

Analytic Combinatorics: Complex-analytic Methods and Applications

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Limiting distributions

Example (Words over a 2-element alphabet)

$\mathcal{W} = \{a, b\}^*$: set of words over $\{a, b\}$, size = word length.

$w_n = 2^n$, $W(z) = 1/(1 - 2z)$.

Let $\alpha(w) :=$ number of initial a 's of w ,
 $\beta(w) :=$ total number of a 's of w .

Set $a_{n,k} = \mathbf{P}\{\alpha(w) = k \mid w \in \mathcal{W}_n\}$,
 $b_{n,k} = \mathbf{P}\{\beta(w) = k \mid w \in \mathcal{W}_n\}$.

Then

$$a_{n,k} = \frac{1}{2^{k+1}}, \text{ for } k < n, \quad a_{n,n} = \frac{1}{2^n}; \quad b_{n,k} = \frac{1}{2^n} \binom{n}{k}.$$

Example (Words over a 2-element alphabet – cont'd)

Distribution: $\lim_{n \rightarrow \infty} a_{n,k} = a_k := \frac{1}{2^{k+1}}$, uniformly in k .

discrete limiting distribution: $\alpha \sim \text{Geom}\left(\frac{1}{2}\right)$, $\mu_n = O(1)$,
 $\sigma_n^2 = O(1)$.

$b_{n,k}$: Set $n = 2\nu$ and apply Stirling's formula, which gives

$$\lim_{n \rightarrow \infty} \frac{\binom{2\nu}{\nu+t\sqrt{\frac{\nu}{2}}}}{\binom{2\nu}{\nu}} = \frac{e^{-t^2/2}}{\sqrt{\pi\nu}},$$

uniformly for $t = o(\nu^{1/6})$.

$$\implies b_{n,k} \sim \frac{2}{\sqrt{n}} f\left(\frac{k - \frac{n}{2}}{\frac{1}{2}\sqrt{n}}\right) \text{ where } f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\implies \beta \sim \mathcal{N}(\mu_n, \sigma_n^2) \text{ with } \mu_n \sim \frac{n}{2} \text{ and } \sigma_n^2 \sim \frac{n}{4}.$$

Limiting distributions

Why do α and β behave in a different way?

Comparison of the analytic behaviour of the bivariate GF:

$$A(z, u) = \frac{1-z}{1-uz} \frac{1}{1-2z}, \quad B(z, u) = \frac{1}{1-z(1+u)}.$$

The probability GF are

$$a_n(u) = \frac{[z^n]A(z, u)}{[z^n]A(z, 1)} = \frac{1}{2} \frac{1}{1 - \frac{u}{2}} + O((2/3)^n), \text{ for } |u| < 3/2,$$

and

$$b_n(u) = \frac{[z^n]B(z, u)}{[z^n]B(z, 1)} = \left(\frac{1+u}{2} \right)^n$$

$$a_n(u) \xrightarrow{\text{pointwise}} \frac{1}{2} \frac{1}{1 - \frac{u}{2}},$$

which is the probability GF of $\mathcal{G}_{\text{geom}}\left(\frac{1}{2}\right)$.

Limiting distributions

On the other hand: $b_n(u)$ not convergent, scaling required!
Compare with the univariate GF:

$$A(z, 1) = B(z, 1) = \frac{1}{1 - 2z}$$

both have a pole at $z = 1/2$.

$$A(z, u) = \frac{1}{1 - uz} \frac{1 - z}{1 - 2z} \sim \frac{\frac{1}{2}}{1 - \frac{u}{2}} A(z, 1), \text{ as } z \rightarrow \frac{1}{2}, |u| < \frac{3}{2}.$$

Singularity unchanged when u varies!

$$B(z, u) = \frac{1}{1 - z(1 + u)}, \text{ singularity } \rho(u) = \frac{1}{1 + u} \text{ depends on } u!$$

$$[z^n]B(z, u) = \rho(u)^{-n} \dots \text{analytic analog of the CLT.}$$

Discrete limit laws

Definition

X_n has a discrete limit law, if there are p_k such that $\sum_{k \geq 0} p_k = 1$ and $p_k = \lim_{n \rightarrow \infty} \mathbf{P}\{X_n = k\}$.

Remark: Convergence to discrete limit laws is always uniform.

Theorem (Continuity theorem)

Let $\Omega \subseteq \{z \in \mathbb{C} : |z| \leq 1\}$ with an accumulation point x with $|x| < 1$. Consider the PGF $p_n(u) = \sum_{k \geq 0} p_{n,k} u^k$ and $p(u) = \sum_{k \geq 0} p_k u^k$ and assume that for all $u \in \Omega$ we have $\lim_{n \rightarrow \infty} p_n(\bar{u}) = p(u)$. Then $\lim_{n \rightarrow \infty} p_{n,k} = p_k$.

Rate of convergence: If $p_n(u)$ and $p(u)$ are analytic in $|u| < r$ and continuous on $|u| = r$, then

$$|p_{n,k} - p_k| \leq r^{-k} \sup_{|u|=r} |p_n(u) - p(u)|.$$

Application: The composition schema $F(z, u) = g(uh(z))$ corresponding to the substitution construction of combinatorial structures.

$g(z)$ and $h(z)$ are GF, $h(0) = 0$, the radii of convergence are ρ_g and ρ_h , and let

$$\tau_g = \lim_{x \rightarrow \rho_g^-} g(x), \quad \tau_h = \lim_{x \rightarrow \rho_h^-} h(x).$$

Definition

The composition schema $g(uh(z))$ is called

- subcritical, if $\tau_h < \rho_g$,
- critical, if $\tau_h = \rho_g$,
- supercritical, if $\tau_h > \rho_g$.

Lemma

Let the composition schema $F(z, u) = g(uh(z))$ be subcritical, ρ_h being the only dominant singularity of $h(z)$ and

$$h(z) = \tau_h - c \left(1 - \frac{z}{\rho_h}\right)^\lambda + o\left(\left(1 - \frac{z}{\rho_h}\right)^\lambda\right), \quad c \in \mathbb{R}^+, \quad 0 < \lambda < 1.$$

Then we have a discrete limit law with

$$\lim_{n \rightarrow \infty} \frac{f_{n,k}}{f_n} = \frac{kg_k \tau_h^{k-1}}{g'(\tau_h)}, \quad p(u) = \frac{ug'(\tau_h u)}{g'(\tau_h)}.$$

Proof: $g(z)$ is analytic at $z = \tau_h$, so $g(h(z))$ has a unique dominant singularity at $z = \rho_h$ and is Δ -analytic.

Discrete limit laws

Therefore

$$g(h(z)) = g(\tau_h) - cg'(\tau_h) \left(1 - \frac{z}{\rho_h}\right)^\lambda (1 + o(1)).$$

A transfer gives then

$$f_n = [z^n]g(h(z)) = -\frac{cg'(\tau_h)}{\Gamma(-\lambda)} \rho_h^{-n} n^{-\lambda-1} (1 + o(1)).$$

Let $r < \rho_g/\tau_h$. Then, for $|u| \leq r$ we obtain

$$\begin{aligned} g(uh(z)) &= g\left(u\tau_h - cu\left(1 - \frac{z}{\rho_h}\right)^\lambda + o\left(\left(1 - \frac{z}{\rho_h}\right)^\lambda\right)\right) \\ &= g(u\tau_h) - cug'(u\tau_h)\left(1 - \frac{z}{\rho_h}\right)^\lambda + o\left(\left(1 - \frac{z}{\rho_h}\right)^\lambda\right) \end{aligned}$$

This implies

$$p(u) = \lim_{n \rightarrow \infty} \frac{[z^n]g(uh(z))}{[z^n]g(h(z))} = \frac{ug'(u\tau_h)}{g'(\tau_h)}$$

Finally, apply the continuity theorem.



Discrete limit laws

Sequences, sets and cycles of labelled objects:

Let

$$Q(w) = \frac{1}{1-w}, \quad E(w) = e^w, \quad L(w) = \log \frac{1}{1-w}.$$

Theorem

Let $g(z) \in \{Q(z), E(z), L(z)\}$ and $F(z, u) = g(uh(z))$ be a subcritical composition schema, i.e.

$\tau_h < 1$ (for sequences and cycles) and $\tau_h < \infty$ (for sets), respectively.

Then the number of components, X_n with $\mathbf{P}\{X_n = k\} = f_{n,k}/f_n$, converges to a discrete limit law. Precisely, p_k equals

- $(1 - \tau_h)^2 k \tau_h^{k-1}$ (sequences; neg. binomial $NB(2, 1 + 2\tau_h)$),
- $e^{-\tau_h} \frac{\tau_h^{k-1}}{(k-1)!}$ (sets; Poisson with mean $1 + \tau_h$),
- $(1 - \tau_h) \tau_h^{k-1}$ (cycles; geometric with mean $1 + \tau_h$).

Proof: We learned that

$$p(u) = \lim_{n \rightarrow \infty} \frac{[z^n]g(uh(z))}{[z^n]g(h(z))} = \frac{ug'(u\tau_h)}{g'(\tau_h)}$$

with $g(z) \in \{Q(z), E(z), L(z)\}$. As

$$Q'(w) = \frac{1}{(1-w)^2}, \quad E'(w) = e^w, \quad L'(w) = \frac{1}{1-w},$$

we get the PGFs

$$p_Q(u) = u \left(\frac{1 - \tau_h}{1 - \tau_h u} \right)^2, \quad p_E(u) = ue^{\tau_h(u-1)}, \quad p_L(u) = u \frac{1 - \tau_h}{1 - \tau_h u}.$$

Finally, $p_k = [u^k]p_g(u)$ for $g(z) \in \{Q(z), E(z), L(z)\}$. □

Example (Root degree in plane trees)

Problem is equivalent to finding the number of **components in sequences of rooted plane trees** (ordered forests).

Set

$$F(z, u) = \frac{1}{1 - uh(z)}, \quad h(z) = \frac{1 - \sqrt{1 - 4z}}{2} \quad \implies \tau_h = \frac{1}{2} < \rho_g = 1$$

with $g(w) = 1/(1 - w)$. $\implies$ subcritical schema

$$\implies \lim_{n \rightarrow \infty} \mathbf{P}\{X_n = k\} = k2^{-k-1}.$$

Similar for Cayley trees: $F(z, u) = e^{uh(z)}$, $h(z) = ze^{h(z)}$

$$\implies \tau_h = 1 < \rho_g = \infty.$$

$$\implies \lim_{n \rightarrow \infty} \mathbf{P}\{X_n = k\} = e^{-1}/(k-1)!$$

Continuous limit laws

Consider combinatorial structures $\mathcal{F}$ with parameter χ inducing a random variable X .

That means

$$\mathbf{P}\{X_n = k\} = \frac{|\{\omega \in \mathcal{F}_n : \chi(\omega) = k\}|}{|\mathcal{F}_n|}$$

$\mu_n = \mathbf{E}X_n \rightarrow \infty$, $\sigma_n^2 = \mathbf{Var}X_n \rightarrow \infty \implies$ continuous limiting distribution.

distribution function $F(x) = \mathbf{P}\{X \leq x\}$,

If $F(x) \in C^1(\mathbb{R})$ then $\exists$ density $f(x) = F'(x)$.

$$\mathbf{E}g(X) = \int_{-\infty}^{\infty} g(t) \, dF(t) = \int_{-\infty}^{\infty} g(t)f(t) \, dt.$$

Definition

Let X be a continuous RV, F_X its distribution function.

Let X_n be RVs with distribution functions F_{X_n} .

Then X_n **converges in distribution** (or **converges weakly**) to X (we write then $X_n \xrightarrow{d} X$), if $\forall x \in \mathbb{R} : \lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x)$.

Definition

If X is a RV, $\mu = \mathbf{E}X$, $\sigma^2 = \mathbf{Var}X$, then

$$X^* = \frac{X - \mu}{\sigma}$$

is the **standardized RV** of X . We have

$$F_{X^*}(x) = F_X(\mu + \sigma x), \quad F_X(x) = F_{X^*}\left(\frac{x - \mu}{\sigma}\right).$$

Continuous limit laws

Further functionals:

Moment generating function (Laplace transform):

$$\lambda_X(s) = \mathbf{E}e^{sX} = \int_{-\infty}^{\infty} e^{sx} dF(x), \quad \lambda_{X^*}(s) = e^{-\mu s/\sigma} \lambda_X\left(\frac{s}{\sigma}\right).$$

Characteristic function (Fourier transform):

$$\phi_X(t) = \mathbf{E}e^{itX} = \int_{-\infty}^{\infty} e^{itx} dF(x), \quad \phi_{X^*}(t) = e^{-i\mu t/\sigma} \phi_X\left(\frac{t}{\sigma}\right).$$

Central moments:

$$\mu_r = \mathbf{E}X^r = \frac{d^r}{ds^r} \lambda_X(s) \Big|_{s=0} = (-i)^r \frac{d^r}{dt^r} \phi_X(t) \Big|_{t=0}.$$

Definition

X_n satisfies a **continuous limit theorem** if there is a continuous RV Y such that $X_n^* \xrightarrow{d} Y$.

