0)x}} + O\left(e^{-x\delta^2}\right) + J$$

with

$$|J| \le 2cx \int_0^\delta t^3 \exp\left(-xh''(t_0)\frac{t^2}{2} + cxt^3\right)\,\mathrm{d}t$$

and thus we get

$$|J| \le 2cx \int_0^\delta t^3 \exp\left(-xt^2\left(\frac{h''(t_0)}{2} - \delta c\right)\right)\,\mathrm{d}t \le 2cx \int_0^\delta t^3 \exp\left(-xt^2 b\right)\,\mathrm{d}t$$

for $b = \frac{h''(t_0)}{2} - \delta c$. Substitution $u = \sqrt{xb}t$ gives

$$|J| \le \frac{2c}{b^2 x}\int_0^\infty u^3 e^{-u^2}\,\mathrm{d}u = O\left(\frac{1}{x}\right)$$

and therefore $I_1(x) = \sqrt{\frac{2\pi}{h''(t_0)x}}\left(1 + O\left(\frac{1}{\sqrt{x}}\right)\right)$. $\square$

Saddle points

Let f be analytic in some region Ω, $z_0 \in \Omega$

z_0 is an ordinary point: $f(z_0) \neq 0$, $f'(z_0) \neq 0$

For $z = z_0 + re^{i\theta}$ and $\frac{f'(z_0)}{f(z_0)} = \lambda e^{i\phi}$ we have

$$|f(z)| = |f(z_0) + re^{i\theta} f'(z_0) + O(r^2)| \sim |f(z_0)| \cdot |1 + r\lambda \cos(\theta + \phi)|$$

Let r be sufficiently small, while θ varies. Then

$$|f(z)| \sim |f(z_0)|(1 \pm \lambda r) \quad \text{for } \theta = -\phi,\ \theta = -\phi + \pi,\ \text{resp.}$$

(line of steepest ascent resp. descent)

$$|f(z)| \sim |f(z_0)| + o(r) \quad \text{for } \theta = -\phi \pm \frac{\pi}{2}$$

(level set, perpendicular to gradient)

z_0 is a zero: $f(z_0) = 0$, $f'(z_0) \neq 0$ if first order.

Then $|f(z)| \sim r|f'(z_0)| = O(r)$. For mth order: $|f(z)| = O(r^m)$.

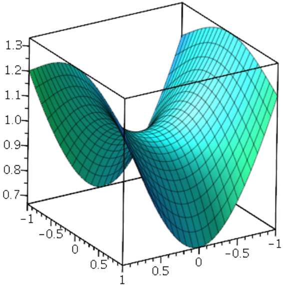
z_0 is a simple saddle point: $f(z_0) \neq 0$, $f'(z_0) = 0$, $f''(z_0) \neq 0$

Set $\lambda e^{i\phi} = \frac{1}{2}\frac{f''(z_0)}{f(z_0)}$.

$$|f(z)| = \left| f(z_0) + \frac{r^2}{2} e^{2i\theta} f''(z_0) + O(r^3) \right|$$

$$= |f(z_0)| \left| 1 + \lambda r^2 e^{i(2\theta+\phi)} + O(r^3) \right|$$

$$\sim |f(z_0)|(1 + \lambda r^2 \cos(2\theta + \phi))$$



$|f(z)|$ maximal for $\theta = -\phi/2$
$|f(z)|$ minimal for $\theta = -\phi/2 + \pi/2$
$|f(z)| \sim |f(z_0)|$ for $\theta = -\phi/2 + \pi/4$
and for $\theta = -\phi/2 + 3\pi/4$

Pattern repeats once. $(0 \leq \theta \leq 2\pi)$

p-fold saddles: $f'(z_0) = f''(z_0) = \cdots = f^{(p)}(z_0) = 0$, $f^{(p+1)}(z_0) \neq 0$

Example: $p = 2$ and $p = 4$: