

Analytic Combinatorics: Complex-analytic Methods and Applications

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Application: meromorphic schema

Theorem

Let $F(z, u)$ be analytic in $\Omega = \{(z, u) \mid |z| < \rho, |u| < 1\}$, having nonnegative coefficients when expanded at $(0, 0)$.

Assume that there is $\varepsilon > 0$ and $r \geq \rho$ such that in

$D = \{(z, u) \mid |z| \leq r, |u - 1| \leq \varepsilon\}$ we have

- $F(z, u) = \frac{B(z, u)}{C(z, u)}$ with $B(z, u)$ and $C(z, u)$ analytic in D ,
- $C(z, 1)$ has the unique (simple) zero $z = \rho$ in $|z| \leq r$,
- $B(\rho, 1) \neq 0$ and $(\frac{\partial}{\partial z} C)(\rho, 1) \neq 0$.

Moreover, assume that the solution $\rho(u)$ of $C(\rho(u), u) = 0$ satisfies **Var** $(\rho(1)/\rho(u)) \neq 0$ where

$$\mathbf{Var} f = \frac{f''(1)}{f(1)} + \frac{f'(1)}{f(1)} - \left(\frac{f'(1)}{f(1)} \right)^2.$$

Theorem (cont'd)

Let X_n be a RV with PGF

$$p_n(u) = \frac{[z^n]F(z, u)}{[z^n]F(z, 1)}.$$

Then $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ with

$$\mu_n = \left[\left(\frac{\rho(1)}{\rho(u)} \right)' \right]_{u=1} \cdot n + O(1)$$

and

$$\sigma_n^2 = \mathbf{Var} \left(\frac{\rho(1)}{\rho(u)} \right) \cdot n + O(1).$$

Proof: Set

$$f_n(u) = [z^n]F(z, u) = \frac{1}{2\pi i} \oint F(z, u) \frac{dz}{z^{n+1}}.$$

By the implicit function theorem, $\rho(u)$ def'd by $C(\rho(u), u) \equiv 0$ is analytic for $|u - 1| < \delta$.

Then

$$f_n(u) = -\text{Res} \left(\frac{B(z, u)}{C(z, u)} z^{-n-1}; z = \rho(u) \right) + \underbrace{\frac{1}{2\pi i} \oint_{|z|=r} F(z, u) \frac{dz}{z^{n+1}}}_I$$

$$|I| \leq \frac{\sup_{|z|=r, |u-1|<\delta} |B(z, u)|}{\inf_{|z|=r, |u-1|<\delta} |C(z, u)|} r^{-n}$$

The domain for sup/inf is compact, $\inf = \min \neq 0$ and $|B(z, u)|$ is bounded.

So,

$$f_n(u) = -\frac{B(\rho(u), u)}{\left(\frac{\partial}{\partial z} C\right)(\rho(u), u)} \rho(u)^{-n-1} + O(r^{-n})$$

uniformly for $|u - 1| < \delta$. Consequently,

$$\begin{aligned}\mu_n &= \left[\left(\frac{\rho(1)}{\rho(u)} \right)' \right]_{u=1} \cdot n + O(1) = -\frac{\rho'(1)}{\rho(1)} n + O(1), \\ \sigma_n^2 &= \mathbf{Var} \left(\frac{\rho(1)}{\rho(u)} \right) \cdot n + O(1).\end{aligned}$$

As $\mathbf{Var} \left(\frac{\rho(1)}{\rho(u)} \right) \neq 0$, the variability condition is satisfied.

$$\xrightarrow{QPT} X_n \sim \mathcal{N}(\mu_n, \sigma_n^2).$$



Example (Number of ascents in permutations)

Let $x_1 x_2 \cdots x_n$ be a permutation of $1, 2, \dots, n$.

Ascent: $x_i x_{i+1}$ with $x_i < x_{i+1}$

$a_{n,m}$ = number of permutations with exactly m ascents.

Insert $n + 1$:

- If $n + 1$ is inserted at the beginning or into an ascent ($x_i < x_{i+1}$, $x_i < n + 1 > x_{i+1}$), then the number of ascents does not change ($m + 1$ cases);
- If $n + 1$ is inserted into a descent or appended at the end ($x_i > x_{i+1}$, $x_i < n + 1 > x_{i+1}$), then the number of ascents is increased by 1 ($n - m$ cases).

$$\implies a_{n+1,m} = (m+1)a_{n,m} + (n+1-m)a_{n,m-1}, \quad (n \geq 1)$$

Example (Number of ascents in permutations – cont'd)

Set $F(z, u) = \sum_{n \geq 1} \sum_{m \geq 0} a_{n,m} u^m \frac{z^n}{n!}$ then

$$\begin{aligned} \sum_{n \geq 1} (n+1) \frac{z^n}{(n+1)!} \sum_{m \geq 0} a_{n+1,m} u^m &= \sum_{n \geq 1} \frac{z^n}{(n+1)!} \sum_{m \geq 0} (m+1) a_{n,m} u^m \\ &\quad + \sum_{n \geq 1} (n+1-m) \frac{z^n}{(n+1)!} \sum_{m \geq 0} a_{n,m-1} u^m \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial z}(F(z, u) - z) &= F_z - 1 = \frac{\partial}{\partial u}(uF) + \frac{\partial}{\partial z}(uzF) - u \frac{\partial}{\partial u}(uF) \\ (uz - 1)F_z + u(1 - u)F_u + F + 1 &= 0, \end{aligned}$$

boundary conditions

$$F(0, u) = \sum_{m \geq 0} a_{0,m} u^m = 0, \quad F(z, 0) = \sum_{n \geq 1} a_{n,0} \frac{z^n}{n!} = e^z - 1.$$

Example (Number of ascents in permutations – cont'd)

Let $v = \log u + z(u - 1)$, $w = \log u - z(u - 1)$. Then

$$F_z = (F_v - F_w)(u - 1), \quad F_u = F_v \left(\frac{1}{u} + z \right) + F_w \left(\frac{1}{u} - z \right).$$

Therefore (note that $(uz - 1)F_z + u(1 - u)F_u + F + 1 = 0$)

$$2(1 - u)F_v + F + 1 = 0$$

and, as $u = e^{(v+w)/2}$,

$$\frac{F_v}{F + 1} = \frac{1}{2(u - 1)} = \frac{1}{2(e^{(v+w)/2} - 1)}$$

Now integrate w.r.t. v :

Example (Number of ascents in permutations – cont'd)

$$\begin{aligned} \log(F + 1) &= \int \frac{dv}{2(e^{(v+w)/2} - 1)} + \tilde{f}(w) \\ &\stackrel{2du/u=dv}{=} \int \frac{du}{u(u-1)} + \tilde{f}(w) = \log \frac{u-1}{u} + \tilde{f}(w) \end{aligned}$$

Hence

$$F + 1 = \frac{u-1}{u} f(w), \quad F(0, u) = 0 \implies 1 = \frac{u-1}{u} f(\log u).$$

So,

$$f(w) = \frac{e^w}{e^w - 1} \implies F = \frac{(1-u)e^{z(1-u)}}{1 - ue^{z(1-u)}} - 1 = \frac{1-u}{e^{z(u-1)} - u} - 1$$

Example (Number of ascents in permutations – cont'd)

$$F(z, u) = \frac{1 - u}{e^{z(u-1)} - u} - 1, \quad \text{Rem.: } F(z, 1) = \frac{1}{1 - z} - 1$$

The poles are

$$\rho_k(u) = \rho(u) + \frac{2k\pi i}{u - 1}, \quad \rho(u) = \frac{\log u}{u - 1}$$

If $u \sim 1$ then $\rho(u) \sim 1$, $\rho_k(u) \rightarrow \infty$.

For $|u| \leq 2$: one pole in $|z| \leq 2$, namely $\rho(u)$. Thus

$$F(z, u) = \frac{1}{\rho(u) - z} + O(2^{-n}), \text{ as } u \rightarrow 1, \quad [z^n]F(z, u) = \rho(u)^{-n-1} + O(2^{-n})$$

Example (Number of ascents in permutations – cont'd)

Variability condition follows from

$$\rho(u) = \frac{\log u}{u-1} = 1 - \frac{1}{2}(u-1) + \frac{1}{3}(u-1)^2 + \dots,$$

from which we get $\rho(1) = 1$, $\rho'(1) = -1/2$, $\rho''(1) = 2/3$ and so

$$\begin{aligned} \text{Var} \left(\frac{\rho(1)}{\rho(u)} \right) &= \left(\left(\frac{1}{\rho} \right)'' + \left(\frac{1}{\rho} \right)' - \left(\left(\frac{1}{\rho} \right)' \right)^2 \right) \\ &= (\rho')^2 - \rho' - \rho'' = \frac{1}{12} > 0 \end{aligned}$$

Finally, $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ with

$$\mu_n = \frac{n}{2} + O(1), \quad \sigma_n^2 = \frac{n}{12} + O(1).$$

The vectorial kernel method

Functional equations and the kernel method

underdetermined equations with **unknown** and **known** functions:

$$K(z, u)F(z, u) = A(z, u) + B(z, u)G(z)$$

$K(z, u)$... kernel

Choose $u = u(z)$ s.t. $K(z, u(z)) = 0$

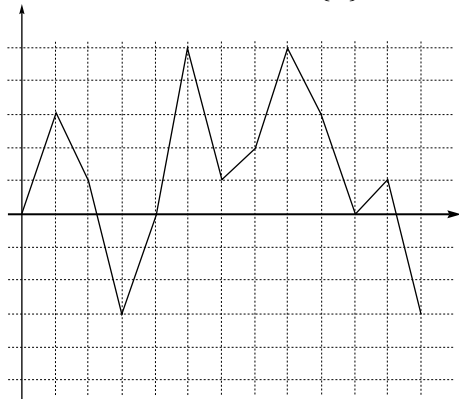
If there is one branch $u_1(z)$, then $G(z) = -\frac{A(z, u_1(z))}{B(z, u_1(z))}$

More generally:

$$K(z, u)F(z, u) = A(z, u) + \sum_{i=1}^m B_i(z, u)G_i(z)$$

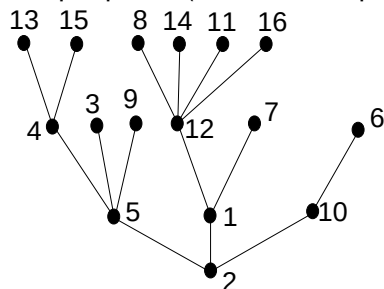
directed lattice path: step set $\mathcal{S} \subseteq \mathbb{N} \times \mathbb{Z}$

“one-dimensional”: $\mathcal{S} \subseteq \{1\} \times \mathbb{Z}$



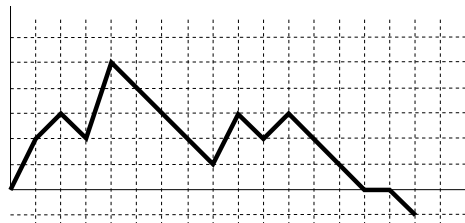
simple or Łukasiewicz paths: $\mathcal{S} = \{1\} \times (\mathbb{N} \cup \{-1\})$
(this corresponds to plane trees)

Simple paths (Łukasiewicz paths): $\mathcal{S} \subseteq \{1\} \times (\mathbb{N} \cup \{-1\})$



degree sequence: 3, 2, 0, 4, 0, 0, 0, 0, 3, 0, 2, 0, 0, 0, 1, 0

jump sequence: 2, 1, -1, 3, -1, -1, -1, -1, 2, -1, 1, -1, -1, -1, 0, -1



Lattice paths staying above x -axis:

Steps encoded by $P(u) = \sum_s u^s$

$$\begin{aligned} F(t, u) &= 1 + tP(u)F(t, u) - t\{u^{<0}\}P(u)F(t, u) \\ &= 1 + tP(u)F(t, u) - t \sum_{k=0}^{c-1} F_k(t) \{u^{<0}\} (P(u)u^k) \end{aligned}$$

Parameters of lattice paths, e.g. the area below the walk

$$F(t, u, q) = 1 + tP(uq)F(t, uq^2, q) - t \sum_{k=0}^{c-1} F_k(t, q) q^k \{u^{<0}\} (P(u)u^k)$$

Directed walks

Step set: $S \subseteq \mathbb{Z}$, finite, contains a positive and a negative element.

lattice path: finite word $w = [v_1, v_2, \dots, v_n]$ with $v_i \in S$

which corresponds to the steps $\begin{pmatrix} 1 \\ v_1 \end{pmatrix}, \begin{pmatrix} 1 \\ v_2 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ v_n \end{pmatrix}$

length of w is the number steps: $|w|$

final altitude: $\text{alt}(w) = v_1 + v_2 + \dots + v_n$

walk starts at the origin and ends at $(|w|, \text{alt}(w))$

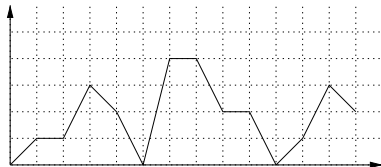
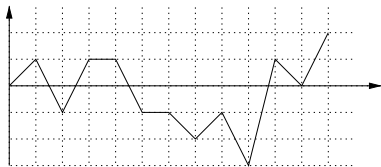
step polynomial (Laurent polynomial):

$$P(u) = \sum_{s \in S} u^s, \quad \min S = -c, \quad \max S = d$$

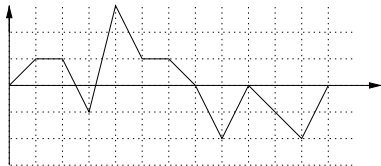
$1 - tP(u) = 0$ has c small roots and d large roots.

Restrictions

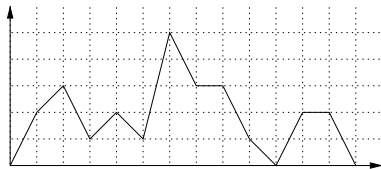
(free) walks: gen. function $W(t, u)$, meanders: $M(t, u)$,



bridges: $B(t)$,



excursions: $E(t)$,



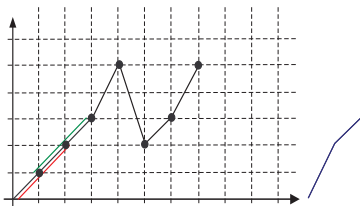
Patterns

pattern: $p = [a_1, \dots, a_\ell]$ with $a_i \in S$

occurrence of p : contiguous substring v of w with $v \equiv p$

Examples:

- Dyck paths ($S = \{-1, 1\}$) avoiding peaks ($p = [1, -1]$)
- Motzkin paths ($S = \{-1, 0, 1\}$) avoiding peaks
- $[1, 1, 1, 2, -3, 1, 2]$:



two occurrences of $[1, 1]$, avoids $[2, 1]$

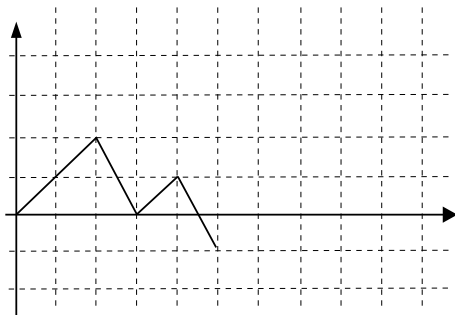
presuffix of p : non-empty word q that is a prefix and a suffix of p

trivial presuffix: p

If there are non-trivial presuffixes: p exhibits **autocorrelation**.

Examples:

- $p_1 = [1, 1, -2, 1 - 2]$: no autocorrelation



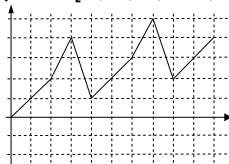
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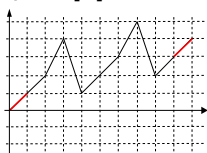
If there are non-trivial presuffixes: p exhibits **autocorrelation**.

Examples:

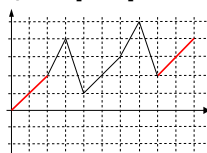
■ $p_2 = [1, 1, 2, -3, 1, 1, 2, -3, 1, 1]$:



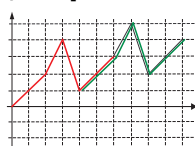
$q_1 = [1]$



$q_2 = [1, 1]$



$q_3 = [1, 1, 2, -3, 1, 1]$



Autocorrelation

complements: $\bar{q}$ is a complement of a presuffix q if $p = q.\bar{q}$

autocorrelation polynomial: Guibas and Odlyzko (1981)

Let $Q := \{q \mid q \text{ is a presuffix of } p\}$. Then

$$R(t, u) := \sum_{q \in Q} t^{|\bar{q}|} u^{\text{alt}(\bar{q})}$$

■ $p_1 = [1, 1, -2, 1 - 2]$: $R(t, u) = 1$

■ $p_2 = [1, 1, 2, -3, 1, 1, 2, -3, 1, 1]$:

q	$\bar{q}$	contribution
p	ε	1
$[1, 1, 2, -3, 1, 1]$	$[2, -3, 1, 1]$	$t^4 u$
$[1, 1]$	$[2, -3, 1, 1, 2, -3, 1, 1]$	$t^8 u^2$
$[1]$	$[1, 2, -3, 1, 1, 2, -3, 1, 1]$	$t^9 u^3$

So, $R(t, u) = 1 + t^4 u + t^8 u^2 + t^9 u^3$.

The kernel

The **kernel** of a walk w.r.t. p :

$$K(t, u) := (1 - tP(u))R(t, u) + t^{|p|}u^{\text{alt}(p)}$$

Compare with the classical lattice path theory
(Banderier, Flajolet (2002)): $K(t, u) := 1 - tP(u)$.

We have

$$K(t, u) = \sum_{i \in I} t^{\alpha_i} u^{\beta_i}$$

Set

$$e := -\min_{i \in I} \beta_i \quad \text{and} \quad f := \max_{i \in I} \beta_i.$$

If there is no autocorrelation, then

$$e = \max\{c, -\text{alt}(p)\}, \text{ and } f = \max\{d, \text{alt}(p)\}.$$

$K(t, u) = 0$ has e small roots and f large roots.

(root: $u(t)$ s.t. $K(t, u(t)) = 0$)

GF for walks with forbidden patterns

Theorem (Free walks)

$W(t, u) \dots$ g.f. of free p -avoiding walks, where $p = [a_1, \dots, a_\ell]$.
Then

$$W(t, u) = \frac{R(t, u)}{K(t, u)}, \quad W(t) := W(t, 1) = \frac{1}{1 - tP(1) + \frac{t^\ell}{R(t, 1)}}$$

Theorem (Bridges)

$B(t) \dots$ g.f. of p -avoiding bridges, where $p = [a_1, \dots, a_\ell]$. Then

$$B(t) = - \sum_{i=1}^e \frac{u'_i(t)}{u_i(t)} \frac{R(t, u_i(t))}{K_t(t, u_i(t))}$$

where $u_1(t), \dots, u_e(t)$ are the small roots of the kernel equation $K(t, u) = 0$ and $K_t = \frac{\partial}{\partial t} K$.

Theorem (Meanders)

$M(t, u) \dots$ g.f. of p -avoiding meanders. Then there is a function $G(t, u) \in \mathbb{R}[[t]][u]$ such that

$$M(t, u) = \frac{G(t, u)}{u^e K(t, u)} \prod_{i=1}^e (u - u_i(t))$$

where $u_1(t), \dots, u_e(t)$ are the small roots of the kernel equation $K(t, u) = 0$.

Theorem (Excursions)

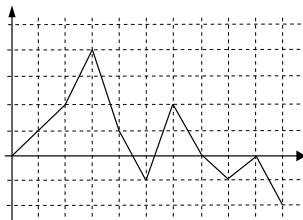
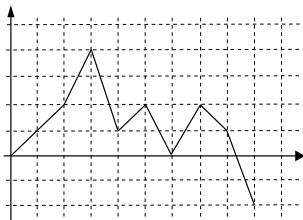
$E(t) \dots$ g.f. of p -avoiding excursions. Then

$$E(t) = M(t, 0) = \frac{G(t, 0)(-1)^e}{\lim_{u \rightarrow 0} u^e K(t, u)} \prod_{i=1}^e u_i(t).$$

Special cases:

Definition

A pattern $p = [a_1, \dots, a_\ell]$ is a **quasi-meander** if for all $1 \leq i \leq \ell - 1$ the inequality $\text{alt}([a_1, \dots, a_i]) \geq 0$ holds.



Definition

A pattern $p = [a_1, \dots, a_\ell]$ is a **reversed meander** if for all $0 \leq i \leq \ell - 1$ the inequality $\text{alt}([a_1, \dots, a_i]) > \text{alt}(p)$ holds. In particular, $\text{alt}(p) < 0$.

Theorem (Meanders avoiding a quasi-meander)

$M(t, u) \dots$ g.f. of p -avoiding meanders

$E(t) \dots$ g.f. of p -avoiding excursions, where p is a quasi-meander. Then $e = c$ and

$$M(t, u) = \frac{R(t, u)}{u^c K(t, u)} \prod_{i=1}^c (u - u_i(t))$$

where $u_1(t), \dots, u_e(t)$ are the small roots of $K(t, u) = 0$.

Moreover,

$$E(t) = \begin{cases} \frac{(-1)^c}{-t} \prod_{i=1}^c u_i(t) & \text{if } \text{alt}(p) > -c, \\ \frac{(-1)^c}{t^\ell - t} \prod_{i=1}^c u_i(t) & \text{if } \text{alt}(p) = -c. \end{cases}$$

Theorem (Meanders avoiding a reversed meander)

$M(t, u)$... g.f. of p -avoiding meanders

$E(t)$... g.f. of p -avoiding excursions, where p is a reversed meander. Then

$$M(t, u) = \frac{1}{u^e K(t, u)} \prod_{i=1}^e (u - u_i(t))$$

where $u_1(t), \dots, u_e(t)$ are the small roots of $K(t, u) = 0$.

Moreover,

$$E(t) = \frac{(-1)^e}{D(t)} \prod_{i=1}^e u_i(t)$$

where $D(t) := [u^0]u^e K(t, u)$ is either some power of t or a difference of two powers of t .