Analytic Combinatorics: Complex-analytic Methods and Applications

Bernhard Gittenberger

TU Wien

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The saddle point method

Typical application in combinatorics:

$$f_n = [z^n]f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z^{n+1}}\,\mathrm{d}z$$

Lemma

If $f(z)$ is a generating function which is not a polynomial, its radius of convergence R is larger than zero, and $\lim_{z\to R-} f(z) = \infty$, then

$$\lim_{z\to 0+}\frac{f(z)}{z^{n+1}} = \lim_{z\to R-}\frac{f(z)}{z^{n+1}} = \infty \implies \exists!\,\zeta \in (0,R)\ :\ g'(\zeta) = 0$$

where $g(z) = f(z)z^{-n-1}$.

Proof: $(f(z)z^{-n-1})'' > 0$ for $0 < z < R$, hence it is convex. □

Corollary (Saddle point bound)

f analytic at $z = 0$, $f_n \geq 0$, radius of convergence R. Assume $\lim_{z \to R-} f(z) = \infty$ and that $f(z)$ is not a polynomial. Let ζ be the unique positive solution of $\frac{xf'(x)}{f(x)} = n + 1$. Then

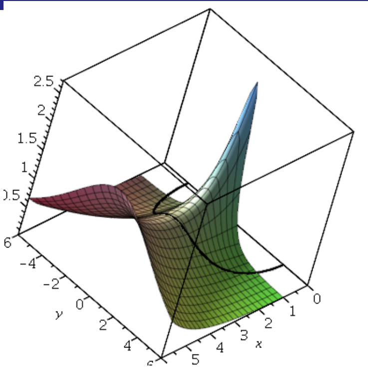
$$f_n = [z^n]f(z) \leq \frac{f(\zeta)}{\zeta^{n+1}}$$

Proof: ζ is saddle point if $\left(\frac{f(z)}{z^{n+1}}\right)' = 0$:

$$\left(\frac{f(z)}{z^{n+1}}\right)' = \frac{f'(z)}{z^{n+1}} - (n+1)\frac{f(z)}{z^{n+2}} = 0 \Longleftrightarrow \frac{zf'(z)}{f(z)} = n+1 \Longleftrightarrow z = \zeta.$$

Choose γ to be the circle $|z| = \zeta$ and use the estimate $\left|\frac{f(z)}{z^{n+1}}\right| \leq \frac{f(\zeta)}{\zeta^{n+1}}$ for the integration. □

Proof idea:

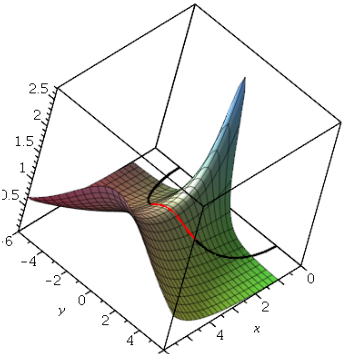


Example

$$f(z) = e^z \implies zf'(z)/f(z) = z = n + 1$$

$$\frac{1}{2\pi i} \oint_{|z|=n+1} \frac{e^z}{z^{n+1}} \,\mathrm{d}z = \frac{e^{n+1}}{(n+1)^n} \asymp \frac{\sqrt{n}}{n!}.$$

Idea of the saddle point method to compute $\int_A^B e^{f(z)}\,\mathrm{d}z$:
(Note: saddle points of $e^{f(z)}$ and of $f(z)$ coincide!)



- Choose integration contour through a saddle point.
- Cut off the tails.
- Approximate the integrand:

$$f(z) = f(\zeta) + \frac{1}{2} f''(\zeta)(z-\zeta)^2 + O(\eta_n)$$

$\eta_n \to 0$ uniformly.

- Add new tails, using the simplified integrand:

$$\int_\gamma \asymp \int_{-\infty}^{\infty} e^{-|f''(\zeta)|z^2/2}\,\mathrm{d}z$$

Example (Stirling's formula revisited)

$$n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}$$

$$\frac{1}{n!} = [x^n]e^x = \frac{1}{2\pi i}\oint \frac{e^x}{x^{n+1}}\,dx = \frac{1}{2\pi r^n}\int_{-\pi}^{\pi} e^{re^{i\theta}} e^{-in\theta}\,d\theta$$

$$re^{i\theta} = r + i\theta r - \frac{\theta^2}{2}r + O\left(\theta^3 r\right)$$

$$e^{re^{i\theta}} \sim e^r \exp\left(i\theta r - \frac{\theta^2}{2}r\right), \text{ as } r \to \infty, \text{ uniformly for } \theta = o(r^{-1/3})$$

Example (Stirling's formula revisited – cont'd)

$$\left|e^{re^{i\theta}}\right| = e^{\Re\left(re^{i\theta}\right)} = e^{r\cos\theta}$$

$$\cos\theta \sim 1 - \frac{\theta^2}{2}, \quad \theta \text{ small}$$

$$\Longrightarrow \left|e^{re^{i\theta}}\right| \leq e^r \exp\left(-\frac{\theta_0^2}{2}r\right), \text{ as } r \to \infty,$$

uniformly for $\theta_0 \leq |\theta| \leq \pi$

Example (Stirling's formula revisited – cont'd)

If $x = re^{i\theta}$, then with $\delta(r) = r^{-2/5}$

$$e^x \sim e^r e^{i\theta r - \theta^2 r/2}, \quad r \to \infty, |\theta| \le \delta(r)$$

and

$$|e^x| = O\left(e^r \exp\left(-\frac{r^{1/5}}{2}\right)\right) = o\left(\frac{e^r}{\sqrt{r}}\right), \quad r \to \infty, |\theta| \ge \delta(r)$$

Hence

$$\int_{-\pi}^{\pi} e^{re^{i\theta}} e^{-in\theta}\, d\theta = \underbrace{\int_{-\delta}^{\delta} e^{re^{i\theta}} e^{-in\theta}\, d\theta}_{(e^r \cdot \sqrt{2\pi/r}) e^{-(r-n)^2/2r}} + \underbrace{\int_{|\theta| \ge \delta} e^{re^{i\theta}} e^{-in\theta}\, d\theta}_{o(e^r/\sqrt{r})}$$

Example (Stirling's formula revisited – cont'd)

We get

$$r^n[x^n]e^x \sim \frac{e^r}{\sqrt{2\pi r}} \exp\left(-\frac{(r-n)^2}{2r}\right)$$

for $r \to \infty$ and uniformly for all integers n.

Inserting $r = n \implies$ Stirling's formula.

Large powers

Large powers appear frequently in combinatorial enumeration:

- $\mathcal{A} = \mathrm{Seq}_k(\mathcal{B})$, then
$$[z^n]A(z) = [z^n]B(z)^k.$$
- If $f(z) = z\phi(f(z))$ then by Lagrange inversion we get
$$[z^n]f(z) = \frac{1}{n}[z^{n-1}]\phi(z)^n.$$

Theorem

Let $f(z) = \sum_{n \geq 0} f_n z^n$, $f_n \geq 0$, *suppose there are* $n_1 < n_2 < n_3$ *such that* $f_{n_1} f_{n_2} f_{n_3} > 0$ *and* $\gcd(n_2 - n_1, n_3 - n_1) = 1$.
Then, for all $\varepsilon > 0$,

$$f_{n,k} = [z^n] f(z)^k = \frac{f(\rho)}{\rho^n \sqrt{2\pi\sigma^2 k}} \left(\exp\left(-\frac{r^2}{2\sigma^2 k} \right) + O\left(k^{-\frac{1}{2}+\varepsilon} \right) \right),$$

as $k \to \infty$ *and uniformly for* μ *and* r *such that* $n = \mu k + r$ *and* $\mu \in [a, b]$, *where* $a > 0$ *and* $b > 0$. ρ *and* σ^2 *are given by*

$$\frac{\rho f'(\rho)}{f(\rho)} = \mu, \qquad \sigma^2 = \mu - \mu^2 + \frac{\rho^2 f''(\rho)}{f(\rho)}.$$

Proof: Use Cauchy's integral formula with $|z| = \rho$, ρ being the saddle point of $\frac{f(z)^k}{z^{\mu k+1}}$.