Theorem (Continuity theorem)

Let X, X_n be RVs with characteristic functions $\phi(t), \phi_n(t)$ and F_X be continuous. Then

$$\begin{aligned} X_n \xrightarrow{d} X &\iff \forall t \in \mathbb{R} : \lim_{n \rightarrow \infty} \phi_n(t) = \phi(t) \\ &\iff \exists \delta > 0 : \forall t \in (-\delta, \delta) : \lim_{n \rightarrow \infty} \phi_n(t) = \phi(t) \end{aligned}$$

Note: X, Y RVs, then $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$.

Theorem (Central limit theorem)

$X_1, X_2, \dots$ i.i.d. RVs, $\mathbf{E}X_i = \mu$, $\mathbf{Var}X_i = \sigma^2$. $S_n = \sum_{i=1}^n X_i$

Then $S_n^* = \frac{S_n - \mu n}{\sigma \sqrt{n}} \rightarrow \mathcal{N}(0, 1)$.

Continuous limit laws

Proof:

$$\begin{aligned}
 \phi_{S_n^*}(t) &= \phi_{X_1 - \mu} \left(\frac{t}{\sigma\sqrt{n}} \right)^n \\
 &= \left(1 + i\mathbf{E}(X_1 - \mu) \frac{t}{\sigma\sqrt{n}} - \frac{\mathbf{E}(X_1 - \mu)^2}{\sigma^2 n} \frac{t^2}{2} + O\left(\frac{t^3}{n^{3/2}}\right) \right)^n \\
 &= \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right) \right)^n \longrightarrow e^{-t^2/2}
 \end{aligned}$$

□

Example (Cycles in permutations)

$$p_n(u) = \binom{n+u-1}{n} = \frac{\Gamma(n+u)}{\Gamma(u)n!}$$

$$p_n(u) = \frac{n^{u-1}}{\Gamma(u)} \left(1 + O\left(\frac{1}{n}\right) \right) \sim (e^{u-1})^{\log n}$$

Large power! $\implies$ CLT

Theorem (Quasi-power theorem, Hwang 1997)

Suppose $X_n \geq 0$ are discrete RVs PGF $p_n(u)$.

Moreover, assume that

- there are sequences β_n and κ_n both tending to ∞ ,
- there are functions $A(u)$ and $B(u)$ which are analytic at $u = 1$ and satisfy $A(1) = B(1) = 1$,
- $B''(1) + B'(1) - B'(1)^2 \neq 0$ (*variability condition*) such that

$$p_n(u) = A(u)B(u)^{\beta_n} \left(1 + O\left(\frac{1}{\kappa_n}\right) \right),$$

uniformly for $|u - 1| < \varepsilon$. Then, X_n is asymptotically normal with mean and variance

$$\begin{aligned} \mathbf{E}X_n &= \beta_n B'(1) + A'(1) + O(\kappa_n^{-1}) \\ \mathbf{Var}X_n &= \beta_n (B''(1) + B'(1) - B'(1)^2) + A''(1) + A'(1) - A'(1)^2 \\ &\quad + O(\kappa_n^{-1}) \end{aligned}$$

Lemma

Let $\lambda_n(s) = \mathbf{E}e^{sX_n}$ be analytic in $|s| < \rho$.

Assume

$$\lambda_n(s) = e^{\beta_n U(s) + V(s)} \left(1 + O\left(\frac{1}{\kappa_n}\right) \right), \text{ as } n \rightarrow \infty,$$

with $\beta_n, \kappa_n \rightarrow \infty$, $U(s), V(s)$ analytic in $|z| \leq \rho$, $U''(0) \neq 0$.

Set

$$\mu_n = \beta_n U'(0) + V'(0) + O\left(\frac{1}{\kappa_n}\right), \quad \sigma_n = \beta_n U''(0) + V''(0) + O\left(\frac{1}{\kappa_n}\right).$$

Then $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ where the speed of convergence is $O\left(\frac{1}{\kappa_n} + \frac{1}{\sqrt{\beta_n}}\right)$.

Proof: $\log \lambda_n(s) = \beta_n U(s) + V(s) + O(1/\kappa_n)$ is analytic at 0

Thus

$$\frac{\lambda'_n(0)}{\lambda_n(0)} = \lambda'_n(0) = \beta_n U'(0) + V'(0) + O\left(\frac{1}{\kappa_n}\right) = \mu_n$$

and similarly for σ_n^2 .