Then, as $\mu k + 1 \sim n$, we get

$$f_{n,k} \sim \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\rho e^{it}\right)^k \rho^{-n} e^{-itn} \, \mathrm{d}t$$

The maximum of $|f\left(\rho e^{it}\right)|$ is at $t = 0$, thus there is $\delta > 0$ and $0 < \alpha < 1$ such that

$$\left| f\left(\rho e^{it}\right) \right| \leq \alpha f(\rho)$$

when $\rho \in [\rho_a, \rho_b]$ and $\delta \leq |t| \leq \pi$.
We choose ρ_a, ρ_b so that $\rho \in [\rho_a, \rho_b]$ is equivalent to $\mu \in [a, b]$.
Moreover,

$$\log f\left(\rho e^{it}\right) = \log f(\rho) + i\mu t - \frac{\sigma^2 t^2}{2} + g(t^3)$$

and choose δ so small that $|g(t^3)| < \frac{\sigma^2 t^2}{3}$, uniformly for $\rho \in [\rho_a, \rho_b]$, if $|t| < \delta$.

$$
\begin{aligned}
f_{n,k} &= \frac{1}{2\pi}\int_{-\delta}^{\delta} f\left(\rho e^{it}\right)^k \rho^{-n} e^{-int}\,\mathrm{d}t + O\left(f(\rho)^k \rho^{-n}\alpha^k\right)\\
&= \frac{f(\rho)^k}{2\pi\rho^n}\left(\int_{-\delta}^{\delta} \exp\left(-irt - \frac{\sigma^2 k}{2}t^2 + kg(t^3)\right)\mathrm{d}t + O\left(\alpha^k\right)\right)\\
&= \frac{f(\rho)^k}{2\pi\rho^n}\left(\int_{-k^{-\frac{1}{2}+\varepsilon}/\sigma}^{k^{-\frac{1}{2}+\varepsilon}/\sigma} \exp\left(-irt - \frac{\sigma^2 k}{2}t^2 + O\left(k^{-\frac{1}{2}+3\varepsilon}\right)\right)\mathrm{d}t\right.\\
&\qquad\left. + O\left(\alpha^k\right) + O\left(\delta \exp\left(-\frac{k^{2\varepsilon}}{6}\right)\right)\right)
\end{aligned}
$$

Now substitute $u = \sqrt{\sigma^2 k}\cdot t$ and obtain

$$
f_{n,k} = \frac{f(\rho)^k}{2\pi\rho^n\sqrt{\sigma^2 k}}\int_{-k^{\varepsilon}}^{k^{\varepsilon}} \exp\left(-i\frac{r}{\sqrt{\sigma^2 k}}u - \frac{u^2}{2}\right)\left(1 + O\left(k^{-\frac{1}{2}+3\varepsilon}\right)\right)\mathrm{d}u.
$$

$$f_{n,k} = \frac{f(\rho)^k}{2\pi\rho^n\sqrt{\sigma^2 k}} \int_{-k^\varepsilon}^{k^\varepsilon} \exp\left(-i\frac{r}{\sqrt{\sigma^2 k}}u - \frac{u^2}{2}\right)\left(1 + O\left(k^{-\frac{1}{2}+3\varepsilon}\right)\right) \mathrm{d}u.$$

Finally, use

$$\int_{-\infty}^{\infty} e^{-Cx^2-Dx}\,\mathrm{d}x = \frac{\exp\left(\frac{D^2}{4C}\right)}{\sqrt{C}} \underbrace{\int_{-\infty}^{\infty} e^{-v^2}\,\mathrm{d}v}_{=\sqrt{\pi}}$$

with

$$C = \frac{1}{2} \text{ and } D = \frac{ir}{\sqrt{\sigma^2 k}}.$$

□