Set

$$X_n^* = \frac{X_n - \beta_n U'(0)}{\sqrt{\beta_n U''(0)}}, \quad \lambda_n^*(s) = \mathbf{E} e^{sX_n^*}.$$

Then

$$\log \lambda_n^*(s) = -\frac{\beta_n U'(0)}{\sqrt{\beta_n U''(0)}} s + \log \lambda_n \left(\frac{s}{\sqrt{\beta_n U''(0)}} \right)$$

We had

$$\log \lambda_n^*(s) = -\frac{\beta_n U'(0)}{\sqrt{\beta_n U''(0)}} s + \log \lambda_n \left(\frac{s}{\sqrt{\beta_n U''(0)}} \right).$$

Now expand $\lambda_n(s) = e^{\beta_n U(s) + V(s)}(1 + E(s))$ locally around 0,

$$\log \lambda_n(s) = (\beta_n U'(0) + V'(0))s + (\beta_n U''(0) + V''(0))\frac{s^2}{2} + \dots$$

then integrate over small circle around origin (to bound $E(s)$):

$$\log \lambda_n^*(s) = \frac{s^2}{2} + O\left(\frac{|s| + |s|^3}{\sqrt{\beta_n}}\right) + O\left(\frac{|s|}{\kappa_n \sqrt{\beta_n}}\right),$$

for $|s| < \sqrt{\beta_n}$.

Speed of convergence by Berry-Esseen inequality.



Example (Number of blocks in ordered set partitions)

$$\mathcal{F} = \text{Seq}(\mu\text{Set}_{\geq 1}(\mathcal{Z})) \quad \implies \quad f(z, u) = \frac{1}{1 - u(e^z - 1)}$$

where $f(z, u) = \sum_{n \geq 0} \sum_{k \geq 0} f_{n,k} \frac{z^n}{n!} u^k$ with
 $f_{n,k} = \#$ ordered set partitions having k blocks
 Set

$$\rho(u) = \log \left(1 + \frac{1}{u} \right),$$

then the singularities are

$$\rho(u) + 2k\pi i.$$

Example (Number of blocks in ordered set partitions—cont'd)

We obtain therefore

$$\begin{aligned} f_n(u) &= [z^n]f(z, u) = -\text{Res} \left(\frac{f(z, u)}{z^{n+1}} \right) + \frac{1}{2\pi i} \oint_{|z|=6} \frac{f(z, u)}{z^{n+1}} dz \\ &= \frac{1}{u\rho(u)e^{\rho(u)}} \rho(u)^{-n} + O(6^{-n}) \end{aligned}$$

uniformly for $|u - 1| < \varepsilon$. Expand

$$\log \left(1 + \frac{1}{u} \right) = \log 2 - \frac{1}{2}(u - 1) + \frac{3}{8}(u - 1)^2 + \dots$$

which implies

$$\frac{\rho(1)}{\rho(u)} = 1 + \frac{1}{2 \log 2}(u - 1) - \frac{3 \log 2 - 2}{8 \log^2 2}(u - 1)^2 + ((u - 1)^3).$$

Example (Number of blocks in ordered set partitions—cont'd)

We had

$$\frac{\rho(1)}{\rho(u)} = 1 + \frac{1}{2 \log 2}(u-1) - \frac{3 \log 2 - 2}{8 \log^2 2}(u-1)^2 + ((u-1)^3). \quad (*)$$

The PGF is

$$p_n(u) = \frac{f_n(u)}{f_n(1)} = \frac{2 \log 2}{u \rho(u) e^{\rho(u)}} \left(\frac{\rho(1)}{\rho(u)} \right)^n \left(1 + O \left(\left(\frac{\log 2}{6} \right)^n \right) \right)$$

and thus of quasi-power shape with $B(u) = \rho(1)/\rho(u)$.

From (*) we get

$$B'(1) = \frac{1}{2 \log 2} \quad \text{and} \quad B''(1) = -\frac{3 \log 2 - 2}{4 \log^2 2}$$

and $B''(1) + B'(1) - B'(1)^2 \approx 0.15967 \neq 0$.

Example (Number of blocks in ordered set partitions—cont'd)

Thus the number of blocks in ordered set partitions is asymptotically normal with mean and variance

$$\mu_n \sim \frac{1}{2 \log 2} n,$$

$$\sigma_n^2 \sim \frac{1 - \log 2}{4 \log^2 2} n.$$

Speed of convergence: $\kappa_n = \left(\frac{\log 2}{6}\right)^n$, $\beta_n = n$. So, we get

$$O\left(\frac{1}{\sqrt{n}}\right).$